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SEVERAL ITERATIVE ALGORITHMS FOR THE MULTIPLE-SETS SPLIT EQUALITY PROBLEM

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Very recently, A. Moudafi and C. Byrne proposed the split equality problem (SEP), and established several iterative algorithms for solving the SEP. In this paper, we consider the multiple-sets split equality problem (MSSEP) which generalizes the multiple-sets split feasibility problem (MSSFP) and the SEP. Some iterative algorithms for solving the MSSEP are proposed in this paper. Two main ideas are stated for solving the MSSEP. On one hand, the MSSEP is transformed to a problem that obtain a common fixed point of finite averaged mappings; on the other hand, the MSSEP is proved to be equivalent to an optimization problem.

Keywords: Multiple-sets split equality problem; Split feasibility problem; Averaged mapping ; Fixed point; Optimization problem

1. Introduction

Let H_1, H_2, H_3 be real Hilbert spaces, let $\{C_i\}_{i=1}^t \subset H_1$, $\{Q_j\}_{j=1}^r \subset H_2$ be nonempty closed convex sets, let $A\colon H_1 \to H_3$, $B\colon H_2 \to H_3$ be two bounded linear operators. The multiple-sets split equality problem (MSSEP), proposed and studied here, is

$$to\ find\ x \in \bigcap_{i=1}^{t} C_i\ and\ y \in \bigcap_{j=1}^{r} Q_j\ such\ that\ Ax = By. \tag{1.1}$$

When $B = I$, the MSSEP reduces to the multiple-sets split feasibility problem (MSSFP), and if we take $t = r = 1$, the MSSEP becomes the split equality problem (SEP). What's more, if we take $B = I$ and $t = r = 1$, the MSSEP becomes the split feasibility problem (SFP) proposed by Censor and Elfving [4]. For information of the SFP and MSSFP, please see [4-20]; For information of the SEP, please see [1-3].

Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A\colon H_1 \to H_3$, $B\colon H_2 \to H_3$ be two bounded linear operators. The SEP proposed by Moudafi [1,2] is that

$$to\ find\ x \in C,\ y \in Q\ such\ that\ Ax = By; \tag{1.2}$$

For solving the SEP, A. Moudafi proposed the alternating CQ-algorithms (ACQA) in [1] and the relaxed CQ-algorithm (RACQA) in [2]; For solving the approximate split equality problem (ASEP), C. Byrne proposed a simutaneous iterative algorithm (SSEA), the relaxed SSEA (RSSEA) and the perturbed version of the SSEA (PSSEA) in [3].

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In this paper, we propose and study the MSSEP which generalizes the MSSFP and the SEP. Some iterative algorithms for solving the MSSEP are proposed and we will establish their convergence.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let K be a nonempty closed convex subset of H. Recall that the projection from H onto K, denoted P_K, is defined in such a way that, for each $x \in H$, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| = min\{\|x - y\| : y \in K\}.$$

The following important properties of projections are useful to our study.

Proposition 2.1. Given $x \in H$ and $z \in K$.

(a) $z = P_K x$ if and only if $\langle x - z, y - z \rangle \leq 0$, $\forall y \in K$.

(b) $\langle P_K u - P_K v, u - v \rangle \geq \|P_K u - P_K v\|^2$, $\forall u, v \in H$.

Definition 2.1. A mapping $T : H \to H$ is said to be:

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. It is well known that projections are nonexpansive and firmly nonexpansive.

Definition 2.2. Let T be a nonlinear operator whose domain is $D(T) \subseteq H$ and whose range is $R(T) \subseteq H$.

(a) T is said to be monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0, \forall x, y \in D(T).$$

(b) Given a number $\beta > 0$, T is said to be β-strongly monotone if

$$\langle Tx - Ty, x - y \rangle \geq \beta \|x - y\|^2, \forall x, y \in D(T).$$

(c) Given a number $\nu > 0$, T is said to be ν-inverse strongly monotone ($\nu - ism$) if

$$\langle Tx - Ty, x - y \rangle \geq \nu \|Tx - Ty\|^2, \forall x, y \in D(T).$$

(d) Given a number $L > 0$, T is said to be L-Lipschitz if

$$\| Tx - Ty \| \leq L \| x - y \|, \forall x, y \in D(T).$$

It is easily seen that projections are 1-ism.

