CONVEX CONTRACTIVE MAPPINGS IN GENERALIZED METRIC SPACES

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This paper presents a study of various classes of convex contractions in the framework of Jleli-Samet generalized metric spaces. Existence and uniqueness results are proved for operators satisfying such kind of inequalities. Examples are provided in order to prove the usability of these outcomes.

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1. Introduction

The importance of classical contractions with regard to the development of modern mathematics has been undeniable in the last century. Starting as a tool in proving various results in differential equations and general topology, the Banach contraction principle became a central object of study in fixed point theory. The search for various extensions of this initial principle is one of the main aims of researchers in nonlinear analysis and applied mathematics. One of the ways of extending the contraction principle is using more general inequalities. Such an extension was the notion of a convex contraction, introduced by Istraţescu in his work [8]. In this paper, the author proved some existence and uniqueness results on various types of convex contractions. More precisely, the right hand side of the contractive inequality consists in convex combinations of distances, and the scalars involved have their sum smaller than one. Changing the distances or adding new terms in the inequality led to new types of contractions. By replacing distances by diameters of sets, the problem can be approached with adequate topological tools. These ideas were extended successfully in various spaces. For instance, in the paper of Alghamdi *et al.* [4], the case of convex contractions in the setting of cone metric spaces was treated. These ideas were further developed by Miandaragh, *et al.* [14], who designed new theorems of fixed points in metric spaces by means of additional properties of the studied operators. More explicitly, they used the notion of α-admissibility and the (H) property in order to establish the existence and the uniqueness of a fixed point.

Important outcomes can be obtained by the use of generalized metric spaces. In this respect, we mention here the b-metric of Bakhtin [5] and Czerwik [7], interesting results being proved in this context by Ali *et al.* [1], or Shatanawi [18]. A generalization of them, introduced by Kamran *et al.* [10] was used in relation with comparison type functions by Samreen *et al.* [16]. The notion of convex contractions of the second order is brought up in

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literature by Khan *et al.* in [13], in which new generalizations are given using the settings of 2-metric spaces and b-metric spaces.

The aim of this paper is to extend the notion of a convex contraction in the context of Jleli-Samet generalized spaces. These spaces appeared in [9], where Jleli and Samet defined a new type of metric. Concisely, the triangle inequality was replaced by a limit type inequality, and the distance between a point and itself might not necessarily be zero. One can prove that the class of Jleli-Samet spaces genuinely includes the class of classical metric spaces, the b-metric spaces or modular vector spaces having the Fatou's property. In [9], Jleli and Samet proved a Banach type result, followed by other developments in this setting, like a Ćirić's contraction principle. Important tools in the study of these spaces were presented by Karapınar *et al.* [11], where the authors gave some directions for further studies of Jleli-Samet spaces. This setting was used also by Senapati *et al.* [17] in the context of implicit kind contractive inequalities, or Altun and Samet [3] for the study of pseudo Picard operators.

Our article is structured in the following way: firstly, some important definitions regarding generalized metric spaces are recalled. The main results are given as a set of theorems related to various types of contractions given by means of convex combinations of various distances and a certain type of mapping. Additional associated properties ensure the uniqueness of fixed points in different situations. The proofs present the obstacles that we have to overcome in this generalized metric setup, as the absence of a triangle type inequality. Examples and comments unify and clarify our results.

2. **Preliminaries**

The outcomes of the paper are obtained in the setting of generalized metric spaces developed by Jleli and Samet [9].

The difference between these new spaces and other classes of metric-type frameworks can be observed right from the third axiom, in which the triangle inequality is replaced by a limit type relation. The unexpected detail is that the idea of convergence is used somehow beforehand, in order to state the replacement of the triangle inequality from the classic context.

Definition 2.1 ([9])**.** Consider the arbitrary set $X \neq \emptyset$, and let $D\colon X \times X \to [0, \infty]$ be a mapping. We say that D is a JS-metric on X if the following statements are satisfied:

($D1$) For all x, $y \in X$, the following implication holds true

$$D(x, y) = 0 \Longrightarrow x = y;$$

($D2$) For every x, $y \in X$, $D(x, y) = D(y, x)$;

($D3$) There can be found a constant $C > 0$ such that for every x, $y \in X$, and for each sequence $\{x_n\}$ with $\lim_{n\to\infty} D(x_n, x) = 0$, the next inequality is fulfilled

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y).$$

In this work, the pair (X, D) will designate a Jleli-Samet metric space (or a JS-space).

As we have already mentioned in the introductory part, the class of generalized spaces contains the class of classical metric spaces, the class of b-metrics, the class of dislocated spaces, or the class of modular vector spaces, endowed with additional properties, for more details, please see [9].

Other useful notions are given in the following paragraphs.

