ON FUZZY $(2,2)$-REGULAR ORDERED Γ-$\mathcal{AG}^{**}$-GROUPOIDS

Faisal[1], Naveed Yaqoob[2], Kostaq Hila[3]

In this paper, we introduced the concept of fuzzy ordered Γ-$\mathcal{AG}$-groupoids and studied some important features of $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoids in terms of fuzzy Γ-left ideals, fuzzy Γ-right ideals, fuzzy Γ-two-sided ideals, fuzzy Γ-generalized bi-ideals, fuzzy Γ-bi-ideals, fuzzy Γ-interior ideals and fuzzy Γ-$(1,2)$-ideals. We proved that the set of all fuzzy Γ-two-sided ideals of a $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoid $\mathcal{K}$ forms a semilattice structure with identity $\mathcal{K}$. Further we characterized all the fuzzy Γ-ideals of a $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoid and we also proved that all fuzzy Γ-ideals coincide in a $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoid. Finally we gave the method to construct a Γ-$\mathcal{AG}$-groupoid.

Keywords: Ordered Γ-$\mathcal{AG}$-groupoids, $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoids and fuzzy Γ-ideals.

MSC2010: 20M10, 20N99.

1. Introduction

The concept of fuzzy sets was first proposed by Zadeh [12] in 1965, which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. It give us a tool to model the uncertainty present in a phenomena that do not have sharp boundaries. Many papers on fuzzy sets have been appeared which shows the importance and its applications to set theory, algebra, real analysis, measure theory and topology etc. (see [1], [5] and [10]).

An Abel-Grassmann's groupoid ($\mathcal{AG}$-groupoid) is a groupoid $\mathcal{K}$ whose elements satisfy the left invertive law $(ab)c = (cb)a$ [2], for all $a, b, c \in \mathcal{K}$. The concept of this algebraic structure was first introduced by Kazim and Naseeruddin in 1972 [2] and they have called it a left almost semigroup (LA-semigroup). In an $\mathcal{AG}$-groupoid, the medial law [2] $(ab)(cd) = (ac)(bd)$ holds for all $a, b, c, d \in \mathcal{K}$. An $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of an $\mathcal{AG}$-groupoid allow us to introduce the inverses of elements in an $\mathcal{AG}$-groupoid. If an $\mathcal{AG}$-groupoid contains a left identity, then it is unique [6]. In an $\mathcal{AG}$-groupoid $\mathcal{K}$ with left identity, the paramedial law [6] $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c, d \in \mathcal{K}$. If an $\mathcal{AG}$-groupoid contains a left identity, then by using medial law [6], we get $a(bc) = b(ac)$, for

[1] Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, K. P. K., Pakistan, E-mail: `yousafzaimath@yahoo.com`

[2] Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan, E-mail: `nayaqoob@ymail.com`

[3] Department of Mathematics & Computer Science, Faculty of Natural Sciences, University of Gjirokastra, Gjirokastra 6001, Albania, E-mail: `kostaq_hila@yahoo.com`

all $a, b, c \in \mathcal{K}$. Several examples and interesting properties of $\mathcal{AG}$-groupoids can be found in [6] and [9].

If an $\mathcal{AG}$-groupoid $\mathcal{K}$ satisfies $a(bc) = b(ac)$, for all $a, b, c \in \mathcal{K}$ without left identity, then it is called an $\mathcal{AG}^{**}$-groupoid. An $\mathcal{AG}^{**}$-groupoid also satisfies paramedial law without left identity. An $\mathcal{AG}^{**}$-groupoid is the generalization of an $\mathcal{AG}$-groupoid with left identity. Every $\mathcal{AG}$-groupoid with left identity is an $\mathcal{AG}^{**}$-groupoid but the converse is not true in general. Let us consider an $\mathcal{AG}$-groupoid with a binary operation $*$ defined in the following table.

*	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

It is easy to see that the above $\mathcal{AG}$-groupoid is an $\mathcal{AG}^{**}$-groupoid but it does not contains a left identity.

An $\mathcal{AG}$-groupoid is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related with a commutative semigroup, because if an $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative semigroup [6]. The connection of a commutative inverse semigroup with an $\mathcal{AG}$-groupoid has been given in [7] as, a commutative inverse semigroup $(\mathcal{K}, \circ)$ becomes an $\mathcal{AG}$-groupoid $(\mathcal{K}, \cdot)$ under $a \cdot b = b \circ a^{-1}$, for all $a, b \in \mathcal{K}$. An $\mathcal{AG}$-groupoid $\mathcal{K}$ with left identity becomes a semigroup under the binary operation "$\circ$" defined as, if for all $x,\ y \in \mathcal{K}$, there exists $a \in \mathcal{K}$ such that $x \circ y = (xa)y$ [9]. An $\mathcal{AG}$-groupoid is the generalization of a semigroup theory [6] and has vast applications in collaboration with semigroup like other branches of mathematics. The connection of $\mathcal{AG}$-groupoids with the vector spaces over finite fields has been investigated in [3].

From the above discussion, we see that $\mathcal{AG}$-groupoids have very closed links with semigroups and vector spaces which shows the importance and applications of $\mathcal{AG}$-groupoids.

The concept of a Γ-semigroup (generalization of semigroup) has been introduced by M. K. Sen [8] in 1981 as follows: A non-empty set $\mathcal{S}$ is called a Γ-semigroup $(\mathcal{S}, \Gamma)$ if $x\alpha y \in \mathcal{S}$ and $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in \mathcal{S}$ and all $\alpha, \beta \in \Gamma$.

For two non-empty subsets $\mathcal{K}$ and Γ, define $\mathcal{K}\Gamma\mathcal{K}$ as the set of elements of the form $k_1\gamma k_2$, where $k_1, k_2 \in \mathcal{K}$ and $\gamma \in \Gamma$, that is

$$\mathcal{K}\Gamma\mathcal{K} = \{k_1\gamma k_2 / k_1, k_2 \in \mathcal{K},\ \gamma \in \Gamma\}.$$

The pair $(\mathcal{K}, \Gamma)$ is called a Γ-groupoid if $x\alpha y \in \mathcal{K}$ for all $x, y \in \mathcal{K}$ and $\alpha \in \Gamma$. A Γ-groupoid $(\mathcal{K}, \Gamma)$ is called a Γ-$\mathcal{AG}$-groupoid if the following Γ-left invertive law holds for all $x, y, z \in \mathcal{K}$ and all $\alpha, \beta \in \Gamma$

$$(x\alpha y)\beta z = (z\alpha y)\beta x. \tag{1}$$

A Γ-$\mathcal{AG}$-groupoid also satisfies the Γ-medial law for all $w, x, y, z \in \mathcal{K}$ and all $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (w\alpha y)\beta(x\gamma z). \tag{2}$$

A Γ-$\mathcal{AG}$-groupoid is called a Γ-$\mathcal{AG}^{**}$-groupoid if it satisfies the following law for all $x, y, z \in \mathcal{K}$ and all $\alpha, \beta \in \Gamma$

$$x\alpha(y\beta z) = y\alpha(x\beta z). \tag{3}$$

A Γ-$\mathcal{AG}^{**}$-groupoid also satisfies the Γ-paramedial law for all $w, x, y, z \in \mathcal{K}$ and all $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (z\alpha y)\beta(x\gamma w). \tag{4}$$

Note that (3) and (4) also hold for a Γ-$\mathcal{AG}$-groupoid with left identity but a Γ-$\mathcal{AG}$-groupoid with left identity becomes an $\mathcal{AG}$-groupoid with left identity. Indeed, if $\mathcal{K}$ is a Γ-$\mathcal{AG}$-groupoid with left identity e and $a,\ b \in \mathcal{K}$, then

$$a\alpha b = a\alpha(e\beta b) = e\alpha(a\beta b) = a\beta b, \text{ where } \alpha, \beta \in \Gamma \Longrightarrow \alpha = \beta.$$

Assume that $(\mathcal{K}, .)$ is an $\mathcal{AG}$-groupoid, and let γ be a symbol ($\gamma \notin \mathcal{K}$). Define $a\gamma b = a.b$ for all $a, b \in \mathcal{K}$, then $\mathcal{K}$ is a $\{\gamma\}$-$\mathcal{AG}$-groupoid. Conversely, if $\mathcal{K}$ is a Γ-$\mathcal{AG}$-groupoid, and define $a.b = a\gamma b$ for all $a, b \in \mathcal{K}$, then $(\mathcal{K}, .)$ is an $\mathcal{AG}$-groupoid. This means that if $\mathcal{K}$ is a $\{\gamma\}$-$\mathcal{AG}$-groupoid, then $(\mathcal{K}, .)$ is an $\mathcal{AG}$-groupoid.

An ordered Γ-$\mathcal{AG}$-groupoid (po-Γ-$\mathcal{AG}$-groupoid) is a structure $(\mathcal{K}, \Gamma, \leq)$ in which the following conditions hold.

(i) $(\mathcal{K}, \Gamma)$ is a Γ-$\mathcal{AG}$-groupoid.