Definition 2.3. A mapping $T\colon H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0,1)$ and $S\colon H \to H$ is nonexpansive. In this case, we also say that T is α-averaged. Averaged mappings are nonexpansive, projections are averaged.

Proposition 2.2 ([6]). We have the following assertions:

(a) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.

(b) If T is ν-ism and $\gamma > 0$, then γT is $\frac{\nu}{\gamma}$-ism.

(c) T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

(d) If T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

Lemma 2.1. Suppose $T\colon C \to C$ is nonexpansive with a fixed point, where C is a closed convex subset of a Hilbert space. Define the sequence $\{x_n\}$ be generated by Mann's algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ n \geq 0 \tag{2.1}$$

if $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (2.1) converges weakly to a fixed point of T.

Lemma 2.2. Suppose $T\colon C \to C$ is nonexpansive with a fixed point, where C is a closed convex subset of a Hilbert space. Define the sequence $\{x_n\}$ be generated by Halpern's algorithm:

$$x_{n+1} = t_n u + (1 - t_n)T x_n,\ n \geq 0 \tag{2.2}$$

where $u, x_0 \in C$. Assume the following conditions are satisfied

(i) $\lim_{n\to\infty} t_n = 0$;

(ii) $\sum_{n=0}^{\infty} t_n = \infty$;

(iii) $\sum_{n=0}^{\infty} \mid t_{n+1} - t_n \mid < \infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Then the sequence $\{x_n\}$ generated by (2.2) converges strongly to the projection of u onto the fixed point set of T.

Lemma 2.3 ([11]). Let $f\colon H \to R$ be a continuously differentiable function such that the gradient ∇f is Lipschitz continuous

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|,\ x, y \in H.$$

Assume the minimization problem

$$\min\{f(x) : x \in K\} \tag{2.3}$$

is consistent, where K is a closed convex subset of H. Then, for $0 < \gamma < 2/L$, the sequence $\{x_n\}$ generated by the gradient-projection algorithm

$$x_{n+1} = P_K(x_n - \gamma \nabla f(x_n)) \tag{2.4}$$

converges weakly to a solution of (2.3).

Lemma 2.4 ([26]). Let H be a real Hilbert space. Then, $\forall x, y \in H$ and $\forall \lambda \in [0,1]$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Lemma 2.5 ([27]). Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:

(i) every weak limit point of $\{x_n\}$ lies in K;

(ii) $\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in K$.

Then $\{x_n\}$ converges weakly to a point in K.

Lemma 2.6 ([6]). If T is an averaged mapping in a Hilbert space with a fixed point, then, for every x, the sequence of iterates of T at x, $\{T^n x\}$, converges weakly to a fixed point of T.

3. Iterative methods for the MSSEP

Let $S = C \times Q$. Define

$$G = [A \ \ -B],$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We have the following proposition.

Proposition 3.1. w^* solves the SEP (1.2) if and only if w^* solves the fixed point equation $P_S(I - \gamma G^* G)w^* = w^*(\gamma > 0)$.

Proof. If the SEP is consistent, it can now be reformulated as

$$finding\ w^* \in S\ with\ minimizing\ the\ function\ \|Gw\|\ over\ w \in S,$$

if and only if $w^* \in S$ satisfies the variational inequality

$$\langle w - w^*, G^* G w^* \rangle \geq 0,\ \forall w \in S,$$

We can rewrite the above variational inequality as, for any $\gamma > 0$

$$\langle w - w^*, w^* - (w^* - \gamma G^* G w^*) \rangle \geq 0,\ \forall w \in S,$$

if and only if

$$P_S(w^* - \gamma G^* G w^*) = w^*.$$

Next we will state the idea for solving the MSSEP.

Without loss of generalization, suppose $t > r$ in (1.1), we can define $Q_{r+1} = Q_{r+2} = \cdots = Q_t = H_2$. Then the MSSEP (1.1) is equivalent to the following problem:

$$to\ find\ x \in \bigcap_{i=1}^{t} C_i\ and\ y \in \bigcap_{j=1}^{t} Q_j\ such\ that\ Ax = By. \tag{3.1}$$

Proposition 3.2. w^* solves the MSSEP (1.1) if and only if

$$w^* \in \bigcap_{i=1}^{t} Fix(P_{S_i}(I - \gamma G^* G)),$$

where $Fix(T)$ denote the fixed point set of T, and $S_i = C_i \times Q_i$ (i=1,2,...,t).