As it naturally arises from the definition of JS-metric spaces, $\{x_n\} \subseteq X$ is convergent if $\lim_{n\to\infty} D(x_n, x) = 0$.

Definition 2.2 ([11]). Let (X, D) be a JS-metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ is a D-Cauchy sequence if

$$\lim_{m,n\to\infty} D(x_m, x_n) = 0.$$

One can see immediately that any convergent sequence in the JS-metric is also D-Cauchy; the converse is no longer true. A JS-space (X, D) in which any D-Cauchy sequence is D-convergent to an element in X is called D-complete.

In order to formulate and prove theorems in this, we consider the following notations:

$$\delta_{n_0}(D, T, w) = \sup(\{D(T^n w, T^m w) : n, m \in \mathbb{N}, n, m \geq n_0\}),$$

where $n_0 \in \mathbb{N}$, and

$$\delta(D, T, w) = \sup(\{D(T^n w, T^m w) : n, m \in \mathbb{N}\}).$$

Let us consider the orbit of an element w by an operator $T\colon X \to X$ using the symbol

$$\mathcal{O}_T(w) = \{T^n w : n \in \mathbb{N}\}.$$

In practice, in order to see whether a space (X, D) checks the third axiom of Jleli and Samet or not, it suffices to find its convergent sequences and check the limit inequality. The next example will be revealing.

Example 2.1. Let us denote by $X = [0, 1] \cup \{5\}$ for which we can define the mapping $D\colon X \times X \to [0, \infty]$ given by:

$$D(x, y) = \begin{cases} \infty, & \text{if } x = y = 5, \\ |x - y|, & \text{otherwise.} \end{cases}$$

We are going to prove that (X, D) is a complete generalized Jleli-Samet metric space. The first two axioms are directly satisfied by the definition of D. If we take $C = 1$, let x, $y \in X$, and $\{x_n\}$ a D-convergent sequence. Let us prove that:

$$D(x, y) \leq \limsup_{n\to\infty} D(x_n, y).$$

If $x, y \in X\backslash\{5\}$ we have $D(x, y) = |x - y|$, and $\{x_n\} \subset [0, 1]$ such that $\lim_{n\to\infty} D(x_n, x) = 0$. Therefore it is clear that we can write:

$$D(x, y) \quad = \quad |x - y| = \lim_{n\to\infty} |x_n - y| = \limsup_{n\to\infty} D(x_n, y).$$

If we take $x \in [0, 1]$ and $y = 5$, one can find $\{x_n\} \subset [0, 1]$ such that $\lim_{n\to\infty} D(x_n, x) = 0$, so we can also say that:

$$D(x, 5) \quad = \quad |x - 5| = \lim_{n\to\infty} |x_n - 5| = \limsup_{n\to\infty} D(x_n, 5).$$

Suppose that there is $\{x_n\} \subset X$ such that $\lim_{n\to\infty} D(x_n, 5) = 0$. It means that there is $n_1 \in \mathbb{N}$ such that $D(x_n, 5) < \frac{1}{2}$ for all $n \geq n_1$, $n \in \mathbb{N}$. In other words, $|x_n - 5| < \frac{1}{2}$ for all $n \geq n_1$, $n \in \mathbb{N}$. Since $x_n \in [0, 1]$, for all $n \in \mathbb{N}$, it implies that $|x_n - 5| \geq 4$, so we obtained a contradiction. All possible cases had been discussed, so the space is a Jleli-Samet one.

For the completeness, let us choose $\{x_n\}$ a D-Cauchy sequence in X. If we use the properties of the above distance, we get, for any m, $n \in \mathbb{N}$:

$$|x_n - x_m| \leq D(x_n, x_m).$$

As a consequence, any D-Cauchy sequence on X is in fact convergent in the space $[0, 1]$ endowed with the Euclidean distance. In conclusion, the space is D-complete.

Let us define the notions of α-admissibility and triangular α-admissibility, which are going to be used in formulating the contractive inequalities which feature the fixed point results.

Definition 2.3 ([2])**.** Let us consider $X \neq \emptyset$, and the mapping $\alpha\colon X \times X \to [0, \infty)$. An operator $T\colon X \to X$ is α-admissible if $\alpha(x, y) \geq 1$ compels $\alpha(Tx, Ty) \geq 1$, for all x, $y \in X$.

Definition 2.4 ([19])**.** Let us consider $X \neq \emptyset$ and the mapping $\alpha\colon X \times X \to [0, \infty)$. We say that $T\colon X \to X$ is triangular α-admissible if $\alpha(x, y) \geq 1$ compels $\alpha(Tx, Ty) \geq 1$, and $\alpha(x, y) \geq 1$, $\alpha(y, z) \geq 1$ compel $\alpha(x, z) \geq 1$ for all x, y, $z \in X$.

