U.P.B. Sci. Bull., Series A, Vol. 86, Iss. 3, 2024 ISSN 1223-7027

A NEW METHOD FOR SOLVING PARETO REAL VECTOR OPTIMIZATION PROBLEMS

Mircea SULARIA [1], Marius BREBENEL [2*]

Any Pareto real vector optimization problem is equivalent with a Pareto Real Vector Maximization (PRVMax) problem. In this paper we present a new method for solving PRVMax problems. Then we attempt to show its practical interest.

Keywords: the real numbers system, the extended real numbers system, real exponential functions, the real unit interval, the real Gödel-Kleene lattice, real optimization problem, Pareto real vector optimization problem.

1. Introduction

Vector Optimization (VO) is called also *Multiple Criteria Optimization* or *Multiobjective Optimization* (MO). This is an important research subject which deals with solving *Multiple Criteria Decision Making* (MCDM) problems.

In this paper, we consider the basic notion of Godel-Kleene Lattice (called shortly GK-lattice). A concrete example of GK-lattice is the structure of GK-lattice with the support set equal to the real closed interval $[0, 1] \subset R$, which will be denoted by GK[0,1].

Then we present a second example of GK-lattice, namely the structure of Real Godel-Kleene Lattice (called shortly RGKL). The notion of RGKL has been introduced in the paper [20]. An RGKL is a special kind of GK-lattice associated to the usual totally ordered real numbers set $(R, \leq)$ having the support set equal to $R \cup \{\bot, \top\}$. We consider a Pareto real vector maximization (P-rvm) problem together with a finite family of real objective functions. We present a new mathematical method to solve this P-rvm problem. The idea of this method has been mentioned in the paper [21]. The consideration of this method is indispensable to solve the Pareto real Vector Optimization using known methods of real scalar optimization problems constructed using real exponential operators.

[1] Lecturer Prof., Department of Mathematical Methods and Models, University POLITEHNICA of Bucharest, Romania, e-mail: m.sularia@gmail.com
[2] Assist. Prof., Department of Aerospace Sciences "Elie Carafoli", University POLITEHNICA of Bucharest, Romania; email: mariusbreb@yahoo.com (corresponding author)

We combine the finite family of real criterion functions in a unique real criterion function of maximization based on the use of a *Partial Universal Algebra* extending the RGKL and which is defined using the basic structure of the *Real Numbers Ordered Field* together with a special finite family of power functions α^x, with $\alpha \in (1, +\infty) \subset$ R. One constructs a special real maximization problem with the unique real criterion function of maximization previously mentioned. The solution of this special real maximization problem is a Pareto maximal point of the initial P-rvm problem. This method to solve P-rvm problems can be easily implemented on computers. We consider a simple example to illustrate the corresponding algorithm using the software Mathematica 6.01.

It is recommended in practice to use Mathematica, which is very efficient. One can realize in a relatively short time different experiments for applications. Some potential practical applications are also considered in the paper.

2. A description of the basic mathematical results

In this paper we suppose known some basic results from real mathematical analysis [3,7,11,15,17,18]. Let $(\mathbf{R}, +, \cdot, 0, 1, \leq)$ be the complete linearly ordered field of real numbers.

We will use the compact real numbers extended chain $(\overline{\mathbf{R}}, \overline{\leq}, \bot, \top)$, where $\bot = -\infty$ (minus infinite) and $\top = +\infty$ (plus infinite) are two constants such that $(\bot \notin \mathbf{R}, \top \notin \mathbf{R}, \bot \neq \top)$, the support set $\overline{\mathbf{R}}$ is defined by the union between the set **R** and the set of two elements $\bot = -\infty$ and $\top = +\infty$ such that

(1.1) $\overline{\leq}$ is an order relation on the set $\overline{\mathbf{R}}$ defined by the following condition:

(1.2) $(\forall u, v \in \overline{\mathbf{R}})\, u \overline{\leq} v \Leftrightarrow (u, v \in \mathbf{R} \text{ and } u \leq v)$ or $(u = \bot)$ or $(v = \top)$.

Let $([0,1], \leq, 0, 1)$ be the unit chain of real numbers, where the support set is the unit real numbers interval:

(1.3) $[0,1] = \{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$.