Proof. Assume that

$$w^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

solves the MSSEP (1.1), by the equivalent definition (3.1) of the MSSEP, for any $1 \leq i \leq t$, we have $x^* \in C_i$, $y^* \in Q_i$ and $Ax^* = By^*$. By proposition 3.1, for any $1 \leq i \leq t$, $w^* \in Fix(P_{S_i}(I - \gamma G^* G))$. So $w^* \in \bigcap_{i=1}^{t} Fix(P_{S_i}(I - \gamma G^* G))$.

Now we assume that $w^* \in \bigcap_{i=1}^t Fix(P_{S_i}(I-\gamma G^*G))$, for any $1 \le i \le t$, we have $w^* \in Fix(P_{S_i}(I-\gamma G^*G))$. By proposition 3.1, for any $1 \le i \le t$, $x^* \in C_i$, $y^* \in Q_i$ and $Ax^* = By^*$. Hence $x^* \in \bigcap_{i=1}^t C_i$, $y^* \in \bigcap_{i=1}^t Q_i$ such that $Ax^* = By^*$, in other words,

$$w^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

solves the MSSEP (1.1). The proof is complete.

By proposition 3.2, to obtain a solution of the MSSEP (1.1), we have to search for a common fixed point of the operators $\{T_i\}_{i=1}^t$, where $T_i = P_{S_i}(I-\gamma G^*G)$ (i=1,2,...,t). Since G^*G is $\|G\|^2$-lipschitz, G^*G is $1/\|G\|^2 - ism$, by proposition 2.2 (b), γG^*G is $1/(\gamma\|G\|^2) - ism$. Hence by proposition 2.2 (c), when $0 < \gamma < 2/\|G\|^2$, $I-\gamma G^*G$ is $(\gamma\|G\|^2)/2-$averaged, where $0 < (\gamma\|G\|^2)/2 < 1$. What's more, we know that P_{S_i} is averaged, so the composite $P_{S_i}(I-\gamma G^*G)$ is also averaged by proposition 2.2 (d).

Through the analysis above, we have to search for a common fixed point of the averaged mappings $\{T_i\}_{i=1}^t$. Since $Fix(T_i)$ (i=1,2,...,t) is convex, we can see this problem as a convex feasibility problem.

First we propose a fully sequential iterative algorithm which generates a sequence $\{w_n\}$ by

$$w_{n+1} = P_{S_{m(n)}}(I-\gamma G^*G)w_n,\ n \ge 0 \tag{3.2}$$

where the initial guess $w_0 \in H_1 \times H_2$ and $m(n) = n \bmod t + 1$.

Theorem 3.1. Assume the MSSEP (1.1) is consistent and $0 < \gamma < 2/\|G\|^2$. Then the sequence $\{w_n\}$ generated by (3.2) converges weakly to a solution of the MSSEP (1.1).

Proof. Since, for each $1 \le i \le t$, $T_i = P_{S_i}(I-\gamma G^*G)$ is averaged, there exist $\alpha_i \in (0,1)$ and nonexpansive mapping N_i such that $T_i = (1-\alpha_i)I + \alpha_i N_i$. Thus, the algorithm (3.2) can be written as

$$w_{n+1} = (1-\alpha_{m(n)})w_n + \alpha_{m(n)}N_{m(n)}w_n,\ n \ge 0$$

where $m(n) = n \bmod t + 1$ for all n. Let Γ denote the solution set of the MSSEP (1.1) which is the set of common fixed points of the mappings $\{T_1, T_2, ..., T_t\}$ (and also of the mappings $\{N_1, N_2, ..., N_t\}$). Take an arbitrary $z \in \Gamma$ to deduce that (by lemma 2.4)

$$\|w_{n+1}-z\|^2 = (1-\alpha_{m(n)})\|w_n-z\|^2 + \alpha_{m(n)}\|N_{m(n)}w_n-z\|^2 - \alpha_{m(n)}(1-\alpha_{m(n)})\|w_n-N_{m(n)}w_n\|^2$$
$$\le \|w_n-z\|^2 - \alpha_{m(n)}(1-\alpha_{m(n)})\|w_n-N_{m(n)}w_n\|^2.$$

However, $\alpha_{m(n)}(1-\alpha_{m(n)}) \ge \alpha_{min}(1-\alpha_{max}) > 0$, where $\alpha_{min} = min\{\alpha_i : 1 \le i \le t\} \in (0,1)$ and $\alpha_{max} = max\{\alpha_i : 1 \le i \le t\} \in (0,1)$. It follows that

$$\|w_n - N_{m(n)}w_n\|^2 \le \frac{1}{\alpha_{min}(1-\alpha_{max})}(\|w_n-z\|^2 - \|w_{n+1}-z\|^2).$$

