# REMARKS ON $d$-ARY PARTITIONS AND AN APPLICATION TO ELEMENTARY SYMMETRIC PARTITIONS

Mircea Cimpoeaş[1] and Roxana Tănase[2]

*We prove new formulas for* $\mathrm{p}_d(n)$*, the number of d-ary partitions of n, and, also, for* $P_d(n)$*, its polynomial part.*

*Given a partition* $\lambda = (\lambda_1, \dots, \lambda_\ell)$*, its associated j-th symmetric elementary partition,* $\mathrm{pre}_j(\lambda)$*, is the partition whose parts are* $\{\lambda_{i_1} \cdots \lambda_{i_j} \ : \ 1 \leq i_1 < \cdots < i_j \leq \ell\}$*. We prove that if* $\lambda$ *and* $\mu$ *are two d-ary partitions of length* $\ell$ *such that* $\mathrm{pre}_j(\lambda) = \mathrm{pre}_j(\mu)$ *and* $\lambda_{i_1} \cdots \lambda_{i_j} = \mu_{i_1} \cdots \mu_{i_j}$*, for all* $1 \leq i_1 < \cdots < i_j \leq \ell$*, then* $\lambda = \mu$*.*

**Keywords:** Restricted partitions, $d$-ary partitions, Elementary symmetric partitions.

**MSC2010:** 11P81, 11P83.

## 1. Introduction

Let $n$ be a positive integer. We denote $[n] = \{1, 2, \dots, n\}$. A partition of $n$ is a non-increasing sequence of positive integers $\lambda_i$ whose sum equals $n$. We define $p(n)$ as the number of partitions of $n$ and we define $p(0) = 1$. We denote $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1$ and $|\lambda| := \lambda_1 + \cdots + \lambda_\ell = n$. We refer to $|\lambda|$ as the size of $\lambda$ and the numbers $\lambda_i$ as parts of $\lambda$. The number $\ell(\lambda) = \ell$ is the number of parts of $\lambda$ and it is called the length of $\lambda$. For more on the theory of partitions, we refer the reader to [1].

Let $d \geq 2$ be an integer. A partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is called *$d$-ary*, if all $\lambda_i$'s are powers of $d$. A 2-ary partition is called binary. In Proposition 3.**3** we establish a natural bijection between the set of all integer partition and the set of $d$-ary partitions, which conserves the length (but not the size).

In Theorem 3.**5** we give a new formula for $p_d(n)$, the number of $d$-ary partitions of $n$, using the fact that a $d$-ary partition is a partition with the parts in $\{1, d, d^2, d^3, \dots\}$. In Theorem 3.**6**, we give a new formula for $W_j(d, n)$'s, the Sylvester waves of $p_d(n)$. Also, in Theorem 3.**7** and Theorem 3.**8** we give new formulas for $P_d(n) = W_1(d, n)$, the polynomial part of $p_d(n)$. In Example 3.**9**, we illustrate these results.

[1]Professor, Faculty of Applied Sciences, National University of Science and Technology Politehnica Bucharest, Romania and Simion Stoilow Institute of Mathematics, Romania, e-mail: `mircea.cimpoeas@imar.ro`

[2]Assistant professor, Faculty of Applied Sciences, National University of Science and Technology Politehnica Bucharest, Romania, e-mail: `roxana_elena.tanase@upb.ro`

Now, let $K$ be an arbitrary field and $S = K[x_1, \dots, x_\ell]$ be the ring of polynomials over $K$ in $\ell$ indeterminates. We recall that the $j^{th}$ elementary symmetric polynomial of $S$ is

$$e_j(x_1, \dots, x_\ell) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le \ell} x_{i_1} x_{i_2} \cdots x_{i_j}, \text{ where } 1 \le j \le \ell.$$

Also, we define $e_0(x_1, \dots, x_\ell) = 1$ and $e_j(x_1, \dots, x_\ell) = 0$ for $j > \ell$.

