U.P.B. Sci. Bull., Series A, Vol. 87, Iss. 1, 2025 ISSN 1223-7027

NEW VISCOSITY ALGORITHM WITH STRONG CONVERGENCE FOR QUASIMONOTONE VARIATIONAL INEQUALITIES

Duong Viet Thong[1]

This paper investigates quasimonotone and Lipschitz continuous variational inequalities in real Hilbert spaces. To address this problem, we propose a new iterative algorithm for finding an element of the solution set of the quasimonotone variational inequality problem. Our approach combines techniques from the inertial modified subgradient extragradient algorithm and the viscosity approximation method. Using a new self adaptive stepsize, we establish a strong convergence theorem for the sequence generated by the proposed algorithm under appropriate conditions. The results presented in this work improve upon and generalize some recent findings in this area.

Keywords: Subgradient extragradient method, viscosity method, variational inequality problem, quasimonotone mapping, strong convergence.

MSC2020: 47H09, 47H10, 47J20, 47J25.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be a single-valued mapping.

We study the classical variational inequality (VI) as formulated by Fichera [18] and Stampacchia [32] (see also Kinderlehrer and Stampacchia [24]) which is: Find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{1}$$

The solution set of VI is denoted by S.

The dual variational inequality of VI is to find $x^* \in C$ such that

$$\langle Fx, x - x^* \rangle \geq 0, \forall x \in C.$$

We denote the solution set of the dual VI by S_D. S_D is clearly a closed convex set (possibly empty). When F is continuous and C is convex, $S_D \subset S$. If F is pseudomonotone and continuous, then $S = S_D$ (see, Lemma 2.1 in [13]). The inclusion $S \subset S_D$ is false if F is quasimonotone and continuous (see, Example 4.2 in [39]).

We also use S_T and S_N for the trivial and nontrivial solution sets of VI; that is,

$$S_T = \{x^* \in C | \langle Fx^*, x - x^* \rangle = 0, \forall x \in C\},$$

$$S_N = S \backslash S_T.$$

We assume that the following conditions hold:

Condition 1.1. $S_D \neq \emptyset$.

[1] Fundamental Sciences Faculty, National Economics University, Hanoi City, Vietnam, e-mail: thongduongviet@neu.edu.vn

Condition 1.2. *The mapping $F : H \to H$ is L-Lipschitz continuous on H. However, the information of L is not necessary to be known.*

Condition 1.3. *The mapping F is sequentially weakly continuous on C, i.e., for each sequence $\{x_n\} \subset C$: $\{x_n\}$ converges weakly to x^* implies $\{Fx_n\}$ converges weakly to Fx^*.*

Condition 1.4. *The mapping F is quasimonotone on H.*

Variational inequality (VI) is a very general mathematical model with numerous applications in fields like economics, engineering, transportation, and more (see [1, 7, 9, 17, 24, 25]). Over the past decades, many algorithms have been developed for solving VIs, including extragradient methods [26], projection and contraction techniques [8, 20, 33], and various splitting methods [36].

Korpelevich [26] (independently of Antipin [4]) introduced the extragradient method for solving monotone variational inequalities, which requires two projections onto the feasible set per iteration. An important extension is the subgradient extragradient method proposed by Censor et al. [10, 11, 12], where the second projection is replaced by one onto a half-space containing the feasible set. Since half-space projections have an explicit form, this reduces computational complexity compared to the extragradient method. Recently, approaches combining the advantages of projection contraction methods [20, 33] and the subgradient extragradient method have been studied [14, 15, 30, 36].

We now discuss an inertial-type algorithm based on a discrete version of a second-order dissipative dynamical system [5, 6]. This approach can be viewed as a procedure for accelerating convergence, as discussed in [3, 29]and references therein. Recently, several authors have studied inertial methods when the operator F is quasimonotone (or non-monotone), see [2, 21, 22, 28, 37, 38, 39] and references therein. These works analyze the convergence properties of inertial-type algorithms and demonstrate their numerical performance on various imaging and data analysis problems.

To the best of the authors' knowledge, the study of strong convergence for solving quasimonotone variational inequalities in the Hilbert space setting remains unexplored. This motivates the following research question: Can we establish strong convergence results for an inertial subgradient extragradient method to solve quasimonotone variational inequalities?

Motivated by the existing literature, this paper aims to address the aforementioned research question. Specifically, we introduce a new inertial subgradient extragradient method for finding an element in the solution set of a quasimonotone, Lipschitz-continuous variational inequality problem.

The first proposed iterative method combines two well-established techniques: the inertial modified subgradient extragradient method [15, 35] and the viscosity approximation method [27]. Our proposed algorithm computes only one projection onto the closed convex set C per iteration and employ self-adaptive step sizes to approximate a solution of the quasimonotone variational inequality problem. Moreover, we use a novel self adaptive stepsize which may increase a positive value and has never been used in literature before. This approach represents a novel and state-of-the-art contribution to the study of inertial extragradient methods for solving variational inequality problems.

In Section 2, we provide some standard definitions and preliminary concepts. We then introduce our proposed algorithm and establish the strong convergence of the iterative sequence to a solution of the considered variational inequality in Section 3. Finally, we present our conclusions in Section 4.

2. **Preliminaries**

The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in H$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\| \; \forall y \in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive. For properties of the metric projection, the interested reader could be referred to [19, Section 3].

We need to recall the following results and properties, which are useful for the later convergence analysis.