(1.4) $\leq$ is the usual order relation on R.

(1.5) **0** is the real number “zero” and **1** is the real number “one”.

Let $m, n \in \mathbf{N}$ be natural numbers such that $m \geq 2$.

Let f be a function, defined on a nonempty compact subset $X \subset \mathbf{R}^n$,

(1.6) $f = (f_1, f_2, ..., f_m): X \to \mathbf{R}^m$, such that for any point $x \in X$, we have

(1.7) $f(x) = (f_1(x), f_2(x), ..., f_m(x)) \in \mathbf{R}^m$.

A Pareto maximum point of f on the set X is any element $x_0 \in X$ such that the corresponding point $f(x_0) \in f(X)$ is a maximal element of the partial ordered subset $(f(X), \leq^m)$ of the usual ordered real vector space $(\mathbb{R}^m, \leq^m)$.

We suppose that the m components $(f_i : X \to \mathbf{R})_{i=\overline{1,m}}$ of the function f are continuous real functions defined on X. The problem to determine a Pareto maximal point of the function f on the set X is called a PRVMax problem.

We present a new method for solving PRVMax problems based on the use of real exponential functions. We define a special family of order isomorphisms between the two chains $(\overline{\mathbf{R}}, \lesssim, \perp, \top)$ and $([0,1], \leq, 0, 1)$. We prove that for any finite sequence of m real parameters,

(1.8) $a = (a_1, a_2 ... a_m) \in (1, +\infty)^m \subset \mathbf{R}^m$,

there exists a real function

(1.9) $\varphi_\alpha(f): X \to \square,\ x \to \varphi_\alpha(f)(x)$

such that any global maximization point of the function (1.9) on X is a solution of the previously mentioned PRVMax problem.

The definition of the function (1.9) is based on the use of m real exponential functions and its expression is the following

(1.10) $\varphi_a(f)(x) = \prod_{i=1}^{m} \left(1 - \frac{1}{1 + a_i^{f_i(x)}}\right), \forall x \in X$

Two interesting applications of this new method are suggested. An application is to solve multiple objectives optimal economic planning problems [12-14]. A second application is to solve multiple criteria discrete optimal control.

For the both previously mentioned applications one can use the software instruments Matlab [23] or Mathematica [24] via global optimization [16]. We present a simple example based on Mathematica 6.0.1.

3. The real Gödel-Kleene lattice

In this section we will consider a special ordered algebraic structure, which will be called *Real Gödel-Kleene Lattice* (RGKL). Then we formulate the notion of Pareto Real Vector Optimization Problem (PRVOP).

Definition 2.1 Let **A** be an algebraic structure with the support set A together with three binary operations $\wedge, \vee, \Rightarrow$ on A,

$\wedge : A \times A \to A, (x, y) \mapsto x \wedge y$, $\vee : A \times A \to A, (x, y) \mapsto x \vee y$,

$\Rightarrow : A \times A \to A$, $(x, y) \mapsto x \Rightarrow y$, a unary operation * on A, $* : A \to A, x \mapsto x*$ and two constants $\{\perp, \top\} \subseteq A$ such that $\perp \neq \top$.

The previous structure **A** is called a ***Gödel-Kleene Lattice*** (*GKL*) or also an ***LK-lattice*** *LK*[*A*] if the following conditions hold:

(i) The structure $(A,,\wedge,\vee,\Rightarrow,\bot,\top)$ is a Gödel Lattice (GL), with the conjunction operation $\wedge$, the disjunction operation $\vee$, the implication operation $\Rightarrow$, the first element $\bot$ and the last element $\top$ i.e. the next properties hold:

($GL1$) The system $(A,,\wedge,\vee,\bot,\top)$ is a bounded distributive lattice with the binary infimum operation $\wedge$, the binary supremum operation $\vee$, the first element $\bot$ and the last element $\top$.

($GL2$) The binary operation $\Rightarrow$ on A satisfies the following condition called the ***Implication Operator Condition*** or the ***Heyting Condition***:

($C1$) for all $x,y,z \in A$, $z \leq x \Rightarrow y$ if and only if $z \wedge x \leq y$,

and the next condition called the ***Gödel Condition*** or the ***Generalized Linearity Condition***,

($C2$) for all $x,y \in A$, $(x \Rightarrow y) \vee (y \Rightarrow x) = \top$.