This implies that

$\lim_{n\to\infty}\|w_n - z\|$ exists for all $z \in \Gamma$. (3.3)

Hence, $\{w_n\}$ is bounded and

$$\lim_{n\to\infty}\|w_n - N_{m(n)}w_n\| = 0. \tag{3.4}$$

Since $\|w_{n+1}-w_n\| = \alpha_{m(n)}\|w_n - N_{m(n)}w_n\|$, we obtain that

$$\lim_{n\to\infty}\|w_{n+1}-w_n\| = 0. \tag{3.5}$$

Since the family of nonexpansive mappings $\{N_{m(n)}\}$ is finite, we get that (3.4) and (3.5) are sufficient to imply that the weak ω-limit set of the sequence $\{w_n\}$ is contained in the set of common fixed points of the mappings $\{N_i\}_{i=1}^t$; that is

$$\omega_w(w_n) \subset \Gamma. \tag{3.6}$$

By lemma 2.5, (3.3) and (3.6) imply that $\{w_n\}$ converges weakly to a member of Γ. The proof is complete.

Remark 3.1. We can give an iterative algorithm like this

$$w_{n+1} = [P_{S_t}(I-\gamma G^*G)][P_{S_{t-1}}(I-\gamma G^*G)]\cdots[P_{S_1}(I-\gamma G^*G)]w_n,\ n \geq 0 \tag{3.7}$$

where the initial guess $w_0 \in H_1 \times H_2$.

Assume the MSSEP (1.1) is consistent and $0 < \gamma < 2/\|G\|^2$. Then the sequence $\{w_n\}$ generated by (3.7) is a sub-sequence of the sequence generated by (3.2), so it converges weakly to a solution of the MSSEP (1.1).

Second we propose a simultaneous algorithm as follows

$$w_{n+1} = \sum_{i=1}^{t} \lambda_i P_{S_i}(I-\gamma G^*G)w_n,\ n \geq 0 \tag{3.8}$$

where the initial guess $w_0 \in H_1 \times H_2$, $\lambda_i > 0$ for all i and $\sum_{i=1}^t \lambda_i = 1$.

Theorem 3.2. Assume the MSSEP (1.1) is consistent and $0 < \gamma < 2/\|G\|^2$. Then the sequence $\{w_n\}$ generated by (3.8) converges weakly to a solution of the MSSEP (1.1).

Proof. Since each $T_i = P_{S_i}(I-\gamma G^*G)$ is averaged, the convex combination $S := \sum_{i=1}^t \lambda_i T_i$ is also averaged. Note that (3.8) can be written as $w_n = S^n w_0$. Hence, by lemma 2.6, the sequence $\{w_n\}$ generated by (3.8) converges weakly to a fixed point of S. But $Fix(S) = \bigcap_{i=1}^t Fix(T_i)$ is the solution set of the MSSEP (1.1). The proof is complete.

We can also use Mann's iterative algorithm to solve the MSSEP (1.1):

$$w_{n+1} = (1-\alpha_n)w_n + \alpha_n[P_{S_t}(I-\gamma G^*G)][P_{S_{t-1}}(I-\gamma G^*G)]\cdots[P_{S_1}(I-\gamma G^*G)]w_n,\ n \geq 0 \tag{3.9}$$

where the initial guess $w_0 \in H_1 \times H_2$.

Theorem 3.3. Assume the MSSEP (1.1) is consistent and the following conditions are satisfied:

$(i) 0 < \gamma < 2/\|G\|^2$;

$(ii) \sum_{n=1}^\infty \alpha_n(1-\alpha_n) = \infty$.

Then the sequence $\{w_n\}$ generated by (3.9) converges weakly to a solution of the MSSEP (1.1).

Proof. By lemma 2.1, the sequence $\{w_n\}$ generated by (3.9) converges weakly to a fixed point of

$$P_{S_t}(I-\gamma G^*G)P_{S_{t-1}}(I-\gamma G^*G)\cdots P_{S_1}(I-\gamma G^*G),$$

which is exactly a common fixed point of $\{P_{S_i}(I-\gamma G^*G)\}_{i=1}^t$, by proposition 3.2, the proof is complete.

