SOME PROPERTIES OF DERIVATIONS AND m-k-HYPERIDEALS IN ORDERED SEMIHYPERRINGS

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In this paper, at first, we state the concept of derivations on ordered semihyperrings and give some examples. Afterward, m-k-hyperideals in ordered semihyperrings are defined and some results in this respect are investigated. Also, some important properties of the derivations and m-k-hyperideals are studied. Finally, we study the kernel of derivations and the relation between m-k-hyperideals and k-hyperideals on ordered semihyperrings.

Keywords: algebraic hyperstructure; ordered semihyperring; derivation; m-k-hyperideal of type 1.

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1. Introduction and Preliminaries

Hyperstructure theory have been used in diverse branches of Mathematics, Physics, Biology and etc. Some applications of hyperstructure theory in mathematics, cryptography, codes and other fields can be found in [5]. In 2016, Davvaz and Musavi [9] defined cyclic codes and linear codes over hyperfields. Derivations have been applied in coding theory [4]. In [4], codes are constructed over skew polynomial rings, where the multiplication is defined by a derivation. In [6], derivations are applied to construct binary codes as images of derivations of group algebras. In this paper, we are interested in derivations of ordered semihyperrings. We show that If d is a homo-derivation on a positive ordered semihyperring R, then $Ker(d)$ is a k-hyperideal of type 1 of R.

The concept of hyperstructure was first introduced in 1934 by Marty [21] at the 8th Congress of Scandinavian Mathematicians. Several books have been written on the hyperstructure theory, for example, see [7, 8]. The notion of hyperrings and hyperfields was introduced by Krasner [20] as a generalization of rings. Hyperrings and hyperfields were introduced by Krasner in connection with his work on valued fields. In [17], Jun studied algebraic and geometric aspects of Krasner hyperrings.

In 2019, Koam et al. [19] discussed the notion of an ordered quasi(bi)-Γ-ideal in an ordered Γ-semiring. In 2017, Zhang and Li [29] studied derivations of partially ordered sets. In [12], Ebrahimi and Pajoohesh studied inner and homo derivations on l-rings. The notion of derivations first appeared in Posner's classic paper [26]. Derivations has been of great interest to different fields of science. The study of derivations is one interesting topic in hyperstructure theory. Asokkumar [3] and Kamali Ardekani and Davvaz [18] initiated the study of derivations on hyperrings and prime hyperrings.

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The notion of ordered semihypergroups has been studied by many authors in various directions like pseudoorder [10], ordered regular equivalence relation [13] and ordered hyperideal [14]. The concept of ordered semihypergroups introduced by Heidari and Davvaz [14] as a generalization of semihypergroups and ordered semigroups. Davvaz et al. constructed in [10] the ordered semigroup from an ordered semihypergroup by using pseudoorder. Then, Gu and Tang in [13] considered the construction of the ordered semihypergroup from an ordered semihypergroup by using ordered regular equivalence relation.

Omidi and Davvaz initiated the study of derivations on ordered hyperrings in [22, 23]. One of the most important research areas in (ordered) semihyperring theory is the investigation of k-hyperideals [2, 24]. For more information and results on k-hyperideals one may see [2, 24]. Further introduction to ordered semihyperrings can be found in [25].

In [24], Omidi and Davvaz studied the concept of 2-prime (of type 1) hyperideals of ordered semihyperrings using k-hyperideals (of type 1). In 2008, Akram and Dudek [1] investigated some properties of intuitionistic fuzzy left k-ideals of semirings. In [11], Dutta et al. studied some properties of k-regularity of semirings in terms of interval-valued fuzzy k-ideals. In [27], Rao et al. studied some properties of left k-bi-quasi hyperideals in ordered semihyperrings.

Though semiring is a generalization of ring, ideals of semiring do not coincide with ring ideals. For example an ideal of semiring need not be the kernel of homomorphism. To solve this problem Henriksen [15] defined k-ideals in semirings to obtain analogues of ring results. Generalization of hyperideals in (ordered) semihyperrings is necessary for further study of (ordered) semihyperrings. Omidi and Davvaz discussed some properties of key hyperideals (briefly, k-hyperideals) of an ordered semihyperring in [24]. We extend the definition in [24] to multiplicative k-hyperideals (briefly, m-k-hyperideals). In this paper, we study some properties of m-k-hyperideals and derivations of ordered semihyperrings.