Lemma 2.1. ([19]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C(x) \Longleftrightarrow \langle x - z, z - y\rangle \geq 0 \;\; \forall y \in C$. Moreover,*

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle \; \forall x, y \in C.$$

Definition 2.1. ([9]) *Let $F : H \to H$ be a mapping. Then the mapping F is said to be:*

(1) *L-Lipschitz continuous with $L > 0$ if*

$$\|F(x) - F(y)\| \leq L\|x - y\| \;\; \forall x, y \in H.$$

(2) *monotone if*

$$\langle F(x) - F(y), x - y\rangle \geq 0 \;\; \forall x, y \in H.$$

(3) *pseudomonotone in the sense of Karamardian* [23] *if*

$$\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0 \;\; \forall x, y \in H.$$

(4) *quasimonotone, if*

$$\langle F(x), y - x\rangle > 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0 \;\; \forall x, y \in H.$$

(5) *δ-strongly pseudomonotone if there exists a constant $\delta > 0$ such that*

$$\langle F(x), x - y\rangle \geq 0 \Longrightarrow \langle F(y), x - y\rangle \geq \delta\|x - y\|^2 \;\; \forall x, y \in H.$$

(6) *sequentially weakly continuous if, for each sequence $\{x_n\}$ in H, $\{x_n\}$ converges weakly to a point $x \in H$ implies $\{F(x_n)\}$ converges weakly to $F(x)$.*

It is easy to see that every (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) but the converse is not true.

The following lemma provides some sufficient conditions for nonemptiness of S_D.

Lemma 2.2. ([39]) *If either*

(1) *F is pseudomonotone on C and $S \neq \emptyset$,*
(2) *F is the gradient of G, where G is a differential quasiconvex function on an open set $K, C \subset K$ and attains its global minimum on C,*
(3) *F is quasi-monotone on C, $F \neq 0$ on C and C is bounded,*
(4) *F is quasi-monotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $v \in C$ with $\|v\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle F(v), y - v\rangle \leq 0$,*
(5) *F is quasimonotone on C and $S_N \neq \emptyset$,*

(6) *F is quasi-monotone on C, $intC$ is nonempty and there exists $v^* \in S$ such that $F(v^*) \neq 0$.*

Then, S_D is nonempty.

Lemma 2.3. ([34]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n.$$

If $\sum_{n=1}^{+\infty} b_n < \infty$, then $\lim_{n\to+\infty} a_n$ exists.

Lemma 2.4. ([31]) *Let $\{a_n\}$ be sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1,$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ then $\lim_{n\to\infty} a_n = 0$.

3. Main results

We now introduce a novel modified extragradient method to solve quasimonotone variational inequalities. The proposed iterative algorithm takes the following form:

Algorithm 3.1.

Initialization: *Let $f : H \to H$ be a contraction mapping with contraction parameter $\kappa \in [0,1)$. Given $\alpha, \tau_1 > 0$ and $\{\theta_n\}$ is a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \theta_n < +\infty$. Let $s_0, s_1 \in H$ be arbitrary. We assume that $\{\beta_n\}, \{\epsilon_n\}$ are two positive sequences such that $\epsilon_n = \circ(\beta_n)$, means $\lim_{n\to\infty} \dfrac{\epsilon_n}{\beta_n} = 0$, where $\{\beta_n\} \subset (0,1)$ satisfies the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty.$$

Iterative Steps: *Given the current iterate s_n, calculate s_{n+1} as follows:*

Step 1. *Compute*

$$w_n = s_n + \alpha_n(s_n - s_{n-1})$$

and

$$v_n = P_C(w_n - \tau_n F w_n),$$

where

$$\alpha_n = \begin{cases} \min\{\alpha, \dfrac{\epsilon_n}{\|s_n - s_{n-1}\|}\} & \text{if } s_n \neq s_{n-1}, \\ \alpha & \text{otherwise,} \end{cases} \tag{3}$$

If $w_n = v_n$ or $Fw_n = 0$ then stop and w_n is a solution of (1). Otherwise go to **Step 2**.

Step 2. *Compute*

$$z_n = P_{T_n}(w_n - \gamma \tau_n \eta_n F v_n),$$

where

$$T_n = \{x \in H \mid \langle w_n - \tau_n F w_n - v_n, x - v_n \rangle \leq 0\},$$

$$\eta_n := \begin{cases} \dfrac{\langle w_n - v_n, \Delta_n \rangle}{\|\Delta_n\|^2} & \text{if } \Delta_n \neq 0, \\ 0 & \text{if } \Delta_n = 0, \end{cases}$$

and

$$\Delta_n := w_n - v_n - \tau_n(Fw_n - Fv_n).$$

Step 3. *Compute*

$$s_{n+1} = (1-\beta_n)z_n + \beta f(z_n).$$

Update

$$\tau_{n+1} := \begin{cases} \min\left\{\mu\dfrac{\|w_n - v_n\|}{\|Fw_n - Fv_n\|}, (1+\theta_n)\tau_n\right\} & \text{if } Fw_n \neq Fv_n, \\ (1+\theta_n)\tau_n & \text{otherwise.} \end{cases} \tag{4}$$

Set $n := n+1$ *and go to* ***Step 1.***

Remark 3.1. *1. From* (3)*, the definition of* $\{\alpha_n\}$ *we have* $\lim_{n\to+\infty} \dfrac{\alpha_n}{\beta_n}\|s_n - s_{n-1}\| = 0.$

We start the convergence analysis by proving the following lemma.