Definition 2.5 ([15])**.** A set $X \neq \emptyset$ has the property (H) if, for each x, $y \in X$ there is a point $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Definition 2.6 ([6])**.** Let $T\colon X \to X$ be a self mapping. For $\varepsilon > 0$, we say that $x_* \in X$ is an ε-fixed point of T if $D(x_*, Tx_*) < \varepsilon$. As a notation, $F_\varepsilon(T)$ is the set of all ε-fixed points of T. Moreover, it is said that T has the the approximate fixed point property if T has an ε-fixed point for all $\varepsilon > 0$.

An important remark should be made: there are such mappings T which have approximate fixed points without fixed points at all.

In practice, working with general Jleli-Samet metric spaces is problematic when it comes to establish uniqueness results for fixed points. A way to avoid such situations is to restrict the values of the metric such that pathological cases with infinite distances do not appear.

Definition 2.7. If (X, D) is a Jleli-Samet metric space such that $D(x, y) < \infty$, for all x, $y \in X$, we say that (X, D) is a strong Jleli-Samet metric space.

3. **Main results**

Taking into consideration the above preparation, we are now able to state and prove our main theorems.

Theorem 3.1. *Let (X, D) be a Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is a α-admissible mapping;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there are a, $b \in [0, 1)$ with $a + b < 1$, such that*

$$\alpha(x, y)D(T^2x, T^2y) \leq aD(Tx, Ty) + bD(x, y),$$

for all x, $y \in \mathcal{O}_T(x_0)$;

Then the mapping T has the approximate fixed point property.

Proof. Let us consider x_0 fulfilling property ii), and denote the corresponding Picard sequence by $\{x_n\}$. Inductively, one can prove that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Let us denote $r = D(x_0, x_1) + D(x_1, x_2)$ and $\mu = a + b < 1$. Using the contractive inequality, we get:

$$\begin{aligned} D(x_2, x_3) &\leq \alpha(x_0, x_1)D(x_2, x_3) \leq aD(x_1, x_2) + bD(x_0, x_1) \\ &\leq (a + b)r = \mu r, \end{aligned}$$

and

$$\begin{aligned} D(x_3, x_4) &\leq \alpha(x_1, x_2)D(x_3, x_4) \leq aD(x_2, x_3) + bD(x_1, x_2) \\ &\leq a\mu r + br = (a\mu + b)r \leq (a + b)r = \mu r. \end{aligned}$$

This kind of procedure will inductively lead to the following relation:

$$D(x_n, x_{n+1}) \leq \mu^k r,$$

where $n = 2k$ or $n = 2k+1$, for all positive integers k. As $\mu < 1$, it can be concluded that $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$.

Now, from the definition of the limit, we get that for each $\varepsilon > 0$ there is an index $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$ we have:

$$D(T^n x_0, T^{n+1} x_0) = D(T^n x_0, TT^n x_0) < \varepsilon.$$

If we set $q = T^n x_0$, it follows that for all $\varepsilon > 0$, there is $q \in \mathcal{O}_T(x_0)$ such that $D(q, Tq) < \varepsilon$. Thus, there exists an ε-fixed point of T, and this completes the proof. □

We are now in a position to provide a result on the existence of fixed points associated with mappings which satisfy a convex type condition.

Theorem 3.2. *Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is a triangular α-admissible mapping;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there are a, $b \in [0, 1)$ with $a + b < 1$, such that*

$$\alpha(x, y) D(T^2 x, T^2 y) \leq aD(Tx, Ty) + bD(x, y),$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) *T is continuous;*

v) *$\alpha(T^n x_0, T^n x_0) \geq 1$, for all $n \in \mathbb{N}$.*

Then the sequence $\{T^n x_0\}$ is convergent to a point x_* in X. Moreover, x_* is a fixed point of T.

Proof. As in the previous case, we denote by $\{x_n\}$ the Picard sequence corresponding to the point x_0.

According to the triangular α-admissibility and hypothesis v), it follows that $\alpha(x_m, x_n) \geq 1$, for all m, $n \in \mathbb{N}$, $n \geq m$. This remark can be justified by means of mathematical induction.

Our contractive relation imposes, for all $n \geq m \geq k$, the next relations:

$$\begin{aligned} D(x_{m+2}, x_{n+2}) &\leq \alpha(x_m, x_n) D(T^2 x_m, T^2 x_n) \\ &\leq aD(x_{m+1}, x_{n+1}) + bD(x_m, x_n) \\ &\leq a\delta_{k+1}(D, T, x_0) + b\delta_k(D, T, x_0). \end{aligned}$$

After passing through the supremum in the last relation, it can be seen that:

$$\delta_{k+2}(D, T, x_0) \leq a\delta_{k+1}(D, T, x_0) + b\delta_k(D, T, x_0). \tag{1}$$

Now, it is worth mentioning that the sequence $\{\delta_k(D, T, x_0)\}$ is non-increasing and all its terms are positive numbers. Thus, the sequence is convergent. Denote by $L \in [0, \infty)$ its limit. By way of contradiction, let us suppose that $L \neq 0$. Passing through the limit over k in relation (1), we obtain

$$L \leq (a + b)L < L,$$

which is a contradiction.