Given a partition $\lambda$, we have $e_j(\lambda) = 0$ if $\ell(\lambda) < j$ and

$$e_j(\lambda) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le \ell(\lambda)} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j}, \text{ if } 1 \le j \le \ell(\lambda).$$

In [2, 3], Ballantine et al. introduced the following definition. Given a partition $\lambda$, the partition $\operatorname{pre}_j(\lambda)$ is the partition whose parts are

$$\{\lambda_{i_1} \cdots \lambda_{i_j} \ : \ 1 \le i_1 < i_2 < \cdots < i_j \le \ell(\lambda)\},$$

and they called it an *elementary symmetric partition*. Note that $\operatorname{pre}_1(\lambda) = \lambda$, but $\operatorname{pre}_j(\lambda) \neq \lambda$, for $j \ge 2$. For example, if $\lambda = (3, 2, 1, 1)$, then $\operatorname{pre}_2(\lambda) = (6, 3, 3, 2, 2, 1)$.

A natural question to ask is the following: If $\lambda$ and $\mu$ are two partitions such that $\operatorname{pre}_j(\lambda) = \operatorname{pre}_j(\mu)$ then is it true that $\lambda = \mu$? Only the following cases are known in literature: (i) $j = 2$ and $m(\lambda), m(\mu) \le 3$, see [3, Proposition 14] and (ii) $j = 2$ and $\lambda$ and $\mu$ are binary partitions; see [3, Proposition 15]. In Theorem 4.**2** we prove that if $\lambda$ and $\mu$ are two $d$-ary partitions of length $\ell$ such that $\operatorname{pre}_j(\lambda) = \operatorname{pre}_j(\mu)$ and $\lambda_{i_1} \cdots \lambda_{i_j} = \mu_{i_1} \cdots \mu_{i_j}$, for all $1 \le i_1 < \cdots < i_j \le \ell$, where $1 \le j \le \ell - 1$, then $\lambda = \mu$.

## 2. **Preliminaries**

Let $\mathbf{a} := (a_1, a_2, \dots, a_r)$ be a sequence of positive integers, where $r \ge 1$. Let $\lambda$ be a partition. We say that $\lambda$ has parts in $\mathbf{a}$ if $\lambda_i \in \{a_1, \dots, a_r\}$ for all $1 \le i \le \ell(\lambda)$.

The *restricted partition function* associated to $\mathbf{a}$ is $p_{\mathbf{a}} : \mathbb{N} \to \mathbb{N}$, $p_{\mathbf{a}}(n) :=$ the number of integer solutions $(x_1, \dots, x_r)$ of $\sum_{i=1}^r a_i x_i = n$ with $x_i \ge 0$. In other words, $p_{\mathbf{a}}(n)$ counts the number of partitions of $n$ with parts in $\mathbf{a}$. Note that the generating function of $p_{\mathbf{a}}(n)$ is

$$\sum_{n=0}^{\infty} p_{\mathbf{a}}(n) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_r})}. \tag{2.1}$$

Let $D$ be a common multiple of $a_1$, $a_2, \dots, a_r$. Bell [5] proved that $p_{\mathbf{a}}(n)$ is a quasi-polynomial of degree $k - 1$, with the period $D$, that is

$$p_{\mathbf{a}}(n) = d_{\mathbf{a},k-1}(n) n^{k-1} + \cdots + d_{\mathbf{a},1}(n) n + d_{\mathbf{a},0}(n), \tag{2.2}$$

where $d_{\mathbf{a},m}(n + D) = d_{\mathbf{a},m}(n)$ for $0 \le m \le k - 1$ and $n \ge 0$, and $d_{\mathbf{a},k-1}(n)$ is not identically zero.

Sylvester [9],[10] decomposed the restricted partition in a sum of "waves":

$$p_{\mathbf{a}}(n) = \sum_{j\geq 1} W_j(n,\mathbf{a}), \tag{2.3}$$

where the sum is taken over all distinct divisors $j$ of the components of $\mathbf{a}$ and showed that for each such $j$, $W_j(n,\mathbf{a})$ is the coefficient of $t^{-1}$ in

$$\sum_{0\leq \nu<j,\ \gcd(\nu,j)=1} \frac{\rho_j^{-\nu n}e^{nt}}{(1-\rho_j^{\nu a_1}e^{-a_1 t})\cdots(1-\rho_j^{\nu a_k}e^{-a_k t})},$$

where $\rho_j = e^{\frac{2\pi i}{j}}$ and $\gcd(0,0)=1$ by convention. Note that $W_j(n,\mathbf{a})$'s are quasi-polynomials of period $j$. Also, $W_1(n,\mathbf{a})$ is called the *polynomial part* of $p_{\mathbf{a}}(n)$ and it is denoted by $P_{\mathbf{a}}(n)$.