Lemma 3.1. *Assume that* F *is* L*-Lipschitz continuous on* H*. Let* $\{\tau_n\}$ *be the sequence generated by* (4)*. Then*

$$\lim_{n\to\infty} \tau_n = \tau \text{ with } \tau \geq \min\left\{\frac{\mu}{L}, \tau_1\right\}.$$

Moreover

$$\|Fw_n - Fv_n\| \leq \frac{\mu}{\tau_{n+1}}\|w_n - v_n\|. \tag{5}$$

Proof. First, we prove that $\lim_{n\to\infty}\tau_n$ exists. Indeed, we show that $\prod_{n=1}^{\infty}(1+\theta_n) < +\infty$. We have

$$\begin{aligned}\prod_{n=1}^{k}(1+\theta_n) &= e^{\ln\prod_{n=1}^{k}(1+\theta_n)} = e^{\sum_{n=1}^{k}\ln(1+\theta_n)} \\ &\leq e^{\sum_{n=1}^{k}\theta_n} \leq e^{\sum_{n=1}^{+\infty}\theta_n} = e^{\theta} < +\infty,\end{aligned}$$

where $\theta := \sum_{n=1}^{+\infty}\theta_n$. Letting $k \to +\infty$ we get $\prod_{n=1}^{\infty}(1+\theta_n) < +\infty$. On the other hand, we have

$$\begin{aligned}\tau_{n+1} &\leq (1+\theta_n)\theta_n \leq (1+\tau_n)(1+\theta_{n-1})\tau_{n-1} \\ &\leq .. \leq (1+\theta_n)(1+\theta_{n-1})...(1+\theta_1)\tau_1 \\ &\leq \prod_{n=1}^{\infty}(1+\theta_n)\tau_1 \leq e^{\theta}\tau_1 = M, \ \forall n.\end{aligned}$$

This implies that $\tau_{n+1} = (1+\theta_n)\tau_n \leq \tau_n + M\theta_n$. Appling Lemma 2.3, we obtain $\lim_{n\to\infty}\tau_n$ exists. Now, we prove that

$$\tau_n \geq \min\left\{\frac{\mu}{L}, \tau_1\right\} \quad \forall n.$$

Indeed, from F is L-Lipschitz continuous on H we get

$$\|Fw_n - Fy_n\| \leq L\|w_n - y_n\|.$$

If $Fw_2 \neq Fy_2$ then

$$\mu\frac{\|w_2 - y_2\|}{\|Fw_2 - Fy_2\|} \geq \frac{\mu}{L}.$$

By the definition of $\{\tau_n\}$, we get $\tau_2 \geq \min\{\frac{\mu}{L}, \tau_1\}$. If $Fw_2 = Fy_2$ then we get $\tau_2 = (1+\theta_1)\tau_1 \geq \tau_1$. Therefore, in both cases, we get

$$\tau_2 \geq \min\{\frac{\mu}{L}, \tau_1\}.$$

Employing mathematical induction, we now see that $\tau_n \geq \min\{\frac{\mu}{L}, \tau_1\}$ for all $n \geq 1$. The proof is completed. □

Remark 3.2. *It should emphasize here that the sequence $\{\tau_n\}$ generated by* (4) *is new and different from* [28].

Next, we need the following lemma as the key to our results.

Lemma 3.2. [16] *Assume that Conditions 1.1–1.4 and the mapping F is quasimonotone on H. Let $\{w_n\}$ be a sequence generated by Algorithm 3.1. If there exists a subsequence $\{w_{n_k}\}$ convergent weakly to $z \in H$ and $\lim_{k\to\infty} \|w_{n_k} - v_{n_k}\| = 0$, then $z \in S_D$ or $Fz = 0$.*

Theorem 3.1. *Assume that Conditions 1.1–1.4 hold and $Fx \neq 0\ \forall x \in C$ then the sequence $\{s_n\}$ is generated by Algorithm 3.1 converges strongly to an element $p \in S_D \subset S$, where $p = P_{S_D} \circ f(p)$.*

Remark 3.3. *Frist, we note that $P_{S_D} \circ f$ is contraction mapping, hence there exists unique p such that $p = P_{S_D} \circ f(p)$.*

Proof. **Claim 1.**

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2 \quad \forall n \geq n_0. \tag{6}$$

Using (5), we have

$$\begin{aligned}\|\Delta_n\| &= \|w_n - v_n - \tau_n(Fw_n - Fv_n)\| \\ &\geq \|w_n - v_n\| - \tau_n\|Fw_n - Fv_n\| \\ &\geq \|w_n - v_n\| - \frac{\mu\tau_n}{\tau_{n+1}}\|w_n - v_n\| \\ &= \left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|.\end{aligned} \tag{7}$$

Since $\lim_{n\to\infty}\left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right) = 1 - \mu > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\mu\tau_n}{\tau_{n+1}} > \frac{1-\mu}{2} \quad \forall n \geq n_0.$$

It follows from (7) that for all $n \geq n_0$ we get

$$\|\Delta_n\| \geq \frac{1-\mu}{2}\|w_n - v_n\| > 0. \tag{8}$$

Since $x^* \in S_D \subset C \subset T_n$, using Lemma 2.1 we have

$$\begin{aligned}\|z_n - x^*\|^2 =& \|P_{T_n}(w_n - \gamma\eta_n\tau_nFv_n) - P_{T_n}x^*\|^2 \\ \leq& \langle z_n - x^*, w_n - \gamma\eta_n\tau_nFv_n - x^*\rangle \\ =& \frac{1}{2}\|z_n - x^*\|^2 + \frac{1}{2}\|w_n - \gamma\eta_n\tau_nFv_n - x^*\|^2 - \frac{1}{2}\|z_n - w_n + \gamma\eta_n\tau_nFv_n\|^2 \\ =& \frac{1}{2}\|z_n - x^*\|^2 + \frac{1}{2}\|w_n - x^*\|^2 + \frac{1}{2}\gamma^2\eta_n^2\tau_n^2\|Fv_n\|^2 - \langle w_n - x^*, \gamma\eta_n\tau_nFv_n\rangle \\ &- \frac{1}{2}\|z_n - w_n\|^2 - \frac{1}{2}\gamma^2\eta_n^2\tau_n^2\|Fv_n\|^2 - \langle z_n - w_n, \gamma\eta_n\tau_nFv_n\rangle \\ =& \frac{1}{2}\|z_n - x^*\|^2 + \frac{1}{2}\|w_n - x^*\|^2 - \frac{1}{2}\|z_n - w_n\|^2 - \langle z_n - x^*, \gamma\eta_n\tau_nFv_n\rangle.\end{aligned}$$