(ii) $(\mathcal{K}, \leq)$ is a poset (reflexive, anti-symmetric and transitive).

(iii) For all a, b and $x \in \mathcal{K}$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$, where $\beta \in \Gamma$.

An ordered Γ-$\mathcal{AG}$-groupoid is the generalization of an ordered Γ-semigroup.

2. **Preliminaries**

In this section, $\mathcal{K}$ will be considered as an ordered Γ-$\mathcal{AG}$-groupoid.

A fuzzy subset or a fuzzy set of a non-empty set $\mathcal{K}$ is an arbitrary mapping $f : \mathcal{K} \to [0, 1]$. A fuzzy subset f is a class of objects with a grades of membership having the form $f = \{(k,\ f(k))/k \in \mathcal{K}\}$.

Let $k \in \mathcal{K}$, then $A_k = \{(y, z) \in \mathcal{K} \times \mathcal{K} \backslash k \leq y\alpha z, \text{ where } \alpha \in \Gamma\}$. The product of any fuzzy subsets f and g of $\mathcal{K}$ is defined by

$$(f \circ_\Gamma g)(k) = \begin{cases} \bigvee\limits_{k \leq y\alpha z} \{f(y) \wedge g(z)\} & \text{if } A_k \neq \emptyset, \text{ where } \alpha \in \Gamma. \\ 0 & \text{if } A_k = \emptyset. \end{cases}$$

The order relation $\subseteq$ between any two fuzzy subsets f and g of $\mathcal{K}$ is defined by

$$f \subseteq g \text{ if and only if } f(k) \leq g(k), \text{ for all } k \in \mathcal{K}.$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of $\mathcal{K}$

$$(f \cap g)(k) = \min\{f(k),\ g(k)\} = f(k) \wedge g(k), \text{ for all } k \text{ in } \mathcal{K},$$

and

$$(f \cup g)(k) = \max\{f(k),\ g(k)\} = f(k) \vee g(k), \text{ for all } k \text{ in } \mathcal{K}.$$

For $\emptyset \neq A \subseteq \mathcal{K}$, we define

$$(A] = \{t \in \mathcal{K} \mid t \leq a \text{ for some } a \in A\}.$$

For $A = \{a\}$, we usually written as $(a]$.

A non-empty subset A of $\mathcal{K}$ is called a Γ-left (Γ-right) ideal of $\mathcal{K}$ if

(i) $\mathcal{K}\Gamma A \subseteq A$ $(A\Gamma\mathcal{K} \subseteq A)$.

(ii) If $a \in A$ and b is in $\mathcal{K}$ such that $b \leq a$, then b is in A.

A subset A of $\mathcal{K}$ is called a Γ-two-sided ideal of $\mathcal{K}$ if it is both a Γ-left and a Γ-right ideal of $\mathcal{K}$.

A fuzzy subset f of $\mathcal{K}$ is called a fuzzy Γ-left (Γ-right) ideal of $\mathcal{K}$ if

(i) $x \leq y \Rightarrow f(x) \geq f(y)$.

(ii) $f(a\beta b) \geq f(b)$ $(f(a\beta b) \geq f(a))$, for all a and b in $\mathcal{K}$, where $\beta \in \Gamma$.

A fuzzy subset f of $\mathcal{K}$ is called a fuzzy Γ-two-sided ideal of $\mathcal{K}$ if it is both a fuzzy Γ-left and a fuzzy Γ-right ideal of $\mathcal{K}$.

A fuzzy subset f of $\mathcal{K}$ is called a fuzzy Γ-$\mathcal{AG}$-subgroupoid of $\mathcal{K}$ if $f(a\beta b) \geq f(a) \wedge f(b)$ for all a and b in $\mathcal{K}$, where $\beta \in \Gamma$.

A fuzzy subset f of $\mathcal{K}$ is called a fuzzy Γ-generalized bi-ideal of $\mathcal{K}$ if

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, for all x and y in $\mathcal{K}$.

(ii) $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$, for all x, y and z in $\mathcal{K}$, where $\alpha, \beta \in \Gamma$.

A fuzzy $\mathcal{AG}$-subgroupoid f of $\mathcal{K}$ is called a fuzzy Γ-bi-ideal of $\mathcal{K}$ if

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, for all x and y in $\mathcal{K}$.

(ii) $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$ for all x, y and z in $\mathcal{K}$, where $\alpha, \beta \in \Gamma$.

A fuzzy subset f of $\mathcal{K}$ is called a fuzzy Γ-interior ideal of $\mathcal{K}$ if

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, for all x and y in $\mathcal{K}$.

(ii) $f((x\alpha y)\beta z) \geq f(y)$ for all x, y and z in $\mathcal{K}$, where $\alpha, \beta \in \Gamma$.

A fuzzy subset f of $\mathcal{K}$ is called a fuzzy Γ-$(1, 2)$ ideal of $\mathcal{K}$ if

(i) $x \leq y \Rightarrow f(x) \geq f(y)$, for all x and y in $\mathcal{K}$.

(ii) $f((x\alpha a)\beta(y\gamma z))\beta z) \geq f(x) \wedge f(y) \wedge f(z)$ for all x, a, y and z in $\mathcal{K}$, where $\alpha, \beta, \gamma \in \Gamma$.

A fuzzy subset f of $\mathcal{K}$ is called Γ-idempotent if $f \circ_\Gamma f = f$.

Example 2.1. *Consider an open interval $\mathbb{R}_{\mathbb{O}} = (0, 1)$ of real numbers under the binary operation of multiplication. Define $a * b = ba^{-1}r^{-1}$, for all $a, b, r \in \mathbb{R}_{\mathbb{O}}$, then it is easy to see that $(\mathbb{R}_{\mathbb{O}}, *, \leq)$ is an ordered $\mathcal{AG}$-groupoid under the usual order "$\leq$" and we have called it a real ordered $\mathcal{AG}$-groupoid. Define $a\xi b = a * b$ for all $a, b \in \mathbb{R}_{\mathbb{O}}$ and $\xi \in \Gamma$, then $\mathbb{R}_{\mathbb{O}}$ is an ordered Γ-$\mathcal{AG}$-groupoid. Thus we have seen that every ordered $\mathcal{AG}$-groupoid is an ordered Γ-$\mathcal{AG}$-groupoid for $\Gamma = \{\xi\}$, that is, an ordered Γ-$\mathcal{AG}$-groupoid is the generalization of an ordered $\mathcal{AG}$-groupoid.*

Example 2.2. *Consider the set $\mathcal{K} = \{a, b, c\}$ and let $\Gamma = \{\alpha, \beta, \gamma\}$ be the set of three binary operations on $\mathcal{K}$ defined in the tables below.*

α	a	b	c
a	a	a	c
b	a	a	a
c	a	a	a

β	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

γ	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Since $(x\delta y)\xi z = (z\delta y)\xi x$ for all $x, y, z \in \mathcal{K}$ and all $\delta, \xi \in \Gamma$, therefore $\mathcal{K}$ is an ordered Γ-$\mathcal{AG}$-groupoid under the following order.

$$\leq := \{(a, a), (b, b), (c, c), (a, c), (a, b)\}.$$

Consider a fuzzy subset f of $\mathcal{K}$ as follows: $f(a) = 0.8$, $f(b) = 0.4$ and $f(c) = 0.6$, then it is easy to see that f is a fuzzy Γ-two-sided ideal of $\mathcal{K}$.

Note that every fuzzy Γ-two-sided ideal of $\mathcal{K}$ is a fuzzy Γ-$\mathcal{AG}$-subgroupoid of $\mathcal{K}$ but the converse is not true in general. For this, let us define a fuzzy subset f

of $\mathcal{K}$ as follows: $f(a) = 0.9$, $f(b) = 0.7$ and $f(c) = 0.5$, then one can easily observe that f is a fuzzy Γ-$\mathcal{AG}$-subgroupoid of $\mathcal{K}$ but f is not a fuzzy Γ-two-sided ideal of $\mathcal{K}$, because $f(b\gamma c) \ngeq f(b)$, $\gamma \in \Gamma$.

We denote by $F(\mathcal{K})$ the set of all fuzzy subsets of an ordered Γ-$\mathcal{AG}$-groupoid $\mathcal{K}$.