The simultaneous algorithm (3.8) has also its Mann's iterative form:

$$w_{n+1} = (1-\alpha_n)w_n + \alpha_n \sum_{i=1}^{t} \lambda_i P_{S_i}(I-\gamma G^*G)w_n,\ n \geq 0 \tag{3.10}$$

where the initial guess $w_0 \in H_1 \times H_2$, $\lambda_i > 0$ for all i and $\sum_{i=1}^t \lambda_i = 1$.

Theorem 3.4. Assume the MSSEP (1.1) is consistent and the following conditions are satisfied:

$(i) 0 < \gamma < 2/\|G\|^2$;

$(ii) \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

Then the sequence $\{w_n\}$ generated by (3.10) converges weakly to a solution of the MSSEP (1.1).

Proof. Since each $T_i = P_{S_i}(I - \gamma G^* G)$ is averaged, the convex combination $S := \sum_{i=1}^{t} \lambda_i T_i$ is also averaged. By lemma 2.1, the sequence $\{w_n\}$ generated by (3.10) converges weakly to a fixed point of S, which is exactly a common fixed point of $\{T_i\}_{i=1}^{t}$. By proposition 3.2, the proof is complete.

Mann's algorithm has only weak convergence in general, some modifications are needed to obtain strong convergence, related information please see [21] and the reference therein.

Halpern's iteration is an important tool for obtaining a fixed point of nonexpansive mappings because of its strong convergence.

Let $\{w_n\}$ be generated by the following Halpern's iterative algorithm:

$$w_{n+1} = t_n u + (1-t_n) P_{S_{m(n)}}(I - \gamma G^* G) w_n, \ n \geq 0 \tag{3.11}$$

where the initial guess $w_0 \in H_1 \times H_2$ and $m(n) = n \bmod t + 1$.

We have the following result.

Theorem 3.5. Given $u \in H_1 \times H_2$. Assume the MSSEP (1.1) is consistent, $0 < \gamma < 2/\|G\|^2$ and the following conditions are satisfied:

$(i) \lim_{n\to\infty} t_n = 0$;

$(ii) \sum_{n=0}^{\infty} t_n = \infty$;

$(iii) \sum_{n=0}^{\infty} | t_{n+t} - t_n | < \infty$ or $\lim_{n\to\infty}(t_n/t_{n+t}) = 1$.

Then the sequence $\{w_n\}$ generated by (3.11) converges strongly to a solution of the MSSEP (1.1) which is nearest to u from the solution set of the MSSEP (1.1). In particular, if we take $u = 0$, then $\{w_n\}$ converges strongly to the minimum-norm solution of the MSSEP (1.1).

Proof. Since each $T_i = P_{S_i}(I - \gamma G^* G)$ is averaged, the common fixed point set of these mappings satisfies the property (proposition 2.2 [6]):

$$\emptyset \neq \bigcap_{i=1}^{t} Fix(T_i) = Fix(T_t T_{t-1} \cdots T_1) = Fix(T_1 T_t \cdots T_2) = \cdots = Fix(T_{t-1} T_{t-2} \cdots T_t).$$

By theorem 4.1 of [23], the sequence $\{w_n\}$ generated by (3.11) converges strongly to the projection of u onto $\bigcap_{i=1}^{t} Fix(T_i)$, but $\bigcap_{i=1}^{t} Fix(T_i)$ is the solution set of the MSSEP (1.1). The proof is complete.

Theorem 3.6. Given $u \in H_1 \times H_2$. Assume the MSSEP (1.1) is consistent, $0 < \gamma < 2/\|G\|^2$ and the following conditions are satisfied:

$(i) \lim_{n\to\infty} t_n = 0$;

$(ii) \sum_{n=0}^{\infty} t_n = \infty$;

$(iii) \sum_{n=0}^{\infty} | t_{n+1} - t_n | < \infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Define $\{w_n\}$ be generated by the algorithm

$$w_{n+1} = t_n u + (1-t_n)[P_{S_t}(I-\gamma G^* G)][P_{S_{t-1}}(I-\gamma G^* G)] \cdots [P_{S_1}(I-\gamma G^* G)] w_n, \ n \geq 0 \tag{3.12}$$

where the initial guess $w_0 \in H_1 \times H_2$.