Definition 1.1. [28] *A semihyperring is an algebraic hypersructure $(R, +, \cdot)$ which satisfies the following axioms:*

(1) *$(R, +)$ is a commutative semihypergroup with a zero element 0 satisfying $x + 0 = 0 + x = \{x\}$, i.e., (i) For all $x, y, z \in R, x + (y + z) = (x + y) + z$, (ii) For all $x, y \in R, x + y = y + x$, (iii) There exists $0 \in R$ such that $x + 0 = 0 + x = \{x\}$ for all $x \in R$;*
(2) *$(R, \cdot)$ is a semihypergroup;*
(3) *The multiplication $\cdot$ is distributive with respect to the hyperoperation $+$, that is, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in R$;*
(4) *The element $0 \in R$ is an absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.*

A non-empty subset A of a semihyperring $(R, +, \cdot)$ is called a *subsemihyperring* of R if for all $x, y \in A$, $x + y \subseteq A$ and $x \cdot y \subseteq A$. The notion of ordered semirings can be extended to the hyper version. Now, we recall the concept of ordered semihyperring from [24].

Definition 1.2. *An ordered semihyperring is a semihyperring $(R, +, \cdot)$ together with a (partial) order relation $\leq$ such that for all $a, b, x \in R$, we have*

(1) *$a \leq b$ implies $a + x \leq b + x$, meaning that for any $u \in a + x$, there exists $v \in b + x$ such that $u \leq v$.*
(2) *$a \leq b$ and $0 \leq x$ imply $a \cdot x \leq b \cdot x$, meaning that for any $u \in a \cdot x$, there exists $v \in b \cdot x$ such that $u \leq v$. The case $x \cdot a \leq x \cdot b$ is defined similarly.*

In ordered hyperstructure theory, the covering relation is the transitive reflexive reduction of a (partial) order relation $\leq$. An element a of an ordered semihyperring $(R, +, \cdot, \leq)$ covers another element b provided that there exists no third element $c \in R$ for which $b \leq c \leq a$. The covering relation is used to graphically express the (partial) order $\leq$ by means of the Hasse diagram.

Definition 1.3. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-empty subset A of R is said to be a left (resp. right) hyperideal of R if it satisfies the following conditions:*

(1) $(A, +)$ *is a semihypergroup;*
(2) $R \cdot A \subseteq A$ *(resp. $A \cdot R \subseteq A$);*
(3) *For any $a \in A$ and $b \in R$, $b \leq a$ implies $b \in A$.*

If A is both a left and a right hyperideal of R, then A is called a *hyperideal* of R.

There are two types of $k-$hyperideals. In fact, we can consider two definitions for a k-hyperideal, by replacing $a + x \cap I \neq \emptyset$ by $a + x \subseteq I$.

Definition 1.4. [24] *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-empty subset I of R is called a left k-hyperideal of R, if I is a left hyperideal of R and for any $a \in I$ and $x \in R$, from $a + x \approx I$ it follows $x \in I$, where we say that $A \approx B$ if $A \bigcap B \neq \emptyset$. A right k-hyperideal is defined similarly. If a hyperideal I is both left and right k-hyperideal, then I is known as a k-hyperideal of R.*

Following [24], a non-empty subset A of an ordered semihyperring R is called a *left k-hyperideal of type 1* of R, if A is a left hyperideal of R and for any $a \in A$ and $x \in R$, from $a + x \subseteq A$ it follows $x \in A$. A right k-hyperideal of type 1 is defined similarly. If a hyperideal A is both left and right k-hyperideal of type 1, then A is known as a *k-hyperideal of type 1* of R.

Clearly, every k-hyperideal is a k-hyperideal of type 1.