Proposition 2.1. *The set* $(F(\mathcal{K}), \circ_\Gamma, \subseteq)$ *is an ordered* Γ-$\mathcal{AG}$*-groupoid.*

Proof. Clearly $F(\mathcal{K})$ is closed. Let f, g and h be in $F(\mathcal{K})$. If $A_x = \emptyset$ for any $x \in \mathcal{K}$, then $((f \circ_\Gamma g) \circ_\Gamma h)(x) = 0 = ((h \circ_\Gamma g) \circ_\Gamma f)(x)$. Let $A_x \neq \emptyset$, then there exist y and z in $\mathcal{K}$ such that $(y, z) \in A_x$, let $\alpha, \beta \in \Gamma$. Therefore by using (1), we have

$$\begin{aligned}
((f \circ_\Gamma g) \circ_\Gamma h)(x) &= \bigvee_{(y,z)\in A_x} \{(f \circ_\Gamma g)(y) \wedge h(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left\{ \bigvee_{(p,q)\in A_y} \{f(p) \wedge g(q)\} \wedge h(z) \right\} \\
&= \bigvee_{x \leq (p\alpha q)\beta z} \{f(p) \wedge g(q) \wedge h(z)\} \\
&= \bigvee_{x \leq (z\alpha q)\beta p} \{h(z) \wedge g(q) \wedge f(p)\} \\
&= \bigvee_{(w,p)\in A_x} \left\{ \bigvee_{(z,q)\in A_w} (h(z) \wedge g(q) \wedge f(p)) \right\} \\
&= \bigvee_{(w,p)\in A_x} \{(h \circ_\Gamma g)(w) \wedge f(p)\} = ((h \circ_\Gamma g) \circ_\Gamma f)(x).
\end{aligned}$$

Hence $(F(\mathcal{K}), \circ_\Gamma)$ is a Γ-$\mathcal{AG}$-groupoid.

Assume that $f \subseteq g$ and let $A_x = \emptyset$ for any $x \in \mathcal{K}$, then $(f \circ_\Gamma h)(x) = 0 = (g \circ_\Gamma h)(x) \Longrightarrow f \circ_\Gamma h \subseteq g \circ_\Gamma h$ and similarly we can show that $h \circ_\Gamma f \subseteq h \circ_\Gamma g$. Let $A_x \neq \emptyset$, then there exist y and z in $\mathcal{K}$ such that $(y, z) \in A_x$, therefore

$$(f \circ_\Gamma h)(x) = \bigvee_{(y,z)\in A_x} \{f(y) \wedge h(z)\} \leq \bigvee_{(y,z)\in A_x} \{g(y) \wedge h(z)\} = (g \circ_\Gamma h)(x).$$

Similarly we can show that $(h \circ_\Gamma f)(x) \leq (h \circ_\Gamma g)(x)$ holds for all $x \in \mathcal{K}$. Thus $(F(\mathcal{K}), \circ_\Gamma, \subseteq)$ is an ordered Γ-$\mathcal{AG}$-groupoid. $\square$

Corollary 2.1. *The medial law holds in* $F(\mathcal{K})$.

The following Corollary is a consequence of the successive use of (1) in Proposition 2.1.

Corollary 2.2. *For any fuzzy subsets* f, g *and* h *of an ordered* Γ-$\mathcal{AG}$*-groupoid* $\mathcal{K}$*, the following conditions are equivalent:*

(i) $(f \circ_\Gamma g) \circ_\Gamma h = g \circ_\Gamma (f \circ_\Gamma h)$,

(ii) $(f \circ_\Gamma g) \circ_\Gamma h = g \circ_\Gamma (h \circ_\Gamma f)$.

Through out in this paper $\mathcal{K}$ will be considered as an ordered Γ-$\mathcal{AG}^{**}$-groupoid.

Theorem 2.1. *In $\mathcal{K}$, the following properties hold.*

(i) $f \circ_\Gamma (g \circ_\Gamma h) = g \circ_\Gamma (f \circ_\Gamma h)$ *for all* f, g *and* h *in* $F(\mathcal{K})$.

(ii) $(f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k) = (k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f)$ *for all* f, g, h *and* k *in* $F(\mathcal{K})$.

Proof. (i) : Let x be an arbitrary element of $\mathcal{K}$. If $A_x = \emptyset$ for $x \in \mathcal{K}$, then $(f \circ_\Gamma (g \circ_\Gamma h))(x) = 0 = (g \circ_\Gamma (f \circ_\Gamma h))(x)$. Let $A_x \neq \emptyset$, then there exist y and z in $\mathcal{K}$ such that $(y, z) \in A_x$, let $\alpha, \beta \in \Gamma$. Now by using (3), we have

$$\begin{aligned}
(f \circ_\Gamma (g \circ_\Gamma h))(x) &= \bigvee_{(y,z)\in A_x} \{f(y) \wedge (g \circ_\Gamma h)(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left\{ f(y) \wedge \bigvee_{(p,q)\in A_z} \{g(p) \wedge h(q)\} \right\} \\
&= \bigvee_{x \leq y\alpha(p\beta q)} \{f(y) \wedge g(p) \wedge h(q)\} \\
&= \bigvee_{x \leq p\alpha(y\beta q)} \{g(p) \wedge f(y) \wedge h(q)\} \\
&= \bigvee_{(p,w)\in A_x} \left\{ g(p) \wedge \bigvee_{(y,q)\in A_w} \{f(y) \wedge h(q)\} \right\} \\
&= \bigvee_{(p,w)\in A_x} \{g(p) \wedge (f \circ_\Gamma h)(w)\} = (g \circ_\Gamma (f \circ_\Gamma h))(x).
\end{aligned}$$

Thus, $(f \circ_\Gamma (g \circ_\Gamma h))(x) = (g \circ_\Gamma (f \circ_\Gamma h))(x)$, for all x in $\mathcal{K}$.

(ii) : If $A_x = \emptyset$ for $x \in \mathcal{K}$, then $((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) = 0 = ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x)$. Let $A_x \neq \emptyset$, then there exist y and z in $\mathcal{K}$ such that $(y, z) \in A_x$, let $\alpha, \beta, \gamma \in \Gamma$. Therefore by using (4), we have

$$\begin{aligned}
((f \circ_\Gamma g) \circ_\Gamma (h \circ_\Gamma k))(x) &= \bigvee_{(y,z)\in A_x} \{(f \circ_\Gamma g)(y) \wedge (h \circ_\Gamma k)(z)\} \\
&= \bigvee_{(y,z)\in A_x} \left\{ \begin{array}{l} \bigvee_{(p,q)\in A_y} \{f(p) \wedge g(q)\} \\ \wedge \bigvee_{(u,v)\in A_z} \{h(u) \wedge k(v)\} \end{array} \right\} \\
&= \bigvee_{x \leq (p\alpha q)\beta(u\gamma v)} \{f(p) \wedge g(q) \wedge h(u) \wedge k(v)\} \\
&= \bigvee_{x \leq (v\alpha u)\beta(q\gamma p)} \{k(v) \wedge h(u) \wedge g(q) \wedge f(p)\} \\
&= \bigvee_{(m,n)\in A_x} \left\{ \begin{array}{l} \bigvee_{(v,u)\in A_m} \{k(v) \wedge h(u)\} \\ \wedge \bigvee_{(q,p)\in A_n} \{g(q) \wedge f(p)\} \end{array} \right\} \\
&= \bigvee_{(m,n)\in A_x} \{(k \circ_\Gamma h)(m) \wedge (g \circ_\Gamma f)(n)\} \\
&= ((k \circ_\Gamma h) \circ_\Gamma (g \circ_\Gamma f))(x).
\end{aligned}$$

Thus, $(f\circ_\Gamma g)\circ_\Gamma(h\circ_\Gamma k)=(k\circ_\Gamma h)\circ_\Gamma(g\circ_\Gamma f)$, for all x in $\mathcal{K}$. □

Resp. keeping the generalization, the following three Lemmas have the same proof as in [4].

Lemma 2.1. *Let f be a fuzzy subset of an ordered Γ-$\mathcal{AG}$-groupoid $\mathcal{K}$, then f is a fuzzy Γ-left ideal of $\mathcal{K}$ if and only if f satisfies the following.*
(i) $x\leq y\Rightarrow f(x)\geq f(y)$ for all x and y in $\mathcal{K}$.
(ii) $\mathcal{K}\circ_\Gamma f\subseteq f$.

Lemma 2.2. *Let f be a fuzzy subset of an ordered Γ-$\mathcal{AG}$-groupoid $\mathcal{K}$, then f is a fuzzy Γ-right ideal of $\mathcal{K}$ if and only if f satisfies the following.*
(i) $x\leq y\Rightarrow f(x)\geq f(y)$ for all x and y in $\mathcal{K}$.
(ii) $f\circ_\Gamma\mathcal{K}\subseteq f$.

Lemma 2.3. *Let f be a fuzzy subset of an ordered Γ-$\mathcal{AG}$-groupoid $\mathcal{K}$, then f is a fuzzy Γ-two-sided ideal of $\mathcal{K}$ if and only if f satisfies the following.*
(i) $x\leq y\Rightarrow f(x)\geq f(y)$ for all x and y in $\mathcal{K}$.
(ii) $\mathcal{K}\circ_\Gamma f\subseteq f$ and $f\circ_\Gamma\mathcal{K}\subseteq f$.

3. Fuzzy $(2,2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoids

An element a of $\mathcal{K}$ is called a $(2,2)$-regular element of $\mathcal{K}$ if there exists $y\in\mathcal{K}$ such that $a\leq((a\alpha a)\beta y)\gamma(a\delta a)$, where $\alpha,\beta,\gamma,\delta\in\Gamma$ and $\mathcal{K}$ is called $(2,2)$-regular if every element of $\mathcal{K}$ is $(2,2)$-regular.