This implies that

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|z_n - w_n\|^2 - 2\gamma\eta_n\tau_n\langle z_n - x^*, Fv_n\rangle. \tag{9}$$

Since $v_n \in C$ and $x^* \in S_D$, we get $\langle Fv_n, v_n - x^*\rangle \geq 0$, which implies

$$\langle Fv_n, z_n - x^*\rangle = \langle Fv_n, z_n - v_n\rangle + \langle Fv_n, v_n - x^*\rangle \geq \langle Fv_n, z_n - v_n\rangle.$$

Thus, we obtain

$$-2\gamma\eta_n\tau_n\langle Fv_n, z_n - x^*\rangle \leq -2\gamma\eta_n\tau_n\langle Fv_n, z_n - v_n\rangle. \tag{10}$$

On the other hand, from $z_n \in T_n$ we have

$$\langle w_n - \tau_n Fw_n - v_n, z_n - v_n\rangle \leq 0.$$

This implies that

$$\langle w_n - v_n - \tau_n(Fw_n - Fv_n), z_n - v_n\rangle \leq \tau_n\langle Fv_n, z_n - v_n\rangle,$$

thus

$$\langle \Delta_n, z_n - v_n\rangle \leq \tau_n\langle Fv_n, z_n - v_n\rangle.$$

Hence

$$-2\gamma\eta_n\tau_n\langle Fv_n, z_n - v_n\rangle \leq -2\gamma\eta_n\langle \Delta_n, z_n - v_n\rangle. \tag{11}$$

On the other hand, we have

$$-2\gamma\eta_n\langle \Delta_n, z_n - v_n\rangle = -2\gamma\eta_n\langle \Delta_n, w_n - v_n\rangle + 2\gamma\eta_n\langle \Delta_n, w_n - z_n\rangle. \tag{12}$$

From (8), we have $\Delta_n \neq 0 \ \ \forall n \geq n_0$, thus $\eta_n = \dfrac{\langle w_n - v_n, \Delta_n\rangle}{\|\Delta_n\|^2}$, which means

$$\langle w_n - v_n, \Delta_n\rangle = \eta_n\|\Delta_n\|^2 \ \ \forall n \geq n_0. \tag{13}$$

Moreover

$$\begin{aligned} 2\gamma\eta_n\langle \Delta_n, w_n - z_n\rangle &= 2\langle \gamma\eta_n\Delta_n, w_n - z_n\rangle \\ &= \|w_n - z_n\|^2 + \gamma^2\eta_n^2\|\Delta_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2. \end{aligned} \tag{14}$$

Substituting (13) and (14) into (12) we get for all $n \geq n_0$ that

$$\begin{aligned} -2\gamma\eta_n\langle \Delta_n, z_n - v_n\rangle &\leq -2\gamma\eta_n^2\|\Delta_n\|^2 + \|w_n - z_n\|^2 + \gamma^2\eta_n^2\|\Delta_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 \\ &= \|w_n - z_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2. \end{aligned} \tag{15}$$

Combining (11) and (15), we obtain

$$\begin{aligned} -2\gamma\eta_n\tau_n\langle Fv_n, z_n - v_n\rangle &\leq -2\gamma\eta_n^2\|\Delta_n\|^2 + \|w_n - z_n\|^2 + \gamma^2\eta_n^2\|\Delta_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 \\ &= \|w_n - z_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2. \end{aligned} \tag{16}$$

Again, combining (10) and (16), we get

$$\begin{aligned} -2\gamma\eta_n\tau_n\langle Fv_n, z_n - x^*\rangle &\leq -2\gamma\eta_n^2\|\Delta_n\|^2 + \|w_n - z_n\|^2 + \gamma^2\eta_n^2\|\Delta_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 \\ &= \|w_n - z_n\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2. \end{aligned} \tag{17}$$

Substituting (17) into (9) we get

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2 \ \ \forall n \geq n_0.$$

Claim 2.

$$\frac{\left(1 - \dfrac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1 + \dfrac{\mu\tau_n}{\tau_{n+1}}\right)^2}\|w_n - v_n\|^2 \leq \eta_n^2\|\Delta_n\|^2 \ \ \forall n \geq n_0.$$

Indeed, we have

$$\|\Delta_n\| \leq \|w_n - v_n\| + \tau_n \|Fw_n - Fv_n\| \leq \left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right) \|w_n - v_n\|.$$

Hence

$$\|\Delta_n\|^2 \leq \left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2 \|w_n - v_n\|^2,$$

or equivalently

$$\frac{1}{\|\Delta_n\|^2} \geq \frac{1}{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2 \|w_n - v_n\|^2}.$$

Again, we find

$$\begin{aligned}\langle w_n - v_n, \Delta_n\rangle &= \|w_n - v_n\|^2 - \tau_n \langle w_n - v_n, Fw_n - Fv_n\rangle \\ &\geq \|w_n - v_n\|^2 - \tau_n \|w_n - v_n\| \|Fw_n - Fv_n\| \\ &\geq \|w_n - v_n\|^2 - \frac{\mu\tau_n}{\tau_{n+1}} \|w_n - v_n\|^2 \\ &= \left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right) \|w_n - v_n\|^2.\end{aligned}$$