2. Main Results

In this section, for the first time we study the concept of derivations of an ordered semihyperring and present some results in this respect. Moreover, we introduce m-k-hyperideals of type 1 in ordered semihyperrings and investigate some of their properties.

Definition 2.1. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A function $d : R \to R$ is called a derivation of R if it satisfies the following conditions:*

(1) $d(x + y) \subseteq d(x) + d(y)$ *for all $x, y \in R$;*
(2) $d(x \cdot y) \subseteq d(x) \cdot y + x \cdot d(y)$ *for all $x, y \in R$;*
(3) *d is isotone, that is, for any $x, y \in R$, $x \leq y$ implies $d(x) \leq d(y)$.*

A function $d : R \to R$ is called *strong derivation* if for all $x, y \in R$, it satisfies (3), (2) $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$ and (1) $d(x+y) = d(x)+d(y)$. Let d be a derivation of an ordered semihyperring R. Then $d(0) = 0$. Indeed: $d(0) = d(0 \cdot 0) \subseteq d(0) \cdot 0 + 0 \cdot d(0) = 0 + 0 = 0$. We continue this section with some examples.

Example 2.1. *Consider the ordered semihyperring $R = \{0, a, b\}$ with the hyperaddition $+$, the hypermultiplication $\cdot$ and the order relation $\leq$ defined as follows:*

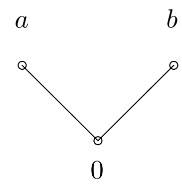
$+$	0	a	b
0	0	a	b
a	a	$\{0, a\}$	R
b	b	R	$\{0, b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	$\{0, a\}$	$\{0, b\}$
b	0	$\{0, b\}$	$\{0, a\}$

$$\leq := \{(0, 0), (a, a), (b, b), (0, a), (0, b)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0, a), (0, b)\}.$$



Define a map $d : R \to R$ *by* $d(0) = 0, d(a) = b$ *and* $d(b) = a$. *Now,* $d(a + a) = d(\{0, a\}) = \{0, b\} = b + b = d(a) + d(a)$ *and* $d(a \cdot a) = d(\{0, a\}) = \{0, b\} = \{0, b\} + \{0, b\} = b \cdot a + a \cdot b = d(a) \cdot a + a \cdot d(a)$. *Also,* $d(a+b) = d(R) = R = b+a = d(a)+d(b)$, $d(a \cdot b) = d(\{0, b\}) = \{0, a\} = \{0, a\}+\{0, a\} = b \cdot b+a \cdot a = d(a) \cdot b+a \cdot d(b)$, $d(b+b) = d(\{0, b\}) = \{0, a\} = a+a = d(b)+d(b)$ *and* $d(b \cdot b) = d(\{0, a\}) = \{0, b\} = \{0, b\} + \{0, b\} = a \cdot b + b \cdot a = d(b) \cdot b + b \cdot d(b)$. *We can easily verify that* $x \leq y$ *implies* $d(x) \leq d(y)$, *for all* $x, y \in R$. *Hence,* d *is a strong derivation on* R.

Example 2.2. *Consider the ordered semihyperring* $R = \{0, a, b, c\}$ *with the hyperaddition* $+$, *the hypermultiplication* $\cdot$ *and the order relation* $\leq$ *defined as follows:*

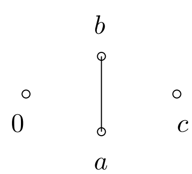
$+$	0	a	b	c
0	0	a	b	c
a	a	$\{a, b\}$	b	c
b	b	b	$\{0, b\}$	c
c	c	c	c	$\{0, c\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