(ii) The system $(A,,\wedge,\vee,*,\bot,\top)$ is a De Morgan lattice with the support set A together with its structure of bounded distributive lattice $(A,,\wedge,\vee,\bot,\top)$ from ($GL1$) and with the negation operator * i.e. the next properties hold:

($DM1$) for all $x,y \in A$, $(x \wedge y)^* = x^* \vee y^*$;

($DM2$) for all $x \in A$, $(x^*)^* = x$;

($DM3$) $\bot^* = \top$.

(iii) The operator $* : A \to A, x \mapsto x*$ satisfies the next condition, called the ***Kleene Condition***:

(KC) for all $x,y \in A$, $x \wedge x^* \leq y \vee y^*$.

Remark 2.2 The notion of *Gödel-Kleene Lattice* (GKL) from Definition 2.1 called also *LK-lattice* has been introduced by Antonio Monteiro in connection with the study of the structure of *symmetric Heyting algebra* (a Heyting algebra together with a De Morgan negation). An LK-lattice is a symmetric Heyting algebra $(A,,\wedge,\vee,\Rightarrow,*,\bot,\top)$ which in addition satisfies *the Generalized Linearity Condition* from Definition 2.1 (i) ($C2$) and *the Kleene Condition* (KC) from Definition 2.1 (iii). A *complete LK-lattice* is an LK-lattice such that its underlying bounded lattice is a complete lattice. An *LK-chain* is any LK-lattice such that it is a totally ordered set with respect to its underlying order.

Remark 2.3 Let $LK(\tau)$ be the class of all LK-lattices considered as algebras of type $\tau = (2,2,2,1,0,0)$. Then $LK(\tau)$ is a variety of algebras and its basic properties are presented in [21]. A basic property is the following theorem: an LK-lattice is a subdirect irreducible algebra of $LK(\tau)$ if and only if it is an LK-chain. It follows the next important property: *any LK-lattice is isomorphic to a subdirect product of LK-chains*. In this paper we consider the problem to solve Pareto real vector optimization problems. A special case of this kind of problem is the optimization problems in fuzzy environments [4]. This kind of problems has been considered by Bellman and Zadeh [2], which proposed a method to obtain their solutions using a max-min operator of aggregation. This method is not satisfactory, because the solutions are not in general Pareto optimal points.

Remark 2.4 In this paper, we will obtain an improvement of the Bellmann-Zadeh method based on the use of a Pareto aggregation operator family of LK-isomorphisms between LK$\left[\overline{\mathbf{R}}\right]$ and LK[0,1], associated to a composition of exponential functions on the real numbers field $\mathbf{R}$.

Remark 2.5 The structure of *Gödel lattice* is a well known structure in algebra of logic. For the first time, in connection with the notion of fuzzy set introduced by Zadeh, Hájek proved that this structure is also of interest in *mathematical foundation of fuzzy inference*.

Remark 2.6 The structure of MV-algebra is the algebraic foundation of many-valued reasoning. The variety of Gödel MV-algebras is introduced. For any complete Gödel MV-algebra, there exists a structure of complete LK-lattice on its underlying De Morgan lattice. This property holds, since the underlying De Morgan lattice of any complete MV-algebra is a complete double Heyting algebra satisfying in addition the Kleene condition (*KC*) from Definition 2.1 (*iii*).

The previous facts shows that the *LK-lattices are important algebraic structures in many-valued logic* [20]. We introduce new basic notions which will be used in the sequel.