Hence for all $n \geq n_0$

$$\eta_n \|\Delta_n\|^2 = \langle w_n - v_n, \Delta_n\rangle \geq \left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right) \|w_n - v_n\|^2 \tag{18}$$

and

$$\eta_n = \frac{\langle w_n - v_n, \Delta_n\rangle}{\|\Delta_n\|^2} \geq \frac{\left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)}{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2}. \tag{19}$$

Combining (18) and (19), we get

$$\eta_n^2 \|\Delta_n\|^2 \geq \frac{\left(1 - \frac{\mu\tau_n}{\tau_{n+1}}\right)^2}{\left(1 + \frac{\mu\tau_n}{\tau_{n+1}}\right)^2} \|w_n - v_n\|^2 \quad \forall n \geq n_0.$$

Claim 3. $\{s_n\}$ is bounded. Indeed, by **Claim 1**, we have

$$\|z_n - p\| \leq \|w_n - p\| \quad \forall n \geq n_0. \tag{20}$$

Moreover, by the definition of w_n we get

$$\begin{aligned}\|w_n - p\| &= \|s_n + \alpha_n(s_n - s_{n-1}) - p\| \\ &\leq \|s_n - p\| + \alpha_n \|s_n - s_{n-1}\| \\ &= \|s_n - p\| + \beta_n \frac{\alpha_n}{\beta_n} \|s_n - s_{n-1}\|.\end{aligned} \tag{21}$$

By condition $\frac{\alpha_n}{\beta_n}\|s_n - s_{n-1}\| \to 0$, there exists a constant $M_1 \geq 0$ such that

$$\frac{\alpha_n}{\beta_n}\|s_n - s_{n-1}\| \leq M_1 \ \forall n. \tag{22}$$

Combining (20), (21) and (25) we obtain

$$\|z_n - p\| \le \|w_n - p\| \le \|s_n - p\| + \beta_n M_1. \tag{23}$$

From the definition of $\{s_n\}$ we get

$$\begin{aligned}
\|s_{n+1} - p\| &= \|\beta_n f(z_n) + (1-\beta_n) z_n - p\| \\
&= \|\beta_n (f(z_n) - p) + (1-\beta_n)(z_n - p)\| \\
&\le \beta_n \|f(z_n) - p\| + (1-\beta_n)\|z_n - p\| \\
&\le \beta_n \|f(z_n) - f(p)\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|z_n - p\| \\
&\le \beta_n \kappa \|z_n - p\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|z_n - p\| \\
&= (1 - (1-\kappa)\beta_n)\|z_n - p\| + \beta_n \|f(p) - p\|.
\end{aligned} \tag{24}$$

Substituting (23) into (24) we obtain

$$\begin{aligned}
\|s_{n+1} - p\| \le& (1 - (1-\kappa)\beta_n)\|s_n - p\| + \beta_n M_1 + \beta_n \|f(p) - p\| \\
=& (1 - (1-\kappa)\beta_n)\|s_n - p\| + (1-\kappa)\beta_n \frac{M_1 + \|f(p) - p\|}{1-\kappa} \\
\le& \max\{\|s_n - p\|, \frac{M_1 + \|f(p) - p\|}{1-\kappa}\} \\
\le& \ldots \le \max\{\|s_0 - p\|, \frac{M_1 + \|f(p) - p\|}{1-\kappa}\}.
\end{aligned}$$

This implies $\{s_n\}$ is bounded. We also get $\{z_n\}, \{f(z_n)\}, \{w_n\}$ are bounded.

Claim 4.

$$\|w_n - z_n - \gamma\eta_n\Delta_n\|^2 + (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2 \le \|s_n - p\|^2 - \|s_{n+1} - p\|^2 + \beta_n M_4 \quad \forall n \ge n_0,$$

for some $M_4 > 0$. Indeed, we get

$$\begin{aligned}
\|s_{n+1} - p\|^2 \le& \beta_n \|f(z_n) - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
\le& \beta_n (\|f(z_n) - f(p)\| + \|f(p) - p\|)^2 + \|z_n - p\|^2 \\
\le& \beta_n (\kappa\|z_n - p\| + \|f(p) - p\|)^2 + (1-\beta_n)\|z_n - p\|^2 \\
\le& \beta_n (\|z_n - p\| + \|f(p) - p\|)^2 + (1-\beta_n)\|z_n - p\|^2 \\
=& \|z_n - p\|^2 + \beta_n (2\|z_n - p\|\|f(p) - p\| + \|f(p) - p\|^2) \\
\le& \|z_n - p\|^2 + \beta_n M_2,
\end{aligned} \tag{25}$$

for some $M_2 > 0$. Substituting (6) into (25) we get

$$\|s_{n+1} - p\|^2 \le \|w_n - x^*\|^2 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2 + \beta_n M_2. \tag{26}$$

It implies from (23) that

$$\begin{aligned}
\|w_n - p\|^2 &\le (\|s_n - p\| + \beta_n M_1)^2 \\
&= \|s_n - p\|^2 + \beta_n (2M_1\|s_n - p\| + \beta_n M_1^2) \\
&\le \|s_n - p\|^2 + \beta_n M_3,
\end{aligned} \tag{27}$$

for some $M_3 > 0$. Combining (26) and (27) we obtain

$$\|s_{n+1} - p\|^2 \le \|s_n - p\|^2 + \beta_n M_3 - \|w_n - z_n - \gamma\eta_n\Delta_n\|^2 - (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2 + \beta_n M_2.$$

This implies that

$$\|w_n - z_n - \gamma\eta_n\Delta_n\|^2 + (2-\gamma)\gamma\eta_n^2\|\Delta_n\|^2 \le \|s_n - p\|^2 - \|s_{n+1} - p\|^2 + \beta_n M_4,$$

where $M_4 := M_2 + M_3$.