$$\leq := \{(0, 0), (a, a), (a, b), (b, b), (c, c)\}.$$

The covering relation and the figure of R *are given by:*

$$\prec = \{(a, b)\}.$$



Define a map $d : R \to R$ *by* $d(0) = 0, d(a) = b, d(b) = b, d(c) = c$. *Now,* $d(a+a) = d(\{a, b\}) = b \in \{0, b\} = b+b = d(a)+d(a)$ *and* $d(a \cdot a) = d(a) = b = b+a = b \cdot a+a \cdot b = d(a) \cdot a+a \cdot d(a)$. *Also,* $d(a + b) = d(b) = b \in \{0, b\} = b + b = d(a) + d(b)$, $d(a \cdot b) = d(a) = b = b + a = b \cdot b + a \cdot b = d(a) \cdot b + a \cdot d(b)$, $d(a + c) = d(c) = c = b + c = d(a) + d(c)$, $d(a \cdot c) = d(a) = b = b + a = b \cdot c + a \cdot c = d(a) \cdot c + a \cdot d(c)$, $d(b + b) = d(\{0, b\}) = \{0, b\} = b + b = d(b) + d(b)$, $d(b \cdot b) = d(b) = b \in \{0, b\} = b + b = b \cdot b + b \cdot b = d(b) \cdot b + b \cdot d(b)$, $d(b + c) = d(c) = c = b + c = d(b) + d(c)$, $d(b \cdot c) = d(b) = b \in \{0, b\} = b + b = b \cdot c + b \cdot c = d(b) \cdot c + b \cdot d(c)$, $d(c + c) = d(\{0, c\}) = \{0, c\} = c + c = d(c) + d(c)$, $d(c \cdot c) = d(c) = c \in \{0, c\} = c + c = c \cdot c+c \cdot c = d(c) \cdot c+c \cdot d(c)$, $d(b \cdot a) = d(b) = b \in \{0, b\} = b+b = b \cdot a+b \cdot b = d(b) \cdot a+b \cdot d(a)$, $d(c \cdot a) = d(c) = c \in \{0, c\} = c + c = c \cdot a + c \cdot b = d(c) \cdot a + c \cdot d(a)$ *and* $d(c \cdot b) = d(c) = c \in \{0, c\} = c + c = c \cdot b + c \cdot b = d(c) \cdot b + c \cdot d(b)$. *By routine checking, we can verify that* $a \leq b$ *implies* $d(a) \leq d(b)$, *for all* $a, b \in R$. *Therefore,* d *is a derivation on* R.

Definition 2.2. *A derivation* d *on an ordered semihyperring* $(R, +, \cdot, \leq)$ *is called positive if* $d(x) \geq 0$ *for* $x \geq 0$. *We say that a positive derivation* d *on* R *is a homo-derivation on* R *if* $d(a \cdot b) = d(a) \cdot d(b)$.

Example 2.3. *Consider the ordered semihyperring $R = \{0, a, b\}$ with the hyperaddition $+$, the hypermultiplication $\cdot$ and the order relation $\leq$ defined as follows:*

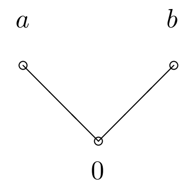
$+$	0	a	b
0	0	a	b
a	a	a	$\{a, b\}$
b	b	$\{a, b\}$	b

$\cdot$	0	a	b
0	0	0	0
a	0	$\{0, a\}$	$\{0, a\}$
b	0	$\{0, b\}$	$\{0, b\}$

$$\leq := \{(0, 0), (a, a), (b, b), (0, a), (0, b)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0, a), (0, b)\}.$$



Define a map $d : R \to R$ by $d(0) = 0, d(a) = b$ and $d(b) = a$. Now, $d(a+a) = d(a) = b = b+b = d(a)+d(a)$ and $d(a\cdot a) = d(\{0, a\}) = \{0, b\} \subseteq \{0, b\}+\{0, a\} = b\cdot a+a\cdot b = d(a)\cdot a+a\cdot d(a)$. Also, $d(a + b) = d(\{a, b\}) = \{a, b\} = b + a = d(a) + d(b)$, $d(a \cdot b) = d(\{0, a\}) = \{0, b\} \subseteq \{0, b\} + \{0, a\} = b \cdot b + a \cdot a = d(a) \cdot b + a \cdot d(b)$, $d(b + b) = d(b) = a = a + a = d(b) + d(b)$ and $d(b \cdot b) = d(\{0, b\}) = \{0, a\} \subseteq \{0, a\} + \{0, b\} = a \cdot b + b \cdot a = d(b) \cdot b + b \cdot d(b)$, $d(b \cdot a) = d(\{0, b\}) = \{0, a\} \subseteq \{0, a\} + \{0, b\} = a \cdot a + b \cdot b = d(b) \cdot a + b \cdot d(a)$ We can easily verify that $x \leq y$ implies $d(x) \leq d(y)$, for all $x, y \in R$. Hence, d is a derivation on R. Now, we have