Definition 2.7 A ***system of data for a multiple-criteria optimization model*** is defined by a quadruple (n,m,X,f), where $n,m \in N$ are natural numbers with $n \geq 1$ and $m \geq 2$, X is a set such that $X \subseteq \mathbf{R}^n$, $X \neq \varnothing$ and $f : X \to \mathbf{R}^m$ is a vector function from *X* to $\mathbf{R}^m$. The set *X* represents a set of alternatives with respect to a decision making problem and $f = (f_1, f_2, \ldots, f_m)$ is a sequence of real functions on *X*, $(f_i : X \to \mathbf{R})_{i=\overline{1,m}}$, called the ***objective functions*** of the optimization model. The number *n* is the dimension of the arithmetical vector space $\mathbf{R}^n$ of alternatives and the number *m* represents the number of objective functions. Let (n,m,X,f) be a system of data for a multiple-criteria optimization model. We consider on the set $\mathbf{R}^m$ an order relation $\leq^m$ defined for any $u = (u_1, u_2, \ldots, u_m) \in \mathbf{R}^m$ and $v = (v_1, v_2, \ldots, v_m) \in \mathbf{R}^n$ by the following condition: $u \leq^m v$ if and only if $u_i \leq v_i, \forall i \in \{1,2,\ldots,m\}$.

(i) A ***Pareto Maximal point*** of the function $f: X \to \mathbf{R}^m$ on the set X is any element $x_{PM} \in X$ such that $f(x_{PM})$ is a maximal element of the partial ordered set $(f(X), \leq^m)$ i.e. the following condition holds:

(PM) $(\forall x \in X)\ [f(x_{PM}) \leq f(x) \Rightarrow f(x_{PM}) = f(x)]$.

(ii) A ***Pareto Maximization problem*** with respect to (n, m, X, f) is the problem to determine a Pareto Maximal point of the function $f: X \to \mathbf{R}^m$ on the set X.

(iii) ***A system of data for an one-criteria maximization model*** is a triple (n, X, φ), where $n \in \mathbf{N}, X \subseteq \mathbf{R}^n$ is a non-empty set and $\varphi: X \to \mathbf{R}$ is a real function called the ***objective function***. An one-criteria global maximization problem with respect to the system of data (n, X, φ) is the problem to maximize the objective function φ on the set X i.e. to determine a point $x_M \in X$ such that $\varphi(x) \leq \varphi(x_M),\ \forall x \in X$.

We present an algebraic structure **S** such that a solution of any Pareto maximization problem with respect to a suitable system of data (n, m, X, f) can be found by solving an one-criteria global maximization problem with respect to (n, X, φ), where the real function φ is obtained from f by aggregation using a special term of **S**. The starting point is the next result, which establishes the existence of a concrete isomorphism between the structures of LK-lattices LK$[\overline{\mathbf{R}}]$ and LK[0,1].

Lemma 2.8 There exists an isomorphism $\theta: \overline{\mathbf{R}} \to [\mathbf{0}, \mathbf{1}]$ from LK$[\overline{\mathbf{R}}]$ onto LK[0,1] defined in terms of the function arctan by the following relation, for all $u \in \overline{\mathbf{R}} = \mathbf{R} \cup \{\perp, \top\}$:

$$\theta(u) = \begin{cases} 0 & \text{if} \quad u = \perp \\ \frac{1}{2}\left(\frac{2}{\pi}\arctan(u) + 1\right) & \text{if} \quad u \in \mathbf{R} \\ 1 & \text{if} \quad u = \top \end{cases}$$

In this paper we will use a family of isomorphisms between LK$[\overline{\mathbf{R}}]$ and LK[0,1] defined in terms of a special concrete family of real exponential functions on the commutative field of real numbers $F[\mathbf{R}] = (\mathbf{R}, +, \cdot, 0, 1)$.

The idea of this definition has been presented in [21]. This fact is essential for the foundation of ***a new method to solve Pareto real maximization problems***. We will use special properties of the real numbers field. Now we present the general notion of exponential function.

Definition 2.9 An exponential function on a commutative field $(K,+,\cdot,0,1)$ is an operator on K, $\exp: K \to K,\ x \mapsto \exp(x)$, such that the next conditions hold, for all $x, y \in K$ **:**

(1) $\exp(x+y) = \exp(x) \cdot \exp(y)$;

(2) $\exp(x) \neq 0$;

(3) $\exp(-x) = \dfrac{1}{\exp(x)}$.