Then the sequence $\{w_n\}$ generated by (3.12) converges strongly to a solution of the MSSEP (1.1) which is nearest to u from the solution set of the MSSEP (1.1). In particular, if we take $u = 0$, then $\{w_n\}$ converges strongly to the minimum-norm solution of the MSSEP (1.1).

Proof. By lemma 2.2, the sequence $\{w_n\}$ generated by (3.12) converges strongly to the projection of u onto the fixed point set of

$$T = T_t T_{t-1} \cdots T_1 = P_{S_t}(I - \gamma G^* G) P_{S_{t-1}}(I - \gamma G^* G) \cdots P_{S_1}(I - \gamma G^* G).$$

But $Fix(T) = \bigcap_{i=1}^{t} Fix(T_i)$ is the solution set of the MSSEP (1.1). The proof is complete.

The simultaneous algorithm (3.8) has also its Halpern's iterative form:

$$w_{n+1} = t_n u + (1 - t_n) \sum_{i=1}^{t} \lambda_i P_{S_i}(I - \gamma G^* G) w_n, \ n \geq 0 \tag{3.13}$$

where the initial guess $w_0 \in H_1 \times H_2$, $\lambda_i > 0$ for all i and $\sum_{i=1}^{t} \lambda_i = 1$.

Theorem 3.7. Given $u \in H_1 \times H_2$. Assume the MSSEP (1.1) is consistent, $0 < \gamma < 2/\|G\|^2$ and the following conditions are satisfied:

(i) $\lim_{n\to\infty} t_n = 0$;

(ii) $\sum_{n=0}^{\infty} t_n = \infty$;

(iii) $\sum_{n=0}^{\infty} \mid t_{n+1} - t_n \mid < \infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Then the sequence $\{w_n\}$ generated by (3.13) converges strongly to a solution of the MSSEP (1.1) which is nearest to u from the solution set of the MSSEP (1.1). In particular, if we take $u = 0$, then $\{w_n\}$ converges strongly to the minimum-norm solution of the MSSEP (1.1).

Proof. By lemma 2.2, the sequence $\{w_n\}$ generated by (3.13) converges strongly to the projection of u onto the fixed point set of $S := \sum_{i=1}^{t} \lambda_i P_{S_i}(I - \gamma G^* G)$. But $Fix(S) = \bigcap_{i=1}^{t} Fix(P_{S_i}(I - \gamma G^* G))$ is the solution set of the MSSEP (1.1). The proof is complete.

Recently, Halpern's iteration has received much attention. Some good results that use weaker condition to obtain strong convergence have been achieved. The following is an example that gets rid of the condition (iii) in theorem 3.6.

Define the sequence $\{w_n\}$ generated by the formula

$$w_{n+1} = t_n(\lambda u + (1-\lambda) w_n) + (1-t_n) P_{S_t}(I-\gamma G^* G) P_{S_{t-1}}(I-\gamma G^* G) \cdots P_{S_1}(I-\gamma G^* G) w_n, \ n \geq 0 \tag{3.14}$$

where the initial guess $w_0 \in H_1 \times H_2$, $\{t_n\} \subset [0, 1]$ and $\lambda \in (0, 1)$.

Theorem 3.8. Given $u \in H_1 \times H_2$. Assume the MSSEP (1.1) is consistent, $0 < \gamma < 2/\|G\|^2$ and $\{t_n\}$ satisfies (i) and (ii) stated in theorem 3.6, then the sequence $\{w_n\}$ generated by (3.14) converges strongly to a solution of the MSSEP (1.1) which is nearest to u from the solution set of the MSSEP (1.1). In particular, if we take $u = 0$, then $\{w_n\}$ converges strongly to the minimum-norm solution of the MSSEP (1.1).

Proof. By theorem 3.1 in [22], the sequence $\{w_n\}$ generated by (3.14) converges strongly to the projection of u onto the fixed point set of

$$S := P_{S_t}(I - \gamma G^* G) P_{S_{t-1}}(I - \gamma G^* G) \cdots P_{S_1}(I - \gamma G^* G) w_n.$$

But $Fix(S) = \bigcap_{i=1}^{t} Fix(P_{S_i}(I - \gamma G^* G))$ is the solution set of the MSSEP (1.1). The proof is complete.

More information about Halpern's iteration please see [22-25] and the reference therein.

We can also transform the MSSEP (1.1) to an optimization problem.

Recall the equivalent definition (3.1) of the MSSEP (1.1), we have the following proposition.