**Theorem 2.1.** *(*[6, Corollary 2.10]*) We have*

$$p_{\mathbf{a}}(n) = \frac{1}{(r-1)!} \sum_{\substack{0\leq j_1\leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1\\ a_1j_1+\cdots+a_rj_r\equiv n(\bmod D)}} \prod_{\ell=1}^{r-1}\left(\frac{n-a_1j_1-\cdots-a_rj_r}{D}+\ell\right).$$

The *unsigned Stirling numbers* are defined by

$$\binom{n+r-1}{r-1} = \frac{1}{n(r-1)!}n^{(r)} = \frac{1}{(r-1)!}\left(\begin{bmatrix} r\\ r\end{bmatrix} n^{r-1}+\cdots\begin{bmatrix} r\\ 2\end{bmatrix} n+\begin{bmatrix} r\\ 1\end{bmatrix}\right). \tag{2.4}$$

**Theorem 2.2.** *(*[7, Proposition 4.2]*) For any positive integer $j$ with $j|a_i$ for some $1\leq i\leq r$, we have that*

$$W_j(n,\mathbf{a}) = \frac{1}{D(r-1)!}\sum_{m=1}^{r}\sum_{\ell=1}^{j}\rho_j^{\ell}\sum_{k=m-1}^{r-1}\begin{bmatrix} r\\ k+1\end{bmatrix}(-1)^{k-m+1}\binom{k}{m-1}\times$$
$$\times \sum_{\substack{0\leq j_1\leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1\\ a_1j_1+\cdots+a_rj_r\equiv \ell(\bmod j)}} D^{-k}(a_1j_1+\cdots+a_rj_r)^{k-m+1}n^{m-1}.$$

**Theorem 2.3.** *(*[6, Corollary 3.6]*) For the polynomial part $P_{\mathbf{a}}(n)$ of the quasi-polynomial $p_{\mathbf{a}}(n)$ we have*

$$P_{\mathbf{a}}(n) = \frac{1}{D(r-1)!} \sum_{0\leq j_1\leq \frac{D}{a_1}-1,\ldots,0\leq j_r\leq \frac{D}{a_r}-1} \prod_{\ell=1}^{r-1}\left(\frac{n-a_1j_1-\cdots-a_rj_r}{D}+\ell\right).$$

The *Bernoulli numbers* $B_\ell$'s are defined by $\frac{t}{e^t-1} = \sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}B_\ell$. We have $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$ and $B_n=0$ is $n$ is odd and $n\geq 1$.

**Theorem 2.4.** *(*[6, Corollary 3.11] *or* [4, page 2]*) The polynomial part of $p_{\mathbf{a}}(n)$ is*

$$P_{\mathbf{a}}(n) := \frac{1}{a_1\cdots a_r}\sum_{u=0}^{r-1}\frac{(-1)^u}{(r-1-u)!}\sum_{i_1+\cdots+i_r=u}\frac{B_{i_1}\cdots B_{i_r}}{i_1!\cdots i_r!}a_1^{i_1}\cdots a_r^{i_r}n^{r-1-u}.$$

## 3. **New formulas for the number of d-ary partitions**

We fix $d \geq 2$ an integer. We denote $\mathcal{P}$, the set of integer partitions, and $\mathcal{P}_d$, the set of $d$-ary partitions. Given a positive integer $n$, we denote $p_d(n)$, the number of $d$-ary partitions of $n$.