Definition 2.10 The real Gödel-Kleene lattice-ordered partial field is the next pair of structures: RGKLordPF$[\overline{\mathbf{R}}] = (LK(\overline{\mathbf{R}}), F[\mathbf{R}])$, where LK$[\overline{\mathbf{R}}]$ is the LK-lattice of the extended real numbers chain $(\overline{\mathbf{R}},, \lesssim, \perp, \top)$ and $F[\mathbf{R}] = (\mathbf{R},+,\cdot,0,1)$ is the commutative field of real numbers. The real Gödel-Kleene lattice-ordered partial field with exponential operators is defined by the real Gödel-Kleene lattice-ordered partial field RGKLordPF$[\overline{\mathbf{R}}]$ previously introduced together with the next family of exponential functions on the field of real numbers $(\mathbf{R},+,\cdot,0,1)$, $(\exp_a : \mathbf{R} \to \mathbf{R})_{a \in (1,\top)}$, where $\forall a \in (1,\top)$, $\exp_a(x) = a^x, \forall x \in \mathbf{R}$ i.e. $\exp_a$ is the exponential function on $\mathbf{R}$ with $a \in (1,\top) = (1,+\infty)$.

Lemma 2.11 Let RGKLordPF$[\overline{\mathbf{R}}]$ be the pair of structures from Definition 2.10 together with the family of exponential functions on $\mathbf{R}$ presented also in Definition 2.10. For every $a \in (1,\top)$, $\exp_a : \mathbf{R} \to \mathbf{R}$ is an exponential function on the field of real numbers $(\mathbf{R},+,\cdot,0,1)$ in the sense of Definition 2.9 and the following properties holds:

(*i*) The image set of the exponential function $\exp_a$ satisfies the next relation: $\exp_a(\mathbf{R}) = (0,\top) = (0,+\infty)$.

(*ii*) The restriction of $\exp_a$ to its image, $\exp_a : \mathbf{R} \to (0,+\infty)$, is an isomorphism from the totally ordered commutative additive group $(\mathbf{R},+,0,\leq)$ onto the totally ordered commutative multiplicative group of real numbers from the positive real axis $((0,+\infty),\cdot, 1, \leq)$ with the inverse isomorphism given by the logarithmic function $\log_a : (0,+\infty) \to \mathbf{R},\ x \mapsto \log_a x$.

(iii) For every exponent $a \in (1,\top)$, the next properties are satisfied in RGKLordPF$[\overline{\mathbf{R}}]$:

(**P1**) Consider the function $E_a : \mathbf{R} \to (0,1),\ x \mapsto E_a(x) = 1 - \dfrac{1}{1+a^x}$. Define also the extended function

$\overline{E_a} : \overline{\mathbf{R}} \to [0,1),\ u \mapsto \overline{E_a}(u)$, where

$$\overline{E_a}(u)=\begin{cases}0 & \text{if } \quad u=\perp\\ E_a(u) & \text{if } \quad u\in\mathbf{R}\\ 1 & \text{if } \quad u=\top\end{cases}$$

Then $\overline{E_a}$ *is an isomorphism from* LK$\left[\overline{\mathbf{R}}\right]$ *onto* LK[0,1].

(**P2**) The function E_a *is a probability distribution function of a random variable* X_a *such that* $E_a\in C^{\infty}(\mathbf{R})$ *and its first derivative,*

$$e_a=E_a^{'}:\mathbf{R}\to\mathbf{R},\ x\mapsto e_a(x)=\frac{d}{dx}\left(1-\frac{1}{1+a^x}\right)=\frac{a^x\ln a}{\left(1+a^x\right)^2},\forall x\in\mathbf{R}$$

is a density function of X_a i.e.

$$P(X_a\le x)=E_a(x)=\int_{\perp}^{x}e_a(t)dt,\ \forall x\in\mathbf{R}\ \text{ and }\ \int_{\perp}^{\top}e_a(t)dt=1$$

We present here a basic mathematical result expressing an efficient method to solve on PC Computers special *Pareto Real Vector Maximization Mathematical Programming Problems*.