Proposition 3.3. Assume the MSSEP (1.1) is consistent, then w^* solves the MSSEP (1.1) if and only if w^* solves the following optimization problem:

$$\min_{w\in\Omega} f(w) := \frac{1}{2}\|Gw\|^2 + \frac{1}{2}\sum_{i=1}^{t}\alpha_i\|P_{C_i\times Q_i}w - w\|^2, \tag{3.15}$$

where $\Omega \subseteq H_1 \times H_2$ is an additional closed convex set and $\alpha_i > 0$ for all i.

Proof. Suppose that

$$w^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

solves the MSSEP (1.1), by the equivalent definition (3.1) of the MSSEP (1.1), we know that $x^* \in C_i$, $y^* \in Q_i$ for any $i = 1, 2, ..., t$ and $Ax^* = By^*$. So $w^* \in C_i \times Q_i$ for any $i = 1, 2, ..., t$ and $Gw^* = 0$, hence $f(w^*) = 0$. In other words, w^* minimizes $f(w)$ in Ω.

Now we assume that

$$w^* = \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

solves the optimization problem (3.15), then $f(w^*) = 0$. So $w^* \in C_i \times Q_i$ for any $i = 1, 2, ..., t$ and $Gw^* = 0$, hence $x^* \in C_i$, $y^* \in Q_i$ for any $i = 1, 2, ..., t$ and $Ax^* = By^*$. We get that $x^* \in \bigcap_{i=1}^{t} C_i$, $y^* \in \bigcap_{i=1}^{t} Q_i$ and $Ax^* = By^*$, by the equivalent definition (3.1) of the MSSEP (1.1), w^* solves the MSSEP (1.1). So we complete the proof.

By proposition 3.3, we formulate the MSSEP (1.1) as a minimization problem. An additional condition like $\sum_{i=1}^{t}\alpha_i = 1$ is sometimes very useful in practical application to real world problems when α_i are weights of importance attached to the constraints. But this condition is not necessary for our analysis below.

The function

$$f(w) = \frac{1}{2}\|Gw\|^2 + \frac{1}{2}\sum_{i=1}^{t}\alpha_i\|P_{C_i\times Q_i}w - w\|^2$$

is continuously differentiable with gradient given by

$$\nabla f(w) = G^*Gw + \sum_{i=1}^{t}\alpha_i(I - P_{C_i\times Q_i})w.$$

Due to the fact that $I - P_{C_i\times Q_i}$ is (firmly) nonexpansive, we get that ∇f is Lipschitz continuous with Lipschitz constant $L = \|G\|^2 + \sum_{i=1}^{t}\alpha_i$, thus the gradient-projection algorithm (2.4) is applicable to solve the minimization problem (3.15). This method generates a sequence $\{w_n\}$ via the procedure:

$$w_{n+1} = P_\Omega\{I - \gamma[G^*G + \sum_{i=1}^{t}\alpha_i(I - P_{C_i\times Q_i})]\}w_n,\ n \geq 0 \tag{3.16}$$

where the initial guess $w_0 \in H_1 \times H_2$, $\gamma > 0$ is a parameter and $\alpha_i > 0$ for all i.

By lemma 2.3, we immediately get the following convergence result.

Theorem 3.9. Assume the MSSEP (1.1) is consistent. If $0 < \gamma < 2/(\|G\|^2 + \sum_{i=1}^{t}\alpha_i)$, then the sequence $\{w_n\}$ generated by the gradient-projection algorithm (3.16) converges weakly to a solution of the MSSEP (1.1).

We can also use the fixed point method to solve the MSSEP (1.1).

By proposition 3.3, we have the following result.

Proposition 3.4. Assume the MSSEP (1.1) is consistent, then w^* solves the MSSEP (1.1) if and only if w^* solves the fixed point equation

$$P_\Omega\{I-\gamma[G^*G+\sum_{i=1}^{t}\alpha_i(I-P_{C_i\times Q_i})]\}w=w. \tag{3.17}$$

Proof. It is well-known that w^* solves the minimization problem (3.15) if and only if $w^*\in\Omega$ satisfies the variational inequality

$$\langle w-w^*,[G^*G+\sum_{i=1}^{t}\alpha_i(I-P_{C_i\times Q_i})]w^*\rangle\geq 0,\ \forall w\in\Omega,$$

We can rewrite the above variational inequality as, for any $\gamma>0$

$$\langle w-w^*,w^*-\{w^*-\gamma[G^*G+\sum_{i=1}^{t}\alpha_i(I-P_{C_i\times Q_i})]w^*\}\rangle\geq 0,\ \forall w\in\Omega,$$

if and only if

$$P_\Omega\{w^*-\gamma[G^*G+\sum_{i=1}^{t}\alpha_i(I-P_{C_i\times Q_i})]w^*\}=w^*.$$

The proof is complete.