$$d(a \cdot a) = d(\{0, a\}) = \{0, b\} = b \cdot b = d(a) \cdot d(a),$$
$$d(a \cdot b) = d(\{0, a\}) = \{0, b\} = b \cdot a = d(a) \cdot d(b),$$
$$d(b \cdot a) = d(\{0, b\}) = \{0, a\} = a \cdot b = d(b) \cdot d(a),$$
$$d(b \cdot b) = d(\{0, b\}) = \{0, a\} = a \cdot a = d(b) \cdot d(b).$$

Therefore, d is a homo-derivation on R.

Theorem 2.1. *Let $(R, +, \cdot, \leq)$ be a positive ordered semihyperring. If d is a homo-derivation on R, then $Ker(d) = \{x \in R \mid d(x) = 0\}$ is a k-hyperideal of type 1 of R.*

Proof. Since $0 \in Ker(d)$, it follows that $Ker(d) \neq \emptyset$. Let $x, y \in Ker(d)$. Then $d(x) = 0 = d(y)$. Since d is a derivation, we have $d(x + y) \subseteq d(x) + d(y) = 0 + 0 = \{0\}$. So, $x + y \subseteq Ker(d)$. Now, let $x \in Ker(d)$ and $r \in R$. Since d is a homo-derivation, we have $d(r \cdot x) = d(r) \cdot d(x) = d(r) \cdot 0 = 0$. Thus, $r \cdot x \subseteq Ker(d)$. Now, let $a \in Ker(d)$, $r \in R$ and $r \leq a$. Since d is a derivation, it follows that $d(r) \leq d(a) = 0$. By hypothesis, $d(r) = 0$ and hence $r \in Ker(d)$. Therefore, $Ker(d)$ is a hyperideal of R. Now, we prove that $Ker(d)$ is a k-hyperideal of type 1 of R. Suppose that $x \in Ker(d)$ and $x + r \subseteq Ker(d)$, where $r \in R$. So, we have

$$0 = d(x + r) \subseteq d(x) + d(r) = 0 + d(r) = d(r).$$

Then $d(r) = 0$ and thus $r \in Ker(d)$. Therefore, $Ker(d)$ is a k-hyperideal of type 1 of R. □

Let d be a derivation of an ordered semihyperring $(R, +, \cdot, \leq)$. Define a set $Fix_d(R)$ on R as following:

$$Fix_d(R) = \{x \in R \mid d(x) = x\}.$$

Recall, an ordered semihyperring R is said to be semiprime if $aRa = 0$ implies $a = 0$ for all $a \in R$. An ordered semihyperring R is *Q-semiprime* if $xR = 0$ or $Rx = 0$ implies $x = 0$ for all $x \in R$.

Theorem 2.2. *If d is a strong homo-derivation on a Q-semiprime ordered semihyperring $(R, +, \cdot, \leq)$, then $Fix_d(R) = \{0\}$.*

Proof. Let $d(x) = x$ for $x \in R$. We show that $x = 0$. For every $y \in R$, we have

$$\begin{aligned} x \cdot d(y) &= d(x) \cdot d(y) \\ &= d(x \cdot y) \\ &= d(x) \cdot y + x \cdot d(y) \\ &= x \cdot y + x \cdot d(y) \end{aligned}$$

It follows that $x \cdot y = 0$ for every $y \in R$. So, we have $x \cdot R = 0$. By the definition of Q-semiprime, $x = 0$. □

Definition 2.3. *Let d be a derivation of an ordered semihyperring $(R, +, \cdot, \leq)$.*