Theorem 2.12 *Let* (n,m,X,f) *be a system of data for a multiple-criteria optimization model given as in Definition 2.10. Suppose that the corresponding set of alternatives* $X\subseteq\mathbf{R}^n$ *is compact and the objective vector function* $f=(f_1,f_2,\ldots,f_m):X\to\mathbf{R}^m$ *is a continuous function on X* i.e. *for every* $i\in\{1,2,...,m)$, *the component function* $f_i:X\to\mathbf{R}$ *is a continuous real function on the set X. For any m-sequence of real numbers* $a=(a_1,a_2,\ldots,a_m)\in(1,\top)^m$ *and for any index* $i\in\{1,2,...,m)$, *let* $E_{a_i}:\mathbf{R}\to(0,1)$ *be the function* E_{a_i} *associated with* $a_i\in(1,\top)$ *introduced in Lemma 2.11* (**P1**). *We define an one-criteria global maximization problem as in Definition 2.7* (*iii*) *by the triple* $(n,X,\varphi_a(f))$, *where* $\varphi_a(f):X\to(0,1),\ x\mapsto\varphi_a(f)(x)$ *is the objective function defined by*

$$\varphi_a(f)(x)=\prod_{i=1}^{m}E_{a_i}(f_i(x)),\forall x\in X\ .$$

Then the following conditions hold:

(*i*) *The function* $\varphi_a(f):X\to(0,1)$ *is a continuous function on the compact set X*.

(*ii*) *Suppose that* $x_M\in X$ *is a solution of the global maximization problem of the function* $\varphi_a(f)$ *on the compact set X* i.e.

(*M*) $(\forall x\in X)\ \varphi_a(f)(x)\le\varphi_a(f)(x_M)$.

Then x_M *is a Pareto maximal point of the vector function* $f : X \to \mathbf{R}^m$ *on the compact set X.*

Proof. (*i*) The property that the objective function $\varphi_a(f)$ is continuous follows from the condition that the function $f : X \to \mathbf{R}^m$ is continuous on *X* and from the fact that the function $\varphi_a(f) : X \to (0,1)$ is defined by the finite product of the next family of *m* continuous functions on *X*:

$\left(E_{a_i} \circ f_i : X \to (0,1)\right)_{i=\overline{1,m}}$, where for all $i \in \{1,2,\ldots,m\}$, we have

$\left(E_{a_i} \circ f_i\right)(x) = E_{a_i}(f_i(x)), \forall x \in X$.

Thus, Theorem 2.12 (*i*) holds.

(*ii*) Regarding the point x_M from the property (ii) satisfying (M), we have to prove also the next condition:

(**P**) The point x_M is a Pareto maximal point of the function *f* on the set *X*.

For this purpose, suppose that the condition (**P**) is false.

From Definition 2.7 (*i*) it follows that the point x_M is not a maximal element of the ordered set $(f(X), \trianglelefteq)$ i.e. the condition (*PM*) from Definition 2.7 (*i*) is false.

Thus, there exist an element $x_0 \in X$ such that $f(x_M) \leq f(x_0)$ and $f(x_M) \neq f(x_0)$.

This implies the relation $\varphi_a(f)(x_M) < \varphi_a(f)(x_0)$ for $x_0 \in X$, but from the property (*M*) of the point x_M, it follows that $\varphi_a(f)(x_0) \leq \varphi_a(f)(x_M)$, for $x_0 \in X$, contradiction. Thus, Theorem 2.18 (*ii*) also holds.

The result obtained by Theorem 2.12 must be verified in practice. One can view that the problem to solve Pareto vector optimization problems has been reduced to the problem to solve concrete sequence of global optimization problems, in order to obtain corresponding sequences of Pareto optimal points.

3. Concluding Remarks

(CR1) Theorem 2.18 shows that in some reasonable conditions, for any *m*-sequence of numbers $a = (a_1, a_2, ..., a_m) \in (1, +\infty)$, there exists an algebraic formula to compose an *m*-sequence of real functions $f = (f_1, f_2, \ldots, f_m) : X \to \mathbf{R}^m$

such that by this composition one obtains a real function $\varphi_a(f) : X \to (0,1)$ with the property that by the global maximization [5, 16] of $\varphi_a(f)$ on the set *X* one obtains a solution of the problem of Pareto maximization of *f* on *X*. We mention that this method can be easily programmed on Personal Computers. For this aim, one can use the software products Matlab [23] or Mathematica [24]. Using Mathematica 6.0.1, we present *a simple example* which justifies this affirmation.