Note that in a $(2,2)$-regular $\mathcal{K}$, the following holds for all $x\in\mathcal{K}$

$$x\leq y\alpha z,\ \text{for some } y,z\in\mathcal{K}\text{ and }\alpha\in\Gamma. \tag{5}$$

Note that $\mathcal{K}$ can be considered as a fuzzy subset of itself and we write $\mathcal{K}(x)=1$, for all $x\in\mathcal{K}$.

Lemma 3.1. *In a $(2,2)$-regular $\mathcal{K}$, $f\circ_\Gamma\mathcal{K}=f$ and $\mathcal{K}\circ_\Gamma f=f$ holds for every fuzzy Γ-two-sided ideal f of $\mathcal{K}$.*

Proof. Let f be a fuzzy Γ-two-sided ideal of a $(2,2)$-regular $\mathcal{K}$. For every $a\in\mathcal{K}$ there exists $y\in\mathcal{K}$ such that $a\leq((a\alpha a)\beta y)\gamma(a\delta a)$, where $\alpha,\beta,\gamma,\delta\in\Gamma$. Let $\xi\in\Gamma$, then by using (5), (4) and (1), we have

$$\begin{aligned} a &\leq ((a\alpha a)\beta y)\gamma(a\delta a)=x\gamma(a\delta a)\leq(a\xi b)\gamma(a\delta a)=(a\xi a)\gamma(b\delta a)\\ &= ((b\delta a)\xi a)\gamma a,\ \text{where } (a\alpha a)\beta y)=x\in\mathcal{K}.\end{aligned}$$

Thus $(((b\delta a)\xi a),a)\in A_a$, since $A_a\neq\emptyset$, therefore

$$\begin{aligned}(f\circ_\Gamma\mathcal{K})(a) &= \bigvee_{(((b\delta a)\xi a),a)\in A_a}\{f((b\delta a)\xi a)\wedge\mathcal{K}(a)\}\\ &\geq f((b\delta a)\xi a)\wedge\mathcal{K}(a)\geq f(a)\wedge 1=f(a).\end{aligned}$$

Now by using Lemma 2.2, $f\circ_\Gamma\mathcal{K}=f$.
Also

$$\begin{aligned}(\mathcal{K}\circ_\Gamma f)(a) &= \bigvee_{(((b\delta a)\xi a),a)\in A_a}\{\mathcal{K}((b\delta a)\xi a)\wedge f(a)\}\\ &\geq \mathcal{K}((b\delta a)\xi a)\wedge f(a)\geq f(a)\wedge 1=f(a).\end{aligned}$$

Now by using Lemma 2.1, $\mathcal{K} \circ_\Gamma f = f$. □

Lemma 3.2. *Let f and g be any fuzzy Γ-two-sided ideals of of a $(2,2)$-regular $\mathcal{K}$, then $f \circ_\Gamma g = f \cap g$.*

Proof. Assume that f and g are any fuzzy Γ-two-sided ideals of a $(2,2)$-regular $\mathcal{K}$, then for every $a \in \mathcal{K}$ there exists $y \in \mathcal{K}$ such that $a \leq ((a\alpha a)\beta y)\gamma(a\delta a)$, where $\alpha, \beta, \gamma, \delta \in \Gamma$. Let $\xi \in \Gamma$, then by using (3), we have

$$a \leq ((a\alpha a)\beta y)\gamma(a\delta a) = x\xi(a\delta a) = a\xi(x\delta a), \text{ where } (a\alpha a)\beta y = x \in \mathcal{K}.$$

Thus $(a, (x\delta a)) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}(f \circ_\Gamma g)(a) &= \bigvee_{(a,(x\delta a)) \in A_a} \{f(a) \wedge g(x\delta a)\} \geq f(a) \wedge g(x\delta a) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a).\end{aligned}$$

Thus by using Lemmas 2.1 and 2.2, we get $f \circ_\Gamma g = f \cap g$. □

Lemma 3.3. *Every fuzzy Γ-two-sided ideal of a $(2,2)$-regular $\mathcal{K}$ is idempotent.*

Proof. Let f be a fuzzy Γ-two-sided ideal of $\mathcal{K}$. For every $a \in \mathcal{K}$ there exists $y \in \mathcal{K}$ such that $a \leq ((a\alpha a)\beta y)\gamma(a\delta a)$, where $\alpha, \beta, \gamma, \delta \in \Gamma$. From Lemma 3.1, $a \leq ((b\delta a)\xi a)\gamma a$. Thus $(((b\delta a)\xi a), a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$(f \circ_\Gamma f)(a) = \bigvee_{(((b\delta a)\xi a), a) \in A_a} \{f((b\delta a)\xi a) \wedge f(a)\} \geq f(a) \wedge f(a) = f(a).$$

Now by using Lemma 2.1, $f \circ_\Gamma f = f$. □

Corollary 3.1. *Every fuzzy Γ-left ideal of a $(2,2)$-regular $\mathcal{K}$ is idempotent.*

Theorem 3.1. *The set of fuzzy Γ-two-sided ideals of a $(2,2)$-regular $\mathcal{K}$ forms a semilattice structure with identity $\mathcal{K}$.*

Proof. Let $\mathcal{F}_{\Gamma_\mathcal{J}}$ be the set of all fuzzy Γ-two-sided ideals of a left regular $\mathcal{K}$ and let f, g and h be in $\mathcal{F}_{\Gamma_\mathcal{J}}$. Clearly $\mathcal{F}_{\Gamma_\mathcal{J}}$ is closed and by Lemma 3.3, we have $f \circ_\Gamma f = f$. Now by using Lemma 3.2, we get $f \circ_\Gamma g = g \circ_\Gamma f$. Therefore by using (1) and commutative law, we have

$$\begin{aligned}(f \circ_\Gamma g) \circ_\Gamma h &= (g \circ_\Gamma f) \circ_\Gamma h = (h \circ_\Gamma f) \circ_\Gamma g = (f \circ_\Gamma h) \circ_\Gamma g \\ &= (g \circ_\Gamma h) \circ_\Gamma f = f \circ_\Gamma (g \circ_\Gamma h).\end{aligned}$$

It is easy to see from Lemma 3.1 that $\mathcal{K}$ is an identity in $\mathcal{F}_{\Gamma_\mathcal{J}}$. □

Lemma 3.4. *A fuzzy subset f of $\mathcal{K}$ is a fuzzy Γ-$\mathcal{AG}$-subgroupoid of $\mathcal{K}$ if and only if $f \circ_\Gamma f \subseteq f$.*

Proof. The proof is straightforward. □

Theorem 3.2. *In a $(2,2)$-regular $\mathcal{K}$, the following statements are equivalent.*

(i) f is a fuzzy Γ-bi-(Γ-generalized bi-) ideal of $\mathcal{K}$.

(ii) $(f \circ_\Gamma \mathcal{K}) \circ_\Gamma f = f$ and $f \circ_\Gamma f = f$.

Proof. $(i) \Longrightarrow (ii)$: Assume that f is a fuzzy Γ-bi-ideal of a $(2,2)$-regular $\mathcal{K}$ and let $a \in \mathcal{K}$, then there exists $y \in \mathcal{K}$ such that $a \leq ((a\alpha a)\beta y)\gamma(a\delta a)$, where $\alpha, \beta, \gamma, \delta \in \Gamma$. Now by using (4), (1) and (3), we have

$$\begin{aligned}
a &\leq ((a\alpha a)\beta y)\gamma(a\delta a) = (a\beta a)\gamma(y\delta(a\alpha a)) = (a\beta a)\gamma x = (x\beta a)\gamma a \\
&\leq (x\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\gamma a = (x\beta((a\beta a)\gamma(y\delta(a\alpha a))))\gamma a \\
&= (x\beta((a\beta a)\gamma x))\gamma a = ((a\beta a)\beta(x\gamma x))\gamma a = ((x\beta x)\beta(a\gamma a))\gamma a \\
&= (((a\gamma a)\beta x)\beta x)\gamma a = (((x\gamma a)\beta a)\beta x)\gamma a \\
&= (((x\gamma(((a\alpha a)\beta y)\gamma(a\delta a)))\beta a)\beta x)\gamma a \\
&= (((x\gamma((a\beta a)\gamma(y\delta(a\alpha a))))\beta a)\beta x)\gamma a \\
&= (((x\gamma((a\beta a)\gamma x))\beta a)\beta x)\gamma a \\
&= ((((a\beta a)\gamma(x\gamma x))\beta a)\beta x)\gamma a \\
&= ((((x\beta x)\gamma(a\gamma a))\beta a)\beta x)\gamma a \\
&= (((a\gamma((x\beta x)\gamma a))\beta a)\beta x)\gamma a, \text{ where } (y\delta(a\alpha a)) = x \in \mathcal{K}.
\end{aligned}$$