**Definition 3.1.** *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{P}$ be a partition. The d-exponential of $\lambda$ is the d-ary partition:*

$$\operatorname{Exp}_d(\lambda) := (d^{\lambda_1 - 1}, \ldots, d^{\lambda_\ell - 1}).$$

**Definition 3.2.** *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{P}_d$ be a d-ary partition. The d-logarithm of $\lambda$ is the partition:*

$$\operatorname{Log}_d(\lambda) := (\log_d(\lambda_1) + 1, \ldots, \log_d(\lambda_\ell) + 1).$$

**Proposition 3.3.** *The maps $\operatorname{Exp}_d : \mathcal{P} \to \mathcal{P}_d$ and $\operatorname{Log}_d : \mathcal{P}_d \to \mathcal{P}$ are bijective and inverse of each other.*

*Proof.* Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{P}$. We have $\operatorname{Exp}_d(\lambda) = (d^{\lambda_1 - 1}, \ldots, d^{\lambda_\ell - 1})$. Since

$$\log_d(d^{\lambda_i - 1}) + 1 = \lambda_i - 1 + 1 = \lambda_i \text{ for all } 1 \leq i \leq \ell,$$

it follows that $\operatorname{Log}_d(\operatorname{Exp}_d(\lambda)) = \lambda$. Similary, if $\mu \in \mathcal{P}_d$ is a $d$-ary partition, then it is easy to see that $\operatorname{Exp}_d(\operatorname{Log}_d(\mu)) = \mu$. Hence, the proof is complete. □

**Lemma 3.4.** *Let $n$ and $k$ be two positive integers such that $n < d^{k+1}$. The number of d-ary partitions of $n$ is*

$$p_d(n) = p_{(1,d,\ldots,d^k)}(n).$$

*In particular, the polynomial part of $p_d(n)$ is $P_d(n) = P_{(1,d,\ldots,d^k)}(n)$.*

*Proof.* Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a $d$-ary partition of $n$, that is $n = |\lambda|$. It follows that $\lambda_i = d^{c_i}$ with $0 \leq c_i$ and $d^{c_i} \leq n$ for all $1 \leq i \leq \ell$. Since $\lambda_1 = d^{c_1} \leq |\lambda| < d^{k+1}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$, it follows that

$$k \geq c_1 \geq c_2 \geq \cdots \geq c_\ell \geq 0,$$

and, therefore, $\lambda$ is a partition with parts in $(1, d, \ldots, d^k)$. On the other hand, any partition with parts in $(1, d, \ldots, d^k)$ is a $d$-ary partition. Hence, the proof is complete. □

**Theorem 3.5.** *Let $n$ and $k$ be two positive integers such that $n < d^{k+1}$. The number of d-ary partitions of $n$ is*

$$p_d(n) = \frac{1}{k!} \sum_{\substack{0 \leq j_1 \leq d^k - 1,\ 0 \leq j_2 \leq d^{k-1} - 1,\ \ldots, 0 \leq j_k \leq d - 1 \\ j_1 + j_2 d + \cdots + j_k d^{k-1} \equiv n (\bmod d^k)}} \prod_{\ell=1}^{k} \left( \frac{n - j_1 - j_2 d - \cdots - j_k d^{k-1}}{d^k} + \ell \right).$$

*Proof.* According to Lemma 3.**4**, we have $p_d(n) = p_{(1,d,\ldots,d^k)}(n)$, where $k = \lfloor \log_d(n) \rfloor$. Hence, the conclusion follows from Theorem 2.**1**, taking $r = k + 1$ and $D = \operatorname{lcm}(1, d, \ldots, d^k) = d^k$. □

From Lemma 3.**4** and (2.3) we can write

$$p_d(n) = \sum_{j\geq 1} W_j(d,n), \text{ where } W_j(d,n) = W_j(n,(1,d,\ldots,d^k)),$$

and $k = \lfloor \log_d(n) \rfloor$. In particular, the polynomial part of $p_d(n)$ is

$$P_d(n) = W_1(d,n).$$

**Theorem 3.6.** *Let $n$ and $k$ be two positive integers such that $n < d^{k+1}$. We have that*

$$W_j(d,n) = \frac{1}{k!d^k} \sum_{m=1}^{k+1} \sum_{\ell=1}^{j} \rho_j^\ell \sum_{s=m-1}^{k} \begin{bmatrix} k+1 \\ s+1 \end{bmatrix} (-1)^{s-m+1} \binom{s}{m-1} \times$$

$$\times \sum_{\substack{0\leq j_1\leq d^k-1,\ldots,0\leq j_k\leq d-1 \\ j_1+dj_2+\cdots+d^{k-1}j_{k-1}\equiv \ell (\bmod j)}} d^{-ks}(j_1+dj_2+\cdots+d^{k-1}j_{k-1})^{s-m+1} n^{m-1}.$$