Example. One can consider the problem to determine in the unity triangle Δ of the positive real plane $\mathbf{R}_+^2$ different Pareto maximal points of the identity vector function, $I_\Delta : \Delta \to \mathbf{R}^2$, defined as follows: $(\forall x = (x_1, x_2) \in \Delta \subset R^2)\ I_\Delta(x) = x$, where $\Delta \subset \mathbf{R}^2$ is defined by the following condition:

$$(x_1, x_2) \in \Delta \text{ if and only if } \begin{cases} x_1 + x_2 \leq 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}.$$

One can view that in this very particular case, we have the next characterization of Pareto maximal points of I_Δ on Δ:

(C) (x_1^*, x_2^*) is a Pareto maximal point of I_Δ on Δ if and only if

(PM) $x_1^* \geq 0, x_2^* \geq 0$ and $x_1^* + x_2^* = 1$.

We present now for this elementary problem, the application of the method expressed by Theorem 2.12. The number of objective functions of maximization is $m = 2$. We must to select different pairs of real exponents (a_1, a_2) such that $a_1 > 1$ and $a_2 > 1$. One determines the solution of the following global maximization problem with a single criteria presented in Theorem 2.12,

$max\left(1 - \frac{1}{1 + a_1^{x_1}}\right) \cdot \left(1 - \frac{1}{1 + a_2^{x_2}}\right)$, $(x_1, x_2) \in \Delta$ (GO). This solution represents just a Pareto maximal point of I_Δ on Δ. The previously mentioned property (C) is confirmed using Mathematica 6.0.1. The next three results are obtained by solving the global maximization problem (GO). The next three particular choices of the parameters are considered:

$a_1 = 3$ and $a_2 = 3$

NMaximize$\left[\left\{\left(1 - \frac{1}{1 + 3^x}\right)\left(1 - \frac{1}{1 + 3^y}\right), x + y \leq 1, x \geq 0, y \geq 0\right\}, \{x, y\}\right]$

Solution: {x = 0.5, y = 0.5}.

$a_1 = 3$ and $a_2 = 7$

NMaximize$\left[\left\{\left(1 - \frac{1}{1 + 3^x}\right)\left(1 - \frac{1}{1 + 7^y}\right), x + y \leq 1, x \geq 0, y \geq 0\right\}, \{x, y\}\right]$

Solution: {x = 0.367544, y = 0.632456}.

$a_1 = 7$ and $a_2 = 3$

$$\textbf{NMaximize}\left[\left\{\left(1-\frac{1}{1+7^x}\right)\left(1-\frac{1}{1+3^y}\right), x+y \le 1, x \ge 0, y \ge 0\right\}, \{x, y\}\right)$$

Solution: {x = 0.632456, y = 0.367544}}

(CR2) In practice, it is also of interest to satisfy some aspiration levels for the values of the objective functions, $f_i(x) \approx f_i^*$, $\forall i \in \{1,2,\ldots,m\}$, which can be represented by a special fuzzy set [22],

$$\psi : X \to [0,1), x \mapsto \psi(x) = \exp\left(-\sum_{j=1}^{n}\left(f_j(x) - f_j\right)^2\right).$$

This kind of problem is important to consider, when it is necessary to realize a realistic compromise between objectives and resources in economical planning. In this special case, one can define a unique objective function $\chi : X \to [0,1)$ by the relation $\chi(x) = \varphi_a(f)(x) \cdot \psi(x), \forall x \in X$.

This manner, to combine probability distribution functions together with membership functions using exponential operators in order to solve multiple-criteria optimization, represents a good improvement of several known methods for decision making in a fuzzy environment [2, 4]. We mention the fact that this procedure of aggregation is very well suited for solving Pareto maximization problems with two criteria. This case is important having practical applications (please see for example [9]). For a number *m* of criteria with $m \ge 3$ this manner to define aggregation operators must be considered in connection with different other useful methods [1,6,8,10,19].

(CR3) A very useful application of the mathematical method presented by Theorem 2.12 is to solve multiple criteria optimization mathematical models in Economical Planning [12–14, 19]. One can extend the previous results to the case of Economic Planning with limited resources formulated as multiple objective discrete optimal control problems. For this purpose, we remark that the use of Matlab is also a good instrument to test and elaborate new useful algorithms based on the application of Theorem 2.12.