Thus $((((a\gamma((x\beta x)\gamma a))\beta a)\beta x), a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}
((f \circ_\Gamma \mathcal{K}) \circ_\Gamma f)(a) &= \bigvee_{((((a\gamma((x\beta x)\gamma a))\beta a)\beta x),a)\in A_a} \{(f \circ_\Gamma \mathcal{K})(((a\gamma((x\beta x)\gamma a))\beta a)\beta x) \\
&\quad \wedge f(a)\} \\
&\geq \bigvee_{(((a\gamma((x\beta x)\gamma a))\beta a),x)\in A_{(((a\gamma((x\beta x)\gamma a))\beta a)\beta x)}} \{f((a\gamma((x\beta x)\gamma a))\beta a) \\
&\quad \wedge \mathcal{K}(x)\} \wedge f(a) \\
&\geq f((a\gamma((x\beta x)\gamma a))\beta a) \wedge 1 \wedge f(a) \\
&\geq f(a) \wedge f(a) \wedge f(a) = f(a).
\end{aligned}$$

Now by using (4), (1) and (3), we have

$$\begin{aligned}
a &\leq ((a\alpha a)\beta y)\gamma(a\delta a) = (a\beta a)\gamma(y\delta(a\alpha a)) = (a\beta a)\gamma x = (x\beta a)\gamma a \\
&\leq (x\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\gamma a = (x\beta((a\beta a)\gamma(y\delta(a\alpha a))))\gamma a \\
&= (x\beta((a\beta a)\gamma x))\gamma a = ((a\beta a)\beta(x\gamma x))\gamma a \\
&= ((x\beta x)\beta(a\gamma a))\gamma a = (a\beta((x\beta x)\gamma a))\gamma a, \text{ where } y\delta(a\alpha a) = x \in \mathcal{K}.
\end{aligned}$$

Thus $((a\beta((x\beta x)\gamma a)),a)\in A_a$, since $A_a\neq\emptyset$, therefore

$$\begin{aligned}((f\circ_\Gamma \mathcal{K})\circ_\Gamma f)(a) &= \bigvee_{((a\beta((x\beta x)\gamma a)),a)\in A_a}\{(f\circ_\Gamma\mathcal{K})(a\beta((x\beta x)\gamma a))\wedge f(a)\}\\ &= \bigvee_{((a\beta((x\beta x)\gamma a)),a)\in A_a}\left\{\begin{array}{c}\bigvee_{(a,(x\beta x)\gamma a)\in A_{a\beta((x\beta x)\gamma a)}} f(a)\\ \wedge\mathcal{K}((x\beta x)\gamma a)\end{array}\right\}\wedge f(a)\\ &= \bigvee_{((a\beta((x\beta x)\gamma a)),a)\in A_a}\left\{\bigvee_{(a,(x\beta x)\gamma a)\in A_{a\beta((x\beta x)\gamma a)}} f(a)\right\}\wedge f(a)\\ &= \bigvee_{((a\beta((x\beta x)\gamma a)),a)\in A_a}\{f(a)\wedge f(a)\}\\ &\leq \bigvee_{((a\beta((x\beta x)\gamma a)),a)\in A_a} f((a\beta((x\beta x)\gamma a))\gamma a)=f(a).\end{aligned}$$

Thus $(f\circ_\Gamma\mathcal{K})\circ_\Gamma f=f$.

Let $\xi\in\Gamma$, then by using (3), $(1),(5)$ and (4), we have

$$\begin{aligned}a &\leq ((a\alpha a)\beta y)\gamma(a\delta a)=(a\beta a)\gamma(y\delta(a\alpha a))=(a\beta a)\gamma x=(x\beta a)\gamma a\\ &\leq (x\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\gamma a=(x\beta((a\beta a)\gamma(y\delta(a\alpha a))))\gamma a\\ &= (x\beta((a\beta a)\gamma x))\gamma a=((a\beta a)\beta(x\gamma x))\gamma a=(((x\gamma x)\beta a)\beta a)\gamma a\\ &\leq (((x\gamma x)\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\beta a)\gamma a\\ &= (((x\gamma x)\beta((a\beta a)\gamma(y\delta(a\alpha a)))\beta a)\gamma a\\ &= (((x\gamma x)\beta((a\beta a)\gamma x)\beta a)\gamma a=(((x\gamma x)\beta((x\beta a)\gamma a))\beta a)\gamma a\\ &\leq (((x\gamma x)\beta((x\beta a)\gamma(b\xi c)))\beta a)\gamma a=(((x\gamma x)\beta((c\beta b)\gamma(a\xi x)))\beta a)\gamma a\\ &= (((x\gamma x)\beta(a\gamma((c\beta b)\xi x)))\beta a)\gamma a=((a\beta((x\gamma x)\gamma((c\beta b)\xi x)))\beta a)\gamma a,\end{aligned}$$

where $y\delta(a\alpha a)=x\in\mathcal{K}$.

Therefore $(((a\beta((x\gamma x)\gamma((c\beta b)\xi x)))\beta a),a)\in A_a$, since $A_a\neq\emptyset$. Thus

$$\begin{aligned}(f\circ_\Gamma f)(a) &= \bigvee_{(((a\beta((x\gamma x)\gamma((c\beta b)\xi x)))\beta a),a)\in A_a}\{f((a\beta((x\gamma x)\gamma((c\beta b)\xi x)))\beta a)\wedge f(a)\}\\ &\geq \{f((a\beta((x\gamma x)\gamma((c\beta b)\xi x)))\beta a)\wedge f(a)\}\\ &\geq f(a)\wedge f(a)\wedge f(a)=f(a),\end{aligned}$$

thus by using Lemma 3.4, $f\circ_\Gamma f=f$.

$(ii)\Longrightarrow(i)$: Let f be a fuzzy subset of a $(2,2)$-regular $\mathcal{K}$ and let $\alpha,\beta\in\Gamma$, then

$$\begin{aligned}f((x\alpha y)\beta z) &= ((f\circ_\Gamma\mathcal{K})\circ_\Gamma f)((x\alpha y)\beta z)\\ &= \bigvee_{(x\alpha y,z)\in A(x\alpha y)\beta z}\{(f\circ_\Gamma\mathcal{K})(x\alpha y)\wedge f(z)\}\\ &\geq \bigvee_{(x,y)\in A_{x\alpha y}}\{f(x)\wedge\mathcal{K}(y)\}\wedge f(z)\\ &\geq f(x)\wedge 1\wedge f(z)=f(x)\wedge f(z).\end{aligned}$$

Since $f\circ_\Gamma f=f$, therefore by Lemma 3.4, f is a fuzzy $\mathcal{AG}$-subgroupoid of $\mathcal{K}$. This shows that f is a fuzzy Γ-bi ideal of $\mathcal{K}$. □

Theorem 3.3. *In a $(2,2)$-regular $\mathcal{K}$, the following statements are equivalent.*

(i) f is a fuzzy Γ-interior ideal of $\mathcal{K}$.

(ii) $(\mathcal{K}\circ_\Gamma f)\circ_\Gamma \mathcal{K}=f$.

Proof. The proof is straightforward. □

Theorem 3.4. *In a $(2,2)$-regular $\mathcal{K}$, the following statements are equivalent.*

(i) f is a fuzzy Γ-$(1,2)$-ideal of $\mathcal{K}$.

(ii) $(f\circ_\Gamma \mathcal{K})\circ_\Gamma (f\circ_\Gamma f)=f$ and $f\circ_\Gamma f=f$.

Proof. $(i)\Longrightarrow(ii)$: Assume that f is a fuzzy Γ-$(1,2)$-ideal of a $(2,2)$-regular $\mathcal{K}$ and let $a\in\mathcal{K}$, then there exists $y\in\mathcal{K}$ such that $a\leq((a\alpha a)\beta y)\gamma(a\delta a)$, where $\alpha,\beta,\gamma,\delta\in\Gamma$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
a &\leq ((a\alpha a)\beta y)\gamma(a\delta a)=(a\beta a)\gamma(y\delta(a\alpha a))=(a\beta a)\gamma x=(x\beta a)\gamma a\\
&\leq (x\beta a)\gamma(((a\alpha a)\beta y)\gamma(a\delta a))=(x\beta a)\gamma((a\beta a)\gamma(y\delta(a\alpha a)))\\
&= (x\beta a)\gamma((a\beta a)\gamma x)=(a\beta a)\gamma((x\beta a)\gamma x)\\
&\leq (a\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\gamma((x\beta a)\gamma x)
\end{aligned}$$

$$\begin{aligned}
&= (a\beta((a\beta a)\gamma(y\delta(a\alpha a))))\gamma((x\beta a)\gamma x)\\
&= (a\beta((a\beta a)\gamma x))\gamma((x\beta a)\gamma x)\\
&= ((a\beta a)\beta(a\gamma x))\gamma((x\beta a)\gamma x)\\
&= (((x\beta a)\gamma x)\beta(a\gamma x))\gamma(a\beta a)\\
&= (a\beta(((x\beta a)\gamma x)\gamma x))\gamma(a\beta a),\ \text{where } y\delta(a\alpha a)=x\in\mathcal{K}.
\end{aligned}$$