Claim 5.

$$\begin{aligned}\|s_{n+1}-p\|^2 \le &(1-(1-\kappa)\beta_n)\|s_n-p\|^2\\ &+(1-\kappa)\beta_n\left[\frac{2}{1-\kappa}\langle f(p)-p, s_{n+1}-p\rangle+\frac{3M}{1-\kappa}\frac{\alpha_n}{\beta_n}\|s_n-s_{n-1}\|\right],\end{aligned}$$

for some $M>0$. Indeed, we have

$$\begin{aligned}\|w_n-p\|^2 &= \|s_n+\alpha_n(s_n-s_{n-1})-p\|^2\\ &= \|s_n-p\|^2+2\alpha_n\langle s_n-p, s_n-s_{n-1}\rangle+\alpha_n^2\|s_n-s_{n-1}\|^2\\ &\le \|s_n-p\|^2+2\alpha_n\|s_n-p\|\|s_n-s_{n-1}\|+\alpha_n^2\|s_n-s_{n-1}\|^2. \end{aligned}\tag{28}$$

Using (2) we have

$$\begin{aligned}\|s_{n+1}-p\|^2 &= \|\beta_n f(z_n)+(1-\beta_n)z_n-p\|^2\\ &= \|\beta_n(f(z_n)-f(p))+(1-\beta_n)(z_n-p)+\beta_n(f(p)-p)\|^2\\ &\le \|\beta_n(f(z_n)-f(p))+(1-\beta_n)(z_n-p)\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle\\ &\le \beta_n\|f(z_n)-f(p)\|^2+(1-\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle\\ &\le \beta_n\kappa^2\|z_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle\\ &\le \beta_n\kappa\|z_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle\\ &= (1-(1-\kappa)\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle\\ &\le (1-(1-\kappa)\beta_n)\|w_n-p\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle. \end{aligned}\tag{29}$$

Substituting (28) into (29)

$$\begin{aligned}\|s_{n+1}-p\|^2 \le &(1-(1-\kappa)\beta_n)\|s_n-p\|^2+2\alpha_n\|s_n-p\|\|s_n-s_{n-1}\|\\ &+\alpha_n^2\|s_n-s_{n-1}\|^2+2\beta_n\langle f(p)-p, s_{n+1}-p\rangle\\ =&(1-(1-\kappa)\beta_n)\|s_n-p\|^2+(1-\kappa)\beta_n\frac{2}{1-\kappa}\langle f(p)-p, s_{n+1}-p\rangle\\ &+\alpha_n\|s_n-s_{n-1}\|\Big(2\|s_n-p\|+\alpha_n\|s_n-s_{n-1}\|\Big)\\ \le&(1-(1-\kappa)\beta_n)\|s_n-p\|^2+(1-\kappa)\beta_n\frac{2}{1-\kappa}\langle f(p)-p, s_{n+1}-p\rangle\\ &+\alpha_n\|s_n-s_{n-1}\|\Big(2\|s_n-p\|+\alpha\|s_n-s_{n-1}\|\Big)\\ \le&(1-(1-\kappa)\beta_n)\|s_n-p\|^2\\ &+(1-\kappa)\beta_n.\frac{2}{1-\kappa}\langle f(p)-p, s_{n+1}-p\rangle+3M\alpha_n\|s_n-s_{n-1}\|\\ \le&(1-(1-\kappa)\beta_n)\|s_n-p\|^2\\ &+(1-\kappa)\beta_n\left[\frac{2}{1-\kappa}\langle f(p)-p, s_{n+1}-p\rangle+\frac{3M}{1-\kappa}\frac{\alpha_n}{\beta_n}\|s_n-s_{n-1}\|\right],\end{aligned}$$

for $M:=\sup_{n\in\mathbb{N}}\{\|s_n-p\|, \alpha\|s_n-s_{n-1}\|\}>0$.

Claim 6. The sequence $\{\|s_n-p\|\}$, $n\ge 0$, converges to zero. To see this, set

$$a_n:=\|s_n-p\|^2$$

and

$$b_n := \frac{2}{1-\kappa}\langle f(p) - p, s_{n+1} - p\rangle + \frac{3M}{1-\kappa}\frac{\alpha_n}{\beta_n}\|s_n - s_{n-1}\|.$$

Then **Claim 5** can be rewritten as follows:

$$a_{n+1} \le (1 - (1-\kappa)\beta_n)a_n + (1-\kappa)\beta_n b_n.$$

By Lemma 2.4, it is sufficient to show that $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0.$$

Since $\lim_{n\to+\infty} \frac{3M}{1-\kappa}\frac{\alpha_n}{\beta_n}\|s_n - s_{n-1}\| = 0$, we only need to show that

$$\limsup_{k\to\infty}\langle f(p) - p, s_{n_k+1} - p\rangle \le 0$$

for every subsequence $\{\|s_{n_k} - p\|\}$ of $\{\|s_n - p\|\}$ satisfying

$$\liminf_{k\to\infty}(\|s_{n_k+1} - p\| - \|s_{n_k} - p\|) \ge 0.$$

Suppose that $\{\|s_{n_k} - p\|\}$ is a subsequence of $\{\|s_n - p\|\}$ such that

$$\liminf_{k\to\infty}(\|s_{n_k+1} - p\| - \|s_{n_k} - p\|) \ge 0.$$

Then

$$\liminf_{k\to\infty}(\|s_{n_k+1}-p\|^2-\|s_{n_k}-p\|^2) = \liminf_{k\to\infty}[(\|s_{n_k+1}-p\|-\|s_{n_k}-p\|)(\|s_{n_k+1}-p\|+\|s_{n_k}-p\|)] \ge 0.$$