Acknowledgment

The first author expresses here his gratitude to Professor Paul Flondor for his suggestion to change the initial title of the paper "A Novel Method …" with the actual title and to present in the paper a definition of the Pareto maximal point of a set in the standard ordered real Euclidean space.

R E F E R E N C E S

[1] *Q. H. Ansari and J.-C. Yao* (eds), Recent Developments in Vector Optimization, Springer, 2012.

[2] *R. Bellman and L.A. Zadeh*, Decision-making in a fuzzy environment, Management Science, 17B(1970)141-164.

[3] *L. Bukovský*, The Structure of the real line, Birkhäuser, 2011.

[4] *C. Carlson and R. Fuller*, Fuzzy multiple criteria decision-making: Recent developments, Fuzzy Sets and Systems, 78 (1996) 139-153.

[5] *U. M. Diwekar*, Introduction to Applied Optimization (Third Edition), Springer, 2020.

[6] *G. Eichfelder and J. Jahn*, Vector Optimization Problems and Their Solution Concepts. In [1, pp. 1-27].

[7] *P. Flondor and O. Stanasila*, Lectii de analiza matematica si exercitii rezolvate, (Mathematical analysis lessons and solved exercises), ALL, 1996. (in Romanian)

[8] *S. Huber, M. J. Geiger and A. Teixeira de Almeida* (eds.), Multiple Criteria Decision Making and Aiding. Cases on Models and Methods with Computer Implementation, Springer, 2019.

[9] *J. Jahn*, Vector Optimization: Theory, Applications and Extensions (second edition), Springer, 2011.

[10] *I. Kaliszewski, J. Mirofiridis and D. Podkopaev*, Multiple Criteria Decision Making by Multiobjective Optimization: A Toolbox, Springer, 2016.

[11] *C. Meghea*, Bazele analizei matematice - Tratat de analiză, Editura Ştiinţifică şi Enciclopedică, 1977.

[12] *C.V. Negoita and M. Sularia*, On fuzzy mathematical programming and tolerances in planning, Econ.Comp.Econom. Cybernetics Stud.Res., no.1 (1976) 3 - 15.

[13] *C.V. Negoita, P. Flondor and M. Sularia*, On fuzzy environment in optimization problems, Econ.Comp.Econom. Cybernetics Stud.Res.,no.1 (1977) 3 - 15.

[14] *C.V. Negoita and M. Sularia*, A selection method of nondominated points in multicriteria decision problems, Econ. Comp. Econom. Cybernetics Stud.Res.,no.1 (1978) 19 - 24.

[15] *V. Olariu, A. Halanay and S. Turbatu*, Analiză matematică, (Mathematical analysis), Editura Didactică şi Pedagogică, 1983. (in Romanian)

[16] *J. A. Snyman and D.N. Wilke*, Practical Mathematical Optimization: Basic Optimization Theory and Gradient-Based Algorithms, Springer, 2018.

[17] *O. Stănăşilă*, Analiză liniară şi geometrie, (Linear analysis and geometry) All, 2000. (in Romanian)

[18] *O. Stănăşilă*, Analiză matematică: ediţie definitivă, (Mathematical Analysis: Definitive Edition) Fundaţia Floarea Darurilor, Bucureşti, 2014. (in Romanian)

[19] *R. E. Steurer*, Multiple criteria optimization: theory, computation and application, Wiley, 2011.

[20] *M. Sularia*, Regarding powers of real Gödel-Kleene lattices, Journal of Multiple-Valued Logic and Soft Computing, vol. 27, issue 4 (2016) 299-352.

[21] *M. Sularia*, Combining probability distribution functions and membership functions using exponential operators, Conference ManyVal 2019 Bucharest - Book of Abstracts (2019) 57-63.

[22] *L. A. Zadeh*, Fuzzy Sets, Information and Control, vol.8, (1965) 338-365.

[23] *W. Y. Yang, Y. K. Choi, J. Kim, M. C. Kim, H. J. Kim and T. Im*, Engineering Mathematics with Matlab, CRC Press Taylor and Francis Group, 2018.

[24] *S. Wolfram*, The Mathematica Book, 5^{th} edition (Wolfram Media 2003).