(1) *The subhyperring A of R is a d-subhyperring of R if $d(a) \in A$, for all $a \in A$.*
(2) *The subhyperring A of R is an injective subhyperring with respect to d if for all $x, y \in R$, $d(x) = d(y)$ and $x \in A$ implies that $y \in A$.*

Example 2.4. *In Example* **2**.*2, $A = \{0, a, b\}$ is a d-subhyperring and injective subhyperring with respect to d.*

Let $(R, +, \cdot, \leq)$ be a positive ordered semihyperring. Clearly, $Ker(d) \neq \emptyset$. Let $x, y \in Ker(d)$. Then $d(x) = 0 = d(y)$. Since d is a derivation, it follows that $d(x+y) \subseteq d(x)+d(y) = 0+0 = \{0\}$. So, $x+y \subseteq Ker(d)$. On the other hand, $d(x\cdot y) \in d(x)\cdot y+x\cdot d(y) = 0\cdot y+x\cdot 0 = 0+0 = \{0\}$. So, we have $x \cdot y \subseteq Ker(d)$. Now, let $x \in Ker(d)$, $r \in R$ and $r \leq x$. Since d is a derivation, it follows that $d(r) \leq d(x) = 0$. By assumption, $d(r) = 0$ and this shows that $r \in Ker(d)$. Hence, $Ker(d)$ is a subhyperring of R.

Theorem 2.3. *Let $(R, +, \cdot, \leq)$ be a positive ordered semihyperring. Then*

(1) *$Ker(d)$ is a d-subhyperring of R.*
(2) *$Ker(d)$ is an injective subhyperring of R.*
(3) *$Ker(d)$ is the smallest injective subhyperring of R.*

Proof. (1): If $a \in Ker(d)$, then $d(a) = 0 \in Ker(d)$. Therefore, $Ker(d)$ is a d-subhyperring of R.

(2): Let $d(x) = d(y)$ and $x \in Ker(d)$. Then $0 = d(x) = d(y)$. So, we have $y \in Ker(d)$. This completes the proof.

(3): By (2), $Ker(d)$ is an injective subhyperring of R. We claim that $Ker(d)$ is the smallest injective subhyperring of R. Let I be an injective subhyperring of R with respect to d. We show that $Ker(d) \subseteq I$. Let $a \in Ker(d)$. Then $d(a) = 0 = d(0)$. Since I is an injective subhyperring, we get $a \in I$. Hence, $Ker(d) \subseteq I$. □

In the following, we discuss m-k-hyperideals of type 1 in ordered semihyperrings, which can be regarded as a generalization of k-hyperideals of type 1.

Definition 2.4. *A hyperideal A of an ordered semihyperring $(R, +, \cdot, \leq)$ is said to be m-k-hyperideal of type 1 if $a \cdot x \subseteq A$, $a \in A$, $x \in R$, then $x \in A$.*

Theorem 2.4. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, every m-k-hyperideal of type 1 of R is a k-hyperideal of type 1 of R.*

Proof. Let A be a m-k-hyperideal of type 1 of an ordered semihyperring R. Consider $a + x \subseteq A$, $a \in A$ and $x \in R$. Since A is a hyperideal of R, we have

$$(a+x)\cdot x \subseteq A\cdot R \subseteq A.$$

So, for any $u \in a+x$, $u\cdot x \subseteq A$. Since A is a m-k-hyperideal of type 1, it follows that $x \in A$. Therefore, A is a k-hyperideal of type 1 of R. □

The following example shows that the converse of Theorem **2**.4 is not true in general.

Example 2.5. *Let $R = \{0, a, b, c\}$ be a set with two hyperoperations $\oplus$ and $\odot$ as follows:*

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	$\{0,b\}$	$\{0,b,c\}$
c	c	a	$\{0,b,c\}$	$\{0,c\}$

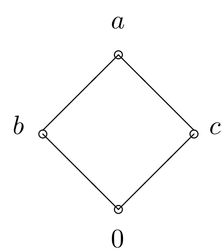
$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	$\{0,b\}$	0
b	0	0	0	0
c	0	$\{0,c\}$	0	0