Using **Claim 4**, we obtain

$$\begin{aligned}
\limsup_{k\to\infty}&\left[\|w_{n_k} - z_{n_k} - \gamma\eta_{n_k}\Delta_{n_k}\|^2 + (2-\gamma)\gamma\eta_{n_k}^2\|\Delta_{n_k}\|^2\right] \\
&\le \limsup_{k\to\infty}\left[\|s_{n_k} - p\|^2 - \|s_{n_k+1} - p\|^2 + \beta_{n_k}M_4\right] \\
&\le \limsup_{k\to\infty}\left[\|s_{n_k} - p\|^2 - \|s_{n_k+1} - p\|^2\right] + \limsup_{k\to\infty}\beta_{n_k}M_4 \\
&= -\liminf_{k\to\infty}\left[\|s_{n_k+1} - p\|^2 - \|s_{n_k} - p\|^2\right] \\
&\le 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|w_{n_k} - z_{n_k} - \gamma\eta_{n_k}\Delta_{n_k}\| = 0 \text{ and } \lim_{k\to\infty}\|\Delta_{n_k}\| = 0. \tag{30}$$

In view of (30), we obtain

$$\lim_{k\to\infty}\|w_{n_k} - z_{n_k}\| = 0. \tag{31}$$

On the other hand, using **Claim 2** we get

$$\lim_{k\to\infty}\|w_{n_k} - v_{n_k}\| = 0. \tag{32}$$

Now, we claim that

$$\|s_{n_k+1} - s_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{33}$$

Indeed, by definition $\{s_{n+1}\}$ we have

$$\|s_{n_k+1} - z_{n_k}\| = \beta_{n_k}\|z_{n_k} - f(z_{n_k})\| \to 0 \text{ as } n \to +\infty. \tag{34}$$

Combining (31) and (34) we deduce

$$\|s_{n_k+1} - w_{n_k}\| \le \|s_{n_k+1} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| \to 0 \text{ as } n \to +\infty. \tag{35}$$

Moreover, we have

$$\|s_{n_k} - w_{n_k}\| = \alpha_{n_k}\|s_{n_k} - s_{n_k-1}\| = \beta_{n_k}\frac{\alpha_{n_k}}{\beta_{n_k}}\|s_{n_k} - s_{n_k-1}\| \to 0. \tag{36}$$

Combining (35) and (36), we get

$$\|s_{n_k+1} - s_{n_k}\| \le \|s_{n_k+1} - w_{n_k}\| + \|w_{n_k} - s_{n_k}\| \to 0.$$

Since the sequence $\{s_{n_k}\}$ is bounded, without any loss of generality we may assume that $\{s_{n_k}\}$ converges weakly to some $z^* \in H$ so that

$$\limsup_{k\to\infty}\langle f(p) - p, s_{n_k} - p\rangle = \langle f(p) - p, z^* - p\rangle. \tag{37}$$

Using (36), we get

$$w_{n_k} \rightharpoonup z^* \text{ as } k \to \infty.$$

Now, using (32), we get $\lim_{k\to\infty}\|w_{n_k} - v_{n_k}\| = 0$, Lemma 3.2 and the assumption $Fx \neq 0$, $\forall x \in C$ we get $z^* \in S_D$.
From (37) and the definition of $p = P_{S_D} \circ f(p)$, and $z^* \in S_D$. we have

$$\limsup_{k\to\infty}\langle f(p) - p, s_{n_k} - p\rangle = \langle f(p) - p, z^* - p\rangle \le 0. \tag{38}$$

Combining (33) and (38), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(p) - p, s_{n_k+1} - p\rangle &= \limsup_{k\to\infty}\langle f(p) - p, s_{n_k+1} - s_{n_k}\rangle + \limsup_{k\to\infty}\langle f(p) - p, s_{n_k} - p\rangle \\ &= \limsup_{k\to\infty}\langle f(p) - p, s_{n_k} - p\rangle \\ &= \langle f(p) - p, z^* - p\rangle \\ &\le 0. \end{aligned} \tag{39}$$

Hence, by (39), $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}\|s_n - s_{n-1}\| = 0$. Combining **Claim 5** and Lemma 2.4, we have $\lim_{n\to\infty}\|s_n - p\| = 0$. The proof is completed.

□

4. **Conclusions**

This paper introduced a novel extragradient method for solving quasimonotone variational inequalities in real Hilbert spaces. The proposed algorithm requires computing only one projection onto the feasible set C per iteration and employs an adaptive stepsize rule. Notably, the convergence of our proposed method does not necessitate prior knowledge of the Lipschitz constant of the variational inequality mapping. Our method represents a novel and state-of-the-art contribution to the study of solving quasimonotone variational inequalities in infinite-dimensional Hilbert spaces.

REFERENCES

[1] *R. Abaidoo and E. K. Agyapong*, Financial development and institutional quality among emerging economies, J. Econ. Dev., **24**(2022), 198-216.

[2] *T. O. Alakoya, O. T. Mewomo and Y. Shehu*, Strong convergence results for quasimonotone variational inequalities, Math. Meth. Oper. Res., **95**(2022), 249-279.

[3] *F. Alvarez and H. Attouch*, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Anal., **9**(2001), 3-11.

[4] *A. S. Antipin*, On a method for convex programs using a symmetrical modification of the Lagrange function, Ekonomika i Mat. Metody, **12**(1976), 1164-1173.