Thus $((a\beta(((x\beta a)\gamma x)\gamma x)),a\beta a)\in A_a$, since $A_a\neq\emptyset$, therefore

$$\begin{aligned}
((f\circ_\Gamma\mathcal{K})\circ_\Gamma(f\circ_\Gamma f))(a) &= \bigvee_{((a\beta(((x\beta a)\gamma x)\gamma x)),a\beta a)\in A_a}\{(f\circ_\Gamma\mathcal{K})(a\beta(((x\beta a)\gamma x)\gamma x))\\
&\quad \wedge(f\circ_\Gamma f)(a\beta a)\}.
\end{aligned}$$

Now let $((x\beta a)\gamma x)\gamma x=c$, then

$$\begin{aligned}
(f\circ_\Gamma\mathcal{K})(a\beta c) &= \bigvee_{(a,c)\in A_{a\beta c}}\{f(a)\wedge\mathcal{K}(c)\}\\
&\geq f(a)\wedge\mathcal{K}(c)=f(a)
\end{aligned}$$

and

$$(f\circ_\Gamma f)(a\beta a)=\bigvee_{(a,a)\in A_{a\beta a}}\{f(a)\wedge f(a)\}\geq f(a).$$

Thus we get

$$((f\circ_\Gamma\mathcal{K})\circ_\Gamma(f\circ_\Gamma f))(a)\geq f(a).$$

Now by using (4), (1) and (3), we have

$$\begin{aligned} a &\leq ((a\alpha a)\beta y)\gamma(a\delta a) = (a\beta a)\gamma(y\delta(a\alpha a)) = (a\beta a)\gamma x \\ &\leq ((((a\alpha a)\beta y)\gamma(a\delta a))\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\gamma x \\ &= (((a\beta a)\gamma(y\delta(a\alpha a)))\beta((a\beta a)\gamma(y\delta(a\alpha a))))\gamma x \\ &= (((a\beta a)\gamma x)\beta((a\beta a)\gamma x))\gamma x = ((a\beta a)\beta(((a\beta a)\gamma x)\gamma x))\gamma x \\ &= ((a\beta a)\beta((x\gamma x)\gamma(a\beta a)))\gamma x = (x\beta((x\gamma x)\gamma(a\beta a)))\gamma(a\beta a) \\ &= (x\beta(a\gamma((x\gamma x)\beta a)))\gamma(a\beta a) = (a\beta(x\gamma((x\gamma x)\beta a)))\gamma(a\beta a) \\ &\leq (a\beta(x\gamma((x\gamma x)\beta(((a\alpha a)\beta y)\gamma(a\delta a)))))\gamma(a\beta a) \\ &= (a\beta(x\gamma((x\gamma x)\beta((a\beta a)\gamma(y\delta(a\alpha a))))))\gamma(a\beta a) \\ &= (a\beta(x\gamma((x\gamma x)\beta((a\beta a)\gamma x))))\gamma(a\beta a) \\ &= (a\beta(x\gamma((a\beta a)\beta((x\gamma x)\gamma x))))\gamma(a\beta a), \text{ where } y\delta(a\alpha a) = x \in \mathcal{K}. \end{aligned}$$

Therefore $((a\beta(x\gamma((a\beta a)\beta((x\gamma x)\gamma x)))), a\beta a) \in A_a$.
Let $(a\beta(x\gamma((a\beta a)\beta((x\gamma x)\gamma x)))) = a\beta t$. Since $A_a \neq \emptyset$. Thus

$$((f \circ_\Gamma \mathcal{K}) \circ_\Gamma (f \circ_\Gamma f))(a) = \bigvee_{(a\beta t, a\beta a)\in A_a} \{(f \circ_\Gamma \mathcal{K})(a\beta t) \wedge (f \circ_\Gamma f)(a\beta a)\}.$$

Now

$$(f \circ_\Gamma \mathcal{K})(a\beta t) = \bigvee_{(a,t)\in A_{a\beta t}} \{f(a) \wedge \mathcal{K}(t)\} = \bigvee_{(a,t)\in A_{a\beta t}} f(a)$$

and

$$(f \circ_\Gamma f)(a\beta a) = \bigvee_{(a,a)\in A_{a\beta a}} \{f(a) \wedge f(a)\} = \bigvee_{(a,a)\in A_{a\beta a}} f(a).$$

Therefore

$$\begin{aligned} (f \circ_\Gamma \mathcal{K})(a\beta t) \wedge (f \circ_\Gamma f)(a\beta a) &= \bigvee_{(a,t)\in A_{a\beta t}} f(a) \wedge \bigvee_{(a,a)\in A_{a\beta a}} f(a) \\ &= \bigvee_{(a,t)\in A_{a\beta t}} \{f(a) \wedge f(a)\}. \end{aligned}$$

Thus from above, we get

$$\begin{aligned} ((f \circ_\Gamma \mathcal{K}) \circ_\Gamma (f \circ_\Gamma f))(a) &= \bigvee_{((a\beta t),a\beta a)\in A_a} \left\{ \bigvee_{(a,t)\in A_{a\beta t}} f(a) \right\} \\ &= \bigvee_{((a\beta t),a\beta a)\in A_a} \{f(a) \wedge f(a) \wedge f(a)\} \\ &\leq \bigvee_{((a\beta t),a\beta a)\in A_a} f((a\beta t)(a\beta a)) = f(a). \end{aligned}$$

Therefore $(f \circ_\Gamma \mathcal{K}) \circ_\Gamma (f \circ_\Gamma f) = f$.

Now by using (1), (4) and (3), we have

$$\begin{aligned} a &\leq ((a\alpha a)\beta y)\gamma(a\delta a) = (a\beta a)\gamma(y\delta(a\alpha a)) = (a\beta a)\gamma x = (x\beta a)\gamma a \\ &\leq (x\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\gamma a = (x\beta((a\beta a)\gamma(y\delta(a\alpha a))))\gamma a \\ &= (x\beta((a\beta a)\gamma x))\gamma a = ((a\beta a)\beta(x\gamma x))\gamma a \\ &\leq ((a\beta(((a\alpha a)\beta y)\gamma(a\delta a)))\beta(x\gamma x))\gamma a \\ &= ((a\beta((a\beta a)\gamma(y\delta(a\alpha a))))\beta(x\gamma x))\gamma a \\ &= ((a\beta((a\beta a)\gamma x))\beta(x\gamma x))\gamma a \\ &= (((a\beta a)\beta(a\gamma x))\beta(x\gamma x))\gamma a \\ &= (((x\gamma x)\beta(a\gamma x))\beta(a\beta a))\gamma a \\ &= ((a\beta((x\gamma x)\gamma x))\beta(a\beta a))\gamma a. \end{aligned}$$

Thus $(((a\beta((x\gamma x)\gamma x))\beta(a\beta a)), a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned} (f \circ_\Gamma f)(a) &= \bigvee_{(((a\beta((x\gamma x)\gamma x))\beta(a\beta a)),a)\in A_a} \{f((a\beta((x\gamma x)\gamma x))\beta(a\beta a)) \wedge f(a)\} \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a). \end{aligned}$$

Now by using Lemma 3.4, $f \circ_\Gamma f = f$.

$(ii) \Longrightarrow (i)$: Let f be a fuzzy subset of a $(2,2)$-regular $\mathcal{K}$. Now since $f \circ_\Gamma f = f$, therefore by Lemma 3.4, f is a fuzzy $\mathcal{AG}$-subgroupoid of $\mathcal{K}$. Let $\alpha, \beta, \gamma \in \Gamma$, then

$$\begin{aligned} f((x\alpha a)\beta(y\gamma z)) &= ((f \circ_\Gamma \mathcal{K}) \circ_\Gamma (f \circ_\Gamma f))((x\alpha a)\beta(y\gamma z)) \\ &= ((f \circ_\Gamma \mathcal{K}) \circ_\Gamma f)((x\alpha a)\beta(y\gamma z)) \\ &= \bigvee_{((x\alpha a),(y\gamma z))\in A_{(x\alpha a)\beta(y\gamma z)}} \{(f \circ_\Gamma \mathcal{K})(x\alpha a) \wedge f(y\gamma z)\}. \\ &\geq (f \circ_\Gamma \mathcal{K})(x\alpha a) \wedge f(y\gamma z) \\ &= \bigvee_{(x,a)\in A_{x\alpha a}} \{f(x) \wedge \mathcal{K}(a)\} \wedge f(y\gamma z) \\ &\geq f(x) \wedge 1 \wedge f(y) \wedge f(z) = f(x) \wedge f(y) \wedge f(z). \end{aligned}$$

This shows that $f((x\alpha a)\beta(y\gamma z)) \geq f(x) \wedge f(y) \wedge f(z)$, therefore f is a fuzzy Γ-$(1,2)$-ideal of $\mathcal{K}$. □

Remark 3.1. *An element a of an ordered Γ-$\mathcal{AG}$-groupoid $\mathcal{K}$ is called a $(2,2)$-regular element of $\mathcal{K}$ if there exist some $x, y, z \in \mathcal{K}$ and $\alpha, \beta, \gamma, \delta, \xi \in \Gamma$, such that*

$$a \leq ((a\alpha a)\beta z)\gamma(a\delta a) \leq (x\beta(a\xi a))\gamma y$$

Indeed by using (5) and (4), we have

$$\begin{aligned} a &\leq ((a\alpha a)\beta z)\gamma(a\delta a) \leq ((a\alpha a)\beta(a\xi b))\gamma(a\delta a) = ((b\alpha a)\beta(a\xi a))\gamma(a\delta a) \\ &= (x\beta(a\xi a))\gamma y, \text{ where } b\alpha a = x \in \mathcal{K} \text{ and } a\delta a = y \in \mathcal{K}. \qquad (6) \end{aligned}$$

$$\Longrightarrow a \leq (x\beta(a\xi a))\gamma y.$$

An element a of an ordered Γ-$\mathcal{AG}$-groupoid $\mathcal{K}$ having the form (6) is called an intra-regular element of $\mathcal{K}$. Similarly $a \leq (x\beta(a\xi a))\gamma y \leq ((a\alpha a)\beta z)\gamma(a\delta a)$ holds for some $x, y, z \in \mathcal{K}$ and $\alpha, \beta, \gamma, \delta, \xi \in \Gamma$. This shows that the concepts of $(2,2)$-regular and an intra-regular coincide in $\mathcal{K}$.