Then, $(R, \oplus, \odot)$ is a semihyperring [16]. *We have $(R, \oplus, \odot, \leq)$ is an ordered semihyperring where the order relation $\leq$ is defined by:*

$$\leq \ := \{(0,0),(a,a),(b,b),(c,c),(0,a),(0,b),(0,c),(b,a),(c,a)\}.$$

The covering relation and the figure of R are given by:

$$\prec \ = \{(0,b),(0,c),(b,a),(c,a)\}.$$



(1) *It is a routine matter to verify that the hyperideal $\{0,b,c\}$ is a k-hyperideal of type 1.*
(2) *$\{0,b,c\}$ is not a m-k-hyperideal of type 1 of R. Indeed:*

$$a\odot b = \{0,b\} \subseteq \{0,b,c\} \text{ and } b \in \{0,b,c\} \text{ but } a \notin \{0,b,c\}.$$

Theorem 2.5. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. If A is a m-k-hyperideal of type 1 of R, then A is a maximal hyperideal of R.*

Proof. Let A be a m-k-hyperideal of type 1 of an ordered semihyperring R. Suppose that B is a hyperideal of R such that $A \subseteq B$, $x \in B$ and $a \in A$. Since A is a hyperideal of R, we have

$$a\cdot x \subseteq A\cdot B \subseteq A\cdot R \subseteq A.$$

Since A is a m-k-hyperideal of type 1 of R, it follows that $x \in A$. Thus $A = B$ and so A is a maximal hyperideal of R. □

A hyperideal I of an ordered semihyperring $(R, +, \cdot, \leq)$ is called an *irreducible hyperideal* if for any hyperideals A and B of R, $A \cap B = I$ implies $A = I$ or $B = I$.

Theorem 2.6. *Let $(R, +, \cdot, \leq)$ ba an ordered semihyperring. If M is a maximal m-k-hyperideal of type 1 of R, then M is irreducible.*

Proof. Suppose that M is not irreducible. Let C and D be two hyperideals of R such that $C \cap D = M$, where $C \neq M$ and $D \neq M$. By hypothesis, we have $C \subset M \subset R$ and $D \subset M \subset R$, which is a contradiction. Therefore, M is an irreducible m-k-hyperideal of type 1 of R. □

Theorem 2.7. *Let A be a m-k-hyperideal of type 1 of an ordered semihyperring $(R, +, \cdot, \leq)$ and $x \in R$ such that $x \notin A$. Then, there exists an irreducible m-k-hyperideal B of R such that $A \subseteq B$ and $x \notin B$.*

Proof. Set

$$\Phi = \{B \mid B \text{ is a } m\text{-}k\text{-hyperideal of type 1 of } R,\ A \subseteq B,\ x \notin B\}.$$

Since $A \in \Phi$, it follows that $\Phi \neq \emptyset$. Also Φ is an ordered set under the usual set inclusion. Suppose that $\{B_\delta \mid \delta \in \Phi\}$ is a chain in Φ. Consider $I = \bigcup\limits_{\delta\in\Phi} B_\delta$. We show that I is a m-k-hyperideal of type 1 of R and $A \subseteq I$. If $a, b \in I$, then $a \in B_\delta$ and $b \in B_\gamma$ for some $\delta, \gamma \in \Phi$. Since $\bigcup\limits_{\delta\in\Phi} B_\delta$ is a totally ordered set, it follows that $B_\delta \subseteq B_\gamma$ or $B_\gamma \subseteq B_\delta$. By Theorem 2.8 of [24], $I = \bigcup\limits_{\delta\in\Phi} B_\delta$ is a m-k-hyperideal of type 1 of R. Since each $B_\delta \in \Phi$ contains A and $a \notin B_\delta$, we have $A \subseteq \bigcup\limits_{\lambda\in\Lambda} B_\delta = I$ and $a \notin I$. Hence $I \in \Phi$ is an upper bound for chain $\{B_\delta \mid \delta \in \Phi\}$. By Zorn's Lemma, there exists a maximal m-k-hyperideal of type 1 say M in Φ. Thus $A \subseteq M$ and $x \notin M$. Now, we prove that M is an irreducible m-k-hyperideal of type 1 of R. Let B_1 and B_2 be any two m-k-hyperideals of type 1 of R such that $B_1 \cap B_2 = M$. Suppose that $M \subsetneqq B_1$ and $M \subsetneqq B_2$. By the maximality of M in Φ, we have $x \in B_1$ and $x \in B_2$. So, $x \in B_1 \cap B_2 = M$, which is a contradiction. Thus $B_1 = M$ or $B_2 = M$. This shows that M is an irreducible m-k-hyperideal of type 1 of R. □