Through the analysis above, we can use the iterative algorithms that get a fixed point of an averaged mapping to solve the MSSEP (1.1).

By lemma 2.6, define $\{w_n\}$ be generated by Picard iteration:

$$w_{n+1}=P_\Omega\{I-\gamma[G^*G+\sum_{i=1}^{t}\alpha_i(I-P_{C_i\times Q_i})]\}w_n,\ n\geq 0 \tag{3.18}$$

where the initial guess $w_0\in H_1\times H_2$, $\gamma>0$ is a parameter and $\alpha_i>0$ for all i. Assume the MSSEP (1.1) is consistent. If $0<\gamma<2/(\|G\|^2+\sum_{i=1}^{t}\alpha_i)$, then the sequence $\{w_n\}$ generated by (3.18) converges weakly to a solution of the MSSEP (1.1). In fact, the algorithm (3.18) is exactly the gradient-projection algorithm (3.16).

We can also use Mann's algorithm to get a fixed point:

$$w_{n+1}=(1-\alpha_n)w_n+\alpha_nP_\Omega\{I-\gamma[G^*G+\sum_{i=1}^{t}\alpha_i(I-P_{C_i\times Q_i})]\}w_n,\ n\geq 0 \tag{3.19}$$

where the initial guess $w_0\in H_1\times H_2$, $\gamma>0$ is a parameter and $\alpha_i>0$ for all i.

By lemma 2.1 and proposition 3.4, we immediately obtain the following result.

Theorem 3.10. Assume the MSSEP (1.1) is consistent and the following conditions are satisfied:

$(i)0<\gamma<2/(\|G\|^2+\sum_{i=1}^{t}\alpha_i)$;

$(ii)\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$.

Then the sequence $\{w_n\}$ generated by (3.19) converges weakly to a solution of the MSSEP (1.1).

If we use Halpern's algorithm, we can obtain the strong convergence.

Define the sequence $\{w_n\}$ be generated by Halpern's iteration:

$$w_{n+1} = t_n u + (1 - t_n)P_\Omega\{I - \gamma[G^*G + \sum_{i=1}^{t} \alpha_i(I - P_{C_i\times Q_i})]\}w_n,\ n \geq 0 \tag{3.20}$$

where the initial guess $w_0 \in H_1 \times H_2$, $\gamma > 0$ is a parameter and $\alpha_i > 0$ for all i.

By lemma 2.2 and proposition 3.4, we immediately obtain the following theorem.

Theorem 3.11. Given $u \in H_1 \times H_2$. Assume the MSSEP (1.1) is consistent, $0 < \gamma < 2/(\|G\|^2 + \sum_{i=1}^{t} \alpha_i)$ and the following conditions are satisfied:

$(i)\ \lim_{n\to\infty} t_n = 0$;

$(ii)\ \sum_{n=0}^{\infty} t_n = \infty$;

$(iii)\ \sum_{n=0}^{\infty} \mid t_{n+1} - t_n \mid < \infty$ or $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Then the sequence $\{w_n\}$ generated by (3.20) converges strongly to a solution of the MSSEP (1.1) which is nearest to u from the solution set of the MSSEP (1.1). In particular, if we take $u = 0$, then $\{w_n\}$ converges strongly to the minimum-norm solution of the MSSEP (1.1).

4. Conclusions

In this paper, the MSSEP is proposed and some iterative algorithms are considered. The MSSEP has abroad applicability in modeling significant real world problem.

In some cases, notably when the convex sets are not linear, the exact computation of the projections onto convex sets calls for the solution of a separate optimization problem for each projection. In such cases the efficiency of methods that use projections onto convex sets is seriously reduced. Yang [7] proposed a relaxed CQ-algorithm where projections onto convex sets are replaced by projections onto, well-defined and easily derived, half-spaces that contain the convex sets, and are, therefore, easily executed. Our algorithm (3.16) has also its relaxed version, more generally, it has its perturbed version which consists in taking approximate sets that involve the $\rho-$distance between two closed convex sets. We will discuss these techniques in another paper.

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