Lemma 3.5. *A fuzzy subset f of a $(2,2)$-regular $\mathcal{K}$ is a fuzzy Γ-right ideal if and only if it is a fuzzy Γ-left ideal.*

Proof. Assume that f is a fuzzy Γ-right ideal of $\mathcal{K}$. Since $\mathcal{K}$ a $(2,2)$-regular, so for each $a \in \mathcal{K}$ there exist $x, y \in \mathcal{K}$ and $\beta, \xi, \gamma \in \Gamma$ such that $a \leq (x\beta(a\xi a))\gamma y$. Now let $\alpha \in \Gamma$, then by using (1), we have

$$f(a\alpha b) \geq f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \geq f(b\gamma y) \geq f(b).$$

Conversely, assume that f is a fuzzy Γ-left ideal of $\mathcal{K}$. Now by using (1), we have

$$\begin{aligned} f(a\alpha b) &\geq f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &\geq f(x\beta(a\xi a)) \geq f(a\xi a) \geq f(a). \end{aligned}$$

□

Theorem 3.5. *For a fuzzy subset f of a $(2,2)$-regular $\mathcal{K}$, the following statements are equivalent.*
(i) f is a fuzzy Γ-two-sided ideal of $\mathcal{K}$.
(ii) f is a fuzzy Γ-interior ideal of $\mathcal{K}$.

Proof. $(i) \Rightarrow (ii)$: Let f be any fuzzy Γ-two- sided ideal of $\mathcal{K}$, then obviously f is a fuzzy Γ-interior ideal of $\mathcal{K}$.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-interior ideal of $\mathcal{K}$ and $a, b \in \mathcal{K}$. Since $\mathcal{K}$ is a $(2,2)$-regular, so for each $a, b \in \mathcal{K}$ there exist $x, y, u, v \in \mathcal{K}$ and $\beta, \xi, \gamma, \delta, \psi, \eta \in \Gamma$ such that $a \leq (x\beta(a\xi a))\gamma y$ and $b \leq (u\delta(b\psi b))\eta v$. Now let $\alpha, \in \Gamma$, thus by using (1), (3) and (2), we have

$$\begin{aligned} f(a\alpha b) &\geq f((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &= f((b\gamma y)\alpha(a\beta(x\xi a))) = f((b\gamma a)\alpha(y\beta(x\xi a))) \geq f(a). \end{aligned}$$

Also by using (3), (4) and (2), we have

$$\begin{aligned} f(a\alpha b) &\geq f(a\alpha((u\delta(b\psi b))\eta v)) = f((u\delta(b\psi b))\alpha(a\eta v)) \\ &= f((b\delta(u\psi b))\alpha(a\eta v)) = f((v\delta a)\alpha((u\psi b)\eta b)) \\ &= f((u\psi b)\alpha((v\delta a)\eta b)) \geq f(b). \end{aligned}$$

Hence f is a fuzzy Γ-two- sided ideal of $\mathcal{K}$. □

Theorem 3.6. *A fuzzy subset f of a $(2,2)$-regular $\mathcal{K}$ is a fuzzy Γ-two-sided ideal if and only if it is a fuzzy Γ-quasi ideal.*

Proof. The proof is straightforward. □

Theorem 3.7. *For a fuzzy subset f of a $(2,2)$-regular $\mathcal{K}$, the following conditions are equivalent.*
(i) f is a fuzzy Γ-bi-ideal of $\mathcal{K}$.
(ii) f is a fuzzy Γ-generalized bi-ideal of $\mathcal{K}$.

Proof. $(i) \Longrightarrow (ii)$: Let f be any fuzzy Γ-bi-ideal of $\mathcal{K}$, then obviously f is a fuzzy Γ-generalized bi-ideal of $\mathcal{K}$.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-generalized bi-ideal of $\mathcal{K}$, and a, $b \in \mathcal{K}$. Since $\mathcal{K}$ is a $(2,2)$-regular so for each $a \in \mathcal{K}$ there exist $x, y \in \mathcal{K}$ and $\beta, \gamma, \xi \in \Gamma$ such that $a \leq (x\beta(a\xi a))\gamma y$. Now let $\alpha, \delta, \in \Gamma$, then by using (5), (4), (2) and (3), we have

$$\begin{aligned} f(a\alpha b) &\geq f(((x\beta(a\xi a))\gamma y)\alpha b) = f(((x\beta(a\xi a))\gamma(u\delta v))\alpha b) \\ &= f(((v\beta u)\gamma((a\xi a)\delta x))\alpha b) = f(((a\xi a)\gamma((v\beta u)\delta x))\alpha b) \\ &= f(((x\gamma(v\beta u))\delta(a\xi a))\alpha b) = f((a\delta((x\gamma(v\beta u))\xi a))\alpha b) \geq f(a) \wedge f(b). \end{aligned}$$

Therefore f is a fuzzy Γ-bi-ideal of $\mathcal{K}$. □

Theorem 3.8. *For a fuzzy subset f of a $(2,2)$-regular $\mathcal{K}$, the following conditions are equivalent.*

(i) f is a fuzzy Γ-two- sided ideal of $\mathcal{K}$.

(ii) f is a fuzzy Γ-bi-ideal of $\mathcal{K}$.

Proof. $(i) \Longrightarrow (ii)$: Let f be any fuzzy Γ-two- sided ideal of $\mathcal{K}$, then obviously f is a fuzzy Γ-bi-ideal of $\mathcal{K}$.

$(ii) \Rightarrow (i)$: Let f be any fuzzy Γ-bi-ideal of $\mathcal{K}$. Since $\mathcal{K}$ is a $(2,2)$-regular so for each $a, b \in \mathcal{K}$ there exist $x, y, u, v \in \mathcal{K}$ and $\beta, \xi, \gamma, \delta, \psi, \eta \in \Gamma$ such that $a \leq (x\beta(a\xi a))\gamma y$ and $b \leq (u\delta(b\psi b))\eta v$. Now let $\alpha \in \Gamma$, then by using (1), (4), (2) and (3), we have

$$\begin{aligned} f(a\alpha b) &\geq f(((x\beta(a\xi a))\gamma y)\alpha b) = f((b\gamma y)\alpha(x\beta(a\xi a))) \\ &= f(((a\xi a)\gamma x)\alpha(y\beta b)) = f(((y\beta b)\gamma x)\alpha(a\xi a)) \\ &= f((a\gamma a)\,\alpha(x\xi(y\beta b))) = f(((x\xi(y\beta b))\gamma a)\,\alpha a) \\ &\geq f(((x\xi(y\beta b))\gamma((x\beta(a\xi a))\gamma y))\alpha a) \\ &= f(((x\beta(a\xi a))\gamma((x\xi(y\beta b))\gamma y))\alpha a) \\ &= f(((y\beta(x\xi(y\beta b)))\gamma((a\xi a)\gamma x))\alpha a) \\ &= f(((a\xi a)\gamma((y\beta(x\xi(y\beta b)))\gamma x))\alpha a) \\ &= f(((x\xi(y\beta(x\xi(y\beta b))))\gamma\,(a\gamma a))\alpha a) \\ &= f((a\gamma\,((x\xi(y\beta(x\xi(y\beta b))))\gamma a))\alpha a) \\ &\geq f(a) \wedge f(a) = f(a). \end{aligned}$$

Now by using (3), (4) and (1), we have

$$\begin{aligned} f(a\alpha b) &\geq f(a\alpha(u\delta(b\psi b))\eta v) = f((u\delta(b\psi b))\alpha(a\eta v)) \\ &= f((v\delta a)\alpha((b\psi b)\eta u)) = f((b\psi b)\alpha((v\delta a)\eta u)) \\ &= f((((v\delta a)\eta u)\psi b)\,\alpha b) \geq f((((v\delta a)\eta u)\psi((u\delta(b\psi b))\eta v))\,\alpha b) \end{aligned}$$

$$\begin{aligned} &= f(((u\delta(b\psi b))\psi(((v\delta a)\eta u)\eta v))\,\alpha b) = f(((v\delta((v\delta a)\eta u))\psi((b\psi b)\eta u))\,\alpha b) \\ &= f(((b\psi b)\psi((v\delta((v\delta a)\eta u))\eta u))\,\alpha b) = f(((u\psi(v\delta((v\delta a)\eta u)))\psi(b\eta b))\,\alpha b) \\ &= f((b\psi((u\psi(v\delta((v\delta a)\eta u)))\eta b))\,\alpha b) \geq f\,(b) \wedge f\,(b) = f\,(b)\,. \end{aligned}$$

□

Theorem 3.9. *For a fuzzy subset f of a $(2,2)$-regular $\mathcal{K}$, the following conditions are equivalent.*

(i) f is a fuzzy Γ-left ideal of $\mathcal{K}$.