[5] *H. Attouch, X. Goudon and P. Redont*, The heavy ball with friction. I. The continuous dynamical system, Commun. Contemp. Math., **2**(2000), 1-34.

[6] *H. Attouch and M. O. Czarnecki*, Asymptotic control and stabilization of nonlinear oscillators with non-isolated equilibria, J. Differential Equations, **179**(2002), 278- 310.

[7] *C. Baiocchi and A. Capelo*, Variational and Quasivariational Inequalities, Applications to Free Boundary Problems, Wiley, New York, 1984.

[8] *X. Cai, G. Gu and B. He*, On the $O(1/t)$ convergence rate of the projection and contraction methods for variational inequalities with Lipschitz continuous monotone operators, Comput. Optim. Appl., **57**(2014), 339-363.

[9] *A. Cegielski*, Iterative Methods for Fixed Point Problems in Hilbert Spaces, Lecture Notes in Mathematics, vol. 2057, Springer, Berlin, 2012.

[10] *Y. Censor, A. Gibali and S. Reich*, The subgradient extragradient method for solving variational inequalities in Hilbert space, J. Optim. Theory Appl., **148**(2011), 318-335.

[11] *Y. Censor, A. Gibali and S. Reich*, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, Optim. Meth. Softw., **26**(2011), 827-845.

[12] *Y. Censor, A. Gibali and S. Reich*, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, Optimization, **61**(2011), 1119-1132.

[13] *R. W. Cottle and J. C. Yao*, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl., **75**(1992), 281-295.

[14] *L. Q. Dong, J. Y. Cho, L. L. Zhong and M. Th. Rassias*, Inertial projection and contraction algorithms for variational inequalities, J. Glob. Optim., **70**(2018), 687-704

[15] *Q. L. Dong, D. Jiang and A. Gibali*, A modified subgradient extragradient method for solving the variational inequality problem, Numer. Algorithms, **79**(2018), 927-940.

[16] *V.T. Dung, P.K. Anh and D.V. Thong*, Convergence of two-step inertial Tseng's extragradient methods for quasimonotone variational inequality problems, Commun. Nonlinear Sci. Numer. Simul. **136**(2024), 108110.

[17] *F. Facchinei and J. S. Pang*, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer Series in Operations Research, vols. I and II. Springer, New York, 2003.

[18] *G. Fichera*, Sul problema elastostatico di Signorini con ambigue condizioni al contorno, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat., **34**(1963), 138-142.

[19] *K. Goebel and S. Reich*, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.

[20] *B. S. He*, A class of projection and contraction methods for monotone variational inequalities, Appl. Math. Optim., **35**(1997), 69-76.

[21] *C. Izuchukwu, Y. Shehu and J. C. Yao*, A simple projection method for solving quasimonotone variational inequality problems, Optim. Eng., **24**(2023), 915-938.

[22] *C. Izuchukwu, Y. Shehu and J. C. Yao*, New inertial forward-backward type for variational inequalities with quasi-monotonicity, J. Glob. Optim., **84**(2022), 441-464.

[23] *S. Karamardian and S. Schaible*, Seven kinds of monotone maps, J. Optim. Theory Appl., **66**(1990), 37-46.

[24] *D. Kinderlehrer and G. Stampacchia*, An introduction to variational inequalities and their applications, Academic, New York, 1980.

[25] *I.V. Konnov*, Combined Relaxation Methods for Variational Inequalities, Springer-Verlag, Berlin, 2001.

[26] *G. M. Korpelevich*, The extragradient method for finding saddle points and other problems. Ekonomika i Mat. Metody, **12**(1976), 747-756

[27] *A. Moudafi*, Viscosity approximation methods for fixed point problems, J. Math. Anal. Appl. **241**(2000), 46-55.

[28] *H. Liu and J. Yang*, Weak convergence of iterative methods for solving quasimonotone variational inequalities, Comput. Optim. Appl., **77**(2020), 491-508.

[29] *B. T. Polyak*, Some methods of speeding up the convergence of iterative methods, Zh. Vychisl. Mat. Mat. Fiz., **4**(1964), 1-17.

[30] *S. Reich, D. V. Thong, P. Cholamjiak and et al.*, Inertial projection-type methods for solving pseudomonotone variational inequality problems in Hilbert space, Numer. Algor., **88**(2021), 813-835.

[31] *S. Saejung and P. Yotkaew*, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal., **75**(2012), 742-750.

[32] *G. Stampacchia*, Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci., **258**(1964), 4413-4416.

[33] *D. F. Sun*, A class of iterative methods for solving nonlinear projection equations, J. Optim. Theory Appl., **91**(1996), 123-140.

[34] *K. K. Tan and H. K. Xu*, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl., **178**(1993), 301-308.

[35] *D. V. Thong and D. V. Hieu*, Modified subgradient extragradient method for variational inequality problems, Numer. Algorithms, **79**(2018), 597-610

[36] *D. V. Thong and P. T. Vuong*, Improved subgradient extragradient methods for solving pseudomonotone variational inequalities in Hilbert spaces, Appl. Numer. Math., **163**(2021), 221-238

[37] *K. Wang, Y. Wang, O. S. Iyiola and Y. Shehu*, Double inertial projection method for variational inequalities with quasi-monotonicity, Optimization, **73**(2024), 707-739.

[38] *Zb. Wang, X. Chen, J. Yi and et al.*, Inertial projection and contraction algorithms with larger step sizes for solving quasimonotone variational inequalities, J. Glob. Optim., **82**(2022), 499522.

[39] *M. Ye and Y. He*, A double projection method for solving variational inequalities without monotonicity, Comput. Optim. Appl., **60**(2015), 141-150.