*Proof.* The conclusion follows from Lemma 3.**4** and Theorem 2.**2**. □

**Theorem 3.7.** *Let $n$ and $k$ be two positive integers such that $n < d^{k+1}$. The polynomial part of $p_d(n)$ is*

$$P_d(n) = \frac{1}{k!d^k} \sum_{0\leq j_1\leq d^k-1,\ 0\leq j_2\leq d^{k-1}-1,\ \ldots,0\leq j_k\leq d-1} \prod_{\ell=1}^{k} \left( \frac{n-j_1-j_2d-\cdots-j_kd^{k-1}}{d^k} + \ell \right).$$

*Proof.* The conclusion follows from Lemma 3.**4** and Theorem 2.**3**. □

**Theorem 3.8.** *Let $n$ and $k$ be two positive integers such that $n < d^{k+1}$. The polynomial part of $p_d(n)$ is*

$$P_d(n) = \frac{1}{d^{\frac{k(k+1)}{2}}} \sum_{u=0}^{k} \frac{(-1)^u}{(k-u)!} \sum_{i_1+\cdots+i_{k+1}=u} \frac{B_{i_1}\cdots B_{i_{k+1}}}{i_1!\cdots i_{k+1}!} d^{i_2+2i_3+\cdots+ki_{k+1}} n^{k-u}.$$

*Proof.* The conclusion follows from Lemma 3.**4** and Theorem 2.**4**. □

**Example 3.9.** (1) Let $n=8$ and $d=3$. Since $n < d^{1+1}$, Theorem 3.**5** implies

$$p_3(8) = \frac{1}{1!} \sum_{0\leq j_1\leq 2,\ j_1\equiv 8 (\bmod 3)} \left( \frac{8-j_1}{3} + 1 \right) = \frac{8-2}{3} + 1 = 3.$$

Also, from Theorem 3.**7** it follows that the polynomial part of $p_3(8)$ is

$$P_3(8) = \frac{1}{1!\cdot 3^1} \sum_{j_1=0}^{2} \left( \frac{8-j_1}{3} + 1 \right) = \frac{1}{9} \sum_{j_1=0}^{2} (11-j_1) = \frac{11+10+9}{9} = \frac{10}{3}.$$

(2) Let $n=20$ and $d=3$. Since $n < d^{2+1}$, Theorem 3.**5** implies

$$p_3(20) = \frac{1}{162} \sum_{\substack{0\leq j_1\leq 8,\ 0\leq j_2\leq 2 \\ j_1+3j_2\equiv 20 (\bmod 9)}} (29-j_1-3j_2)(38-j_1-3j_2).$$

Since the set of pairs $(j_1,j_2)$ which satisfy the above conditions is $\{(2,0),(8,1),(5,2)\}$, it follows that $p_3(20) = \frac{1}{162}(28\cdot 36 + 18\cdot 27 + 18\cdot 27) = 12$.

## 4. **An application to elementary symmetric partitions**

Given $n \geq 2$ an integer, we denote by $\{e_1, \ldots, e_n\}$, the standard basis of the vector space $\mathbb{R}^n$, i.e. $e_i$ is the vector with 1 in the $i$-th position and zeros everywhere else.

Let $1 \leq j \leq n-1$ be an integer. We consider the vectors:

$$c_i = \begin{cases} e_1 + e_2 + \cdots + e_j, & i = 1 \\ e_1 + e_2 + \cdots + e_{j+1} - e_{i-1}, & 2 \leq i \leq j+1 \\ e_{i-j+1} + e_{i-j+2} + \cdots + e_i, & j+2 \leq i \leq n \end{cases}$$

Let $C$ be the $n \times n$ matrix whose columns are $c_1, c_2, \ldots, c_n$.

To better illustrate the structure of the matrix $C$, we present the case $n = 6$ and $j = 3$:

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Lemma 4.1.** *With the above notations, we have that* $\det(C) = j$.