Theorem 2.8. *Suppose that d is a homo-derivation on an ordered semihyperring $(R, +, \cdot, \leq)$. If A is a m-k-hyperideal of type 1 of R, then $d^{-1}(A) = \{x \in R \mid d(x) \in A\}$ is a m-k-hyperideal of type 1 of R such that $Ker(d) \subseteq d^{-1}(A)$.*

Proof. Since $d(0) = 0$, we have $0 \in d^{-1}(A)$ and thus $d^{-1}(A) \neq \emptyset$. Let $x, y \in d^{-1}(A)$. Then $d(x), d(y) \in A$. Since A is a hyperideal of R, we have $d(x + y) \subseteq d(x) + d(y) \subseteq A$. So, $x + y \subseteq d^{-1}(A)$. Let $r \in R$ and $a \in d^{-1}(A)$. Then $d(a) \in A$. Since d is a homo-derivation, it follows that $d(x \cdot a) = d(x) \cdot d(a) \subseteq A$. So, $x \cdot a \subseteq d^{-1}(A)$. Similarly, $a \cdot x \subseteq d^{-1}(A)$. Now, suppose that $a \in d^{-1}(A)$ and $r \in R$ such that $r \leq a$. Then $d(a) \in A$. Since $r \leq a$ and d is a derivation, we have $d(r) \leq d(a) \in A$. Since A is a hyperideal of R, we get $d(r) \in A$. So, $r \in d^{-1}(A)$. Therefore, $d^{-1}(A)$ is a hyperideal of R. Now, we show that $d^{-1}(A)$ is a m-k-hyperideal of type 1 of R. Suppose that $a \in d^{-1}(A)$ and $a \cdot x \subseteq d^{-1}(A)$, where $x \in R$. Then $d(a) \in A$, $d(x) \in R$ and $d(a \cdot x) \subseteq d(d^{-1}(A)) \subseteq A$. So, $d(a \cdot x) = d(a) \cdot d(x) \subseteq A$. Since A is a m-k-hyperideal of type 1 of R, we get $d(x) \in A$. This implies that $x \in d^{-1}(A)$. Therefore, $d^{-1}(A)$ is a m-k-hyperideal of type 1 of R. Moreover, we have $Ker(d) = d^{-1}(\{0\}) \subseteq d^{-1}(A)$. □

3. Conclusion

In this paper, we have described derivations on ordered semihyperrings. We have also proved some results in this respect. Furthermore, we have studied some properties of m-k-hyperideals of type 1 in ordered semihyperrings. We have shown that If d is a homo-derivation on a positive ordered semihyperring R, then $Ker(d)$ is a k-hyperideal of type 1 of R. Furthermore, we have proven that if d is a homo-derivation on an ordered semihyperring R, then $d^{-1}(A)$ is a m-k-hyperideal of type 1 of R, where A is a m-k-hyperideal of type 1. In this study, we have investigated ordered semihyperrings by using m-k-hyperideals and it will give a new direction in the further study of (m-)k-hyperideals of (ordered) semihyperrings. (fuzzy) k-Hyperideals play an essential role in semihyperrings. We can characterize (ordered) semihyperrings in terms of fuzzy k-hyperideals. According to the research results, it is suggested to define and investigate some properties of fuzzy m-k-hyperideals (of type 1), interval-valued fuzzy m-k-hyperideals, idempotent m-k-hyperideals and derivations of fuzzy

m-k-hyperideals in ordered semihyperrings. We hope that this work would offer foundation for further study of the derivations on ordered hyperstructures.

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