(ii) f is a fuzzy Γ-right ideal of $\mathcal{K}$.

(iii) f is a fuzzy Γ-two-sided ideal of $\mathcal{K}$.
(iv) f is a fuzzy Γ-bi-ideal of $\mathcal{K}$.
(v) f is a fuzzy Γ-generalized bi- ideal of $\mathcal{K}$.
(vi) f is a fuzzy Γ-interior ideal of $\mathcal{K}$.
(vii) f is a fuzzy Γ-quasi ideal of $\mathcal{K}$.
(viii) $\mathcal{K} \circ_\Gamma f = f = f \circ_\Gamma \mathcal{K}$.

Proof. $(i) \Longrightarrow (viii)$: It follows from Lemma 3.1.

$(viii) \Longrightarrow (vii)$: It is obvious.

$(vii) \Longrightarrow (vi)$: Let f be a fuzzy Γ-quasi ideal of a $(2,2)$-regular $\mathcal{K}$ and let $a \in \mathcal{K}$, then there exist $b, c \in \mathcal{K}$ and $\beta, \gamma, \xi \in \Gamma$ such that $a \leq (b\beta(a\xi a))\gamma c$. Let $\delta, \eta \in \Gamma$, then by using (3), (4) and (1), we have

$$\begin{aligned}(x\delta a)\eta y &\leq (x\delta(b\beta(a\xi a))\gamma c)\eta y = ((b\beta(a\xi a))\delta(x\gamma c))\eta y \\ &= ((c\beta x)\delta((a\xi a)\gamma b))\eta y = ((a\xi a)\delta((c\beta x)\gamma b))\eta y \\ &= (y\delta((c\beta x)\gamma b))\eta(a\xi a) = a\eta((y\delta((c\beta x)\gamma b))\xi a)\end{aligned}$$

and from above

$$(x\delta a)\eta y \leq (y\delta((c\beta x)\gamma b))\eta(a\xi a) = (a\delta a)\eta(((c\beta x)\gamma b)\xi y) = ((((c\beta x)\gamma b)\xi y)\delta a)\eta a.$$

Now by using Lemma 3.1, we have

$$\begin{aligned}f((x\delta a)\eta y) &= ((f \circ_\Gamma \mathcal{K}) \cap (\mathcal{K} \circ_\Gamma f))((x\delta a)\eta y) \\ &= (f \circ_\Gamma \mathcal{K})((x\delta a)\eta y) \wedge (\mathcal{K} \circ_\Gamma f)((x\delta a)\eta y).\end{aligned}$$

Now

$$(f \circ_\Gamma \mathcal{K})((x\delta a)\eta y) = \bigvee_{(a,(y\delta((c\beta x)\gamma b))\xi a) \in A_{(x\delta a)\eta y}} \{f(a) \wedge \mathcal{K}((y\delta((c\beta x)\gamma b))\xi a)\} \geq f(a)$$

and

$$(\mathcal{K} \circ_\Gamma f)((x\delta a)\eta y) = \bigvee_{((y\delta((c\beta x)\gamma b))\xi a),a) \in A_{(x\delta a)\eta y}} \{\mathcal{K}((((c\beta x)\gamma b)\xi y)\delta a) \wedge f(a)\} \geq f(a).$$

This implies that $f((x\delta a)\eta y) \geq f(a)$ and therefore f is a fuzzy Γ-interior ideal of $\mathcal{K}$.

$(vi) \Longrightarrow (v)$: It follows from Theorems 3.5, 3.8 and 3.7.

$(v) \Longrightarrow (iv)$: It follows from Theorem 3.7.

$(iv) \Longrightarrow (iii)$: It follows from Theorem 3.8.

$(iii) \Longrightarrow (ii)$: It is obvious and $(ii) \Longrightarrow (i)$ can be followed from Lemma 3.5. □

4. The construction of Γ-$\mathcal{AG}$-groupoids

Let us consider an $\mathcal{AG}$-groupod $\mathcal{K} = \{a, b, c\}$ under the binary operation "$*$" in the table below.

$*$	a	b	c
a	c	c	c
b	c	c	a
c	c	c	a

$*$-table

Let $\Gamma = \{\alpha, \beta, \gamma\}$ be the set of three operations defined on $\mathcal{K}$ in the following tables.

α	a	b	c
a	c	c	c
b	c	c	a
c	c	c	a

α-table

β	a	b	c
a	c	c	a
b	c	c	a
c	c	c	c

β-table

γ	a	b	c
a	c	c	a
b	c	c	c
c	c	c	a

γ-table

To form an α-table, replace "$*$" by "α" in $*$-table. Now to form the β-table, copy the c-row ($c\ c\ a$) from α-table in to the b-row of α-table, copy the b-row ($c\ c\ a$) from α-table in to the a-row of α-table, and similarly copy the a-row ($c\ c\ c$) from α-table in to the c-row of α-table. The resulting table is β-table. In a similar way, we can get the γ-table. If we repeat the same process again, we will get back an α-table. It is easy to see that $\mathcal{K} = \{a, b, c\}$ is a Γ-$\mathcal{AG}$-groupod with $\Gamma = \{\alpha, \beta, \gamma\}$.

Note that in the above example of a Γ-$\mathcal{AG}$-groupod, both $\mathcal{K}$ and Γ have the same number of elements. In this method, the number of operations (at most) in Γ will be the same as that of the number of elements in $\mathcal{K}$.

In general, if $\mathcal{K}$ is an $\mathcal{AG}$-groupoid having n elements ($n > 2$), then the possible number of operations in Γ will also be n. The same method can also be applied in a similar way to get a Γ-semigroup.

5. **Conclusions**

In the structural theory of fuzzy algebraic systems, fuzzy ideals with special properties always play an important role. In this paper, we applied the fuzzy set theory to ordered Γ-$\mathcal{AG}$-groupoids and studied some important features of $(2, 2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoids in terms of fuzzy Γ-left ideals, fuzzy Γ-right ideals, fuzzy Γ-two-sided ideals, fuzzy Γ-generalized bi-ideals, fuzzy Γ-bi-ideals, fuzzy Γ-interior ideals and fuzzy Γ-$(1, 2)$-ideals. Further we characterized all the fuzzy Γ-ideals of a $(2, 2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoid and we also proved that all fuzzy Γ-ideals coincide in a $(2, 2)$-regular ordered Γ-$\mathcal{AG}^{**}$-groupoid. Finally we proposed the method to construct a Γ-$\mathcal{AG}$-groupoid.

REFERENCES

[1] *C.L. Chang*, Fuzzy topological spaces, J. Math. Anal. Appl. **24** (1968), 182-190.

[2] *M.A. Kazim and M. Naseeruddin*, On almost semigroups, The Alig. Bull. Math. **2** (1972), 1-7.

[3] *M. Khan*, Some studies in AG*-groupoids, Ph. D., thesis, Quaid-i-Azam University, Islamabad, Pakistan, 2008.

[4] *M. Khan and Faisal*, On fuzzy ordered Abel-Grassmann's groupoids. Journal of Mathematics Research, **3** (2011), 27-40.

[5] *N. Kuroki*, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli, **27** (1979), 17-21.

[6] *Q. Mushtaq and S.M. Yusuf*, On LA-semigroups, The Alig. Bull. Math., **8** (1978), 65-70.

[7] *Q. Mushtaq and S.M. Yusuf*, On AG-groupoid defined by a commutative inverse semigroup. Math. Bech., **40** (1988), 59-62.

[8] *M.K. Sen*, On Γ-semigroups, Proceeding of International Symposium on Algebra and Its Applications, Decker Publication, New York, (1981), 301-308.

[9] *N. Stevanović and P.V. Protić*, Composition of Abel-Grassmann's 3-bands, Novi Sad, J. Math., **2(34)** (2004), 175-182.

[10] *A. Rosenfeld*, Fuzzy groups, J. Math. Anal. Appl., **35** (1971), 512-517.

[11] *X.Y. Xie and J. Tang*, Fuzzy radicals and prime fuzzy ideals of ordered semigroups, Inform. Sci., **178** (2008), 4357-4374.

[12] *L.A. Zadeh*, Fuzzy sets. Inform. Control, **8** (1965), 338-353.