*Proof.* From the definition of $C$, we easily note that $\det(C) = \det(A)$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

is a $(j+1) \times (j+1)$ circulant matrix with the associated polynomial

$$f(x) = 1 + x + x^2 + \cdots + x^{j-1}.$$

For more details on circulant matrices, we refer the reader to [8].

Let $\omega = e^{\frac{2\pi i}{j+1}}$ be a primitive $(j+1)$-th root of unity. Using a basic result on circulant matrices, we have that $\det(A) = \prod_{k=0}^{j} f(\omega^k)$.

It is clear that $f(\omega^0) = f(1) = j$. On the other hand, for $1 \leq k \leq j$, we have that $f(\omega^k) = 1 + \omega^k + \cdots + \omega^{k(j-1)} = -\omega^{kj}$. Therefore, it follows that

$$\det(A) = (-1)^j j \omega^{\frac{j^2(j+1)}{2}}.$$

If $j$ is even, then

$$\omega^{\frac{j^2(j+1)}{2}} = (\omega^{j+1})^{\frac{j^2}{2}} = 1^{\frac{j^2}{2}} = 1.$$

On the other hand, if $j$ is odd, then

$$\omega^{\frac{j^2(j+1)}{2}} = (\omega^{\frac{(j+1)}{2}})^{j^2} = (-1)^{j^2} = -1.$$

Hence, in both cases, we have that $\det(A) = j$. Thus, the proof is complete. $\square$

**Theorem 4.2.** *Let $\lambda$ and $\mu$ be two $d$-ary partitions with $\ell$ parts and let $1 \leq j \leq \ell - 1$ be an integer. If $\operatorname{pre}_j(\lambda) = \operatorname{pre}_j(\mu)$ and and $\lambda_{i_1} \cdots \lambda_{i_j} = \mu_{i_1} \cdots \mu_{i_j}$, for all $1 \leq i_1 < \cdots < i_j \leq \ell$, then $\lambda = \mu$.*

*Proof.* Since $\lambda$ is a $d$-ary partition, it follows that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_i = d^{c_i}$, for all $1 \leq i \leq \ell$, and $c_1 \geq c_2 \geq \cdots \geq c_\ell$. Similarly, $\mu = (\mu_1, \ldots, \mu_\ell)$ with $\mu_i = d^{c'_i}$, for all $1 \leq i \leq \ell$, and $c_1 \geq c_2 \geq \cdots \geq c_\ell$.

From the definition, $\operatorname{pre}_j(\lambda)$ is the partition whose parts are:

$$\{d^{c_{i_1}+c_{i_2}+\cdots+c_{i_j}} \ : \ 1 \leq i_1 < i_2 < \cdots < i_j \leq \ell\}.$$

Similarly, $\operatorname{pre}_j(\mu)$ is the partition whose parts are:

$$\{d^{c'_{i_1}+c'_{i_2}+\cdots+c'_{i_j}} \ : \ 1 \leq i_1 < i_2 < \cdots < i_j \leq \ell\}.$$

Since $\operatorname{pre}_j(\lambda) = \operatorname{pre}_j(\mu)$ and $\lambda_{i_1} \cdots \lambda_{i_j} = \mu_{i_1} \cdots \mu_{i_j}$, for all $1 \leq i_1 < \cdots < i_j \leq \ell$, it follows that

$$c'_{i_1} + c'_{i_2} + \cdots + c'_{i_j} = c_{i_1} + c_{i_2} + \cdots + c_{i_j}, \text{for all } 1 \leq i_1 < i_2 < \cdots < i_j \leq \ell.$$

For convenience, we denote

$$c_{i_1,\ldots,i_j} := c_{i_1} + c_{i_2} + \cdots + c_{i_j}, \text{for all } 1 \leq i_1 < i_2 < \cdots < i_j \leq \ell.$$

From Proposition 3.**3**, in order to prove that $\lambda = \mu$, it suffices to show that $(c_1, \ldots, c_\ell) = (c'_1, \ldots, c'_\ell)$. In order to do that, it is enough to prove that the linear system

$$\begin{cases} x_{i_1} + x_{i_2} + \cdots + x_{i_j} = c_{i_1,\ldots,i_j} \end{cases}, \text{where } 1 \leq i_1 < i_2 < \cdots < i_j \leq \ell, \tag{4.1}$$

has a unique solution. Since $(c_1, \ldots, c_\ell)$ is already a solution of (4.1), it is enough to prove that the matrix associated to (4.1) has the rank $n$. We consider the following subsystem of (4.1):

$$\begin{cases} x_1 + x_2 + \cdots + x_j = c_{1,2,\ldots,j} \\ x_2 + x_3 + \cdots + x_{j+1} = c_{2,\ldots,j+1} \\ x_1 + x_3 + \cdots + x_{j+1} = c_{1,3,\ldots,j+1} \\ \vdots \\ x_1 + \cdots + x_{j-1} + x_{j+1} = c_{1,\ldots,j-1,j+1} \\ x_3 + x_4 + \cdots + x_{j+2} = c_{3,\ldots,j+2} \\ x_4 + x_5 + \cdots + x_{j+3} = c_{4,\ldots,j+3} \\ \vdots \\ x_{\ell-j+1} + \cdots + x_\ell = c_{\ell-j+1,\ldots,\ell} \end{cases}. \tag{4.2}$$

Note that the matrix associated to (4.2) is $C^T$, where $C$ was defined at the beginning of this section.

According to Lemma 4.**1** we have $\det(C^T) = \det(C) = j \neq 0$. Hence, (4.2) has a unique solution. Thus (4.1) has also a unique solution, as required. □

## 5. **Conclusions**

Let $n \geq 1$ and $d \geq 2$ be two integers. We proved new formulas for $\mathrm{p}_d(n)$, the number of $d$-ary partitions of $n$, and, also, for $P_d(n)$, its polynomial part.

Given $\lambda$ a partition of length $\ell$ and $1 \leq j \leq \ell - 1$, we denote $\mathrm{pre}_j(\lambda)$, its associated $j$-th elementary symmetric partition; see [2, 3]. Given $\lambda$ and $\mu$ two $d$-ary partitions of length $\ell$ and $1 \leq j \leq \ell - 1$, we proved that if $\mathrm{pre}_j(\lambda) = \mathrm{pre}_j(\mu)$ and $\lambda_{i_1} \cdots \lambda_{i_j} = \mu_{i_1} \cdots \mu_{i_j}$, for all $1 \leq i_1 < \cdots < i_j \leq \ell$, then $\lambda = \mu$, thus giving a partial positive answer to a problem raised in [2].

## REFERENCES


[1] *G. E. Andrews*, The Theory of Partitions, Cambridge Mathematical Library, Cambridge University Press, Cambridge 1998.

[2] *C. Ballantine, G. Beck, M. Merca et al.*, Elementary Symmetric Partitions, Ann. Comb. (2024), https://doi.org/10.1007/s00026-024-00731-0.

[3] *C. Ballantine, G. Beck, M. Merca*, Partitions and elementary symmetric polynomials: an experimental approach, Ramanujan J. **66**, 34 (2025), https://doi.org/10.1007/s11139-024-01001-6.

[4] *M. Beck, I. M. Gessel, T. Komatsu*, The polynomial part of a restricted partition function related to the Frobenius problem, Electron. J. Comb. **8(1)** (2001), N 7 (5 pages).

[5] *E. T. Bell*, Interpolated denumerants and Lambert series, Am. J. Math. **65** (1943), 382–386.

[6] *M. Cimpoeaş, F. Nicolae*, On the restricted partition function, Ramanujan J. **47(3)** (2018), 565–588.

[7] *M. Cimpoeaş*, Remarks on the restricted partition function, Math. Reports **23(73)(4)** (2021), 425–436.

[8] *A. W. Ingleton*, The Rank of Circulant Matrices, J. London Math. Soc. **31(4)** (1956), 445–460.

[9] *J. J. Sylvester*, On the partition of numbers, Quart. J. Pure Appl. Math. **1** (1857), 81–85.

[10] *J. J. Sylvester*, On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order with an excursus on rational fractions and partitions, Am. J. Math. **5(1)** (1882), 79–136.