Now, one can conclude that $\lim_{m,n\to\infty} D(x_n, x_m) = 0$, and the Picard sequence is D-Cauchy. Taking into consideration that we work in a D-complete space, there exists $x_* \in X$ such that $\lim_{n\to\infty} D(x_n, x_*) = 0$.

By the continuity of T, we find that $\lim_{n\to\infty} D(Tx_n, Tx_*) = 0$. As the limit of a sequence is unique in our framework, it follows that $Tx_* = x_*$ and the theorem is proved. □

As a straight forward remark, if there are two fixed points of T, forming the pair (x_*, y_*), at which α is not smaller than one and the distance between them is finite, and the contractive inequality holds, then the uniqueness property of the fixed point follows.

In order to state a result of the uniqueness of fixed points of mappings which fulfill the convexity condition, regardless of the value of the function α at the pair formed by two presumed fixed points, we have to impose an additional property to these operators, which instead allows the removal of some of those already used.

Theorem 3.3. *Let (X, D) be a strong Jleli-Samet metric space and $T\colon X \to X$ be a self-mapping. Presume that the following assertions hold true:*

i) *T is a α-admissible mapping for some $\alpha\colon X \times X \to [0, \infty)$;*

ii) *there are a, $b \in [0, 1)$ with $a + b < 1$, such that*

$$\alpha(x, y)D(T^2x, T^2y) \leq aD(Tx, Ty) + bD(x, y),$$

for all x, $y \in X$;

iii) *T has property (H);*

Then, if T has a fixed point, it is unique.

Proof. Let us take x_*, $y_* \in X$ two fixed points of T. Using the property (H), there is $w \in X$ such that $\alpha(x_*, w) \geq 1$ and $\alpha(y_*, w) \geq 1$. Recalling the α-admissibility, it is immediate that $\alpha(x_*, T^n w) \geq 1$ and $\alpha(y_*, T^n w) \geq 1$, for all $n \in \mathbb{N}^*$. Let us denote $r = \max\{D(x_*, Tw), D(x_*, T^2w)\}$ and $\mu = a + b$. Using the contractive relation, we get:

$$\begin{aligned} D(x_*, T^3w) &= D(T^2x_*, T^2Tw) \leq \alpha(x_*, Tw)D(T^2x_*, T^2Tw) \\ &\leq aD(x_*, T^2w) + bD(x_*, Tw) \leq (a + b)r = \mu r \end{aligned}$$

and also

$$\begin{aligned} D(x_*, T^4w) &= D(T^2x_*, T^2T^2w) \leq \alpha(x_*, T^2w)D(T^2x_*, T^2T^2w) \\ &\leq aD(x_*, T^3w) + bD(x_*, T^2w) \leq (a\mu + b)r \\ &\leq (a + b)r = \mu r. \end{aligned}$$

Using an induction on k, it can be proved that $D(x_*, T^n w) \leq \mu^k r$, where k is the integer part of $\frac{n}{2}$, for all $k \in \mathbb{N}^*$.

It follows that $\lim_{n\to\infty} D(x_*, T^n w) = 0$. Following the same steps, one can prove that $\lim_{n\to\infty} D(y_*, T^n w) = 0$ and, by the uniqueness of the limit of a sequence in JS spaces, it follows that $x_* = y_*$. □

As consequences of these outcomes, we mention that a contraction principle type result can be obtained by taking $b = 0$ in Theorem 3.2.

Example 3.1. One can choose the space $X = [0, 1] \cup \{5\}$ endowed with the metric $D\colon X \times X \to [0, \infty]$ described by:

$$D(x, y) = \begin{cases} 2, & \text{if } x = y = 5, \\ |x - y|, & \text{otherwise.} \end{cases}$$

Similar to the first example, one can show that (X, D) is a complete Jleli-Samet metric space.

One can choose

$$\alpha(x,y)=\begin{cases}1, & \text{if } x,y\in\left[0,\frac{1}{2}\right],\\ \dfrac{1}{2}, & \text{otherwise.}\end{cases}$$

Let us define

$$T\colon X\to X,\qquad Tx=\begin{cases}\dfrac{x^2}{3}+\dfrac{x}{9}, & \text{if } x\in[0,1],\\ 5, & \text{if } x=5.\end{cases}$$

Let us choose $x_0\in\left(0,\frac{1}{2}\right)$. Then, for all x, $y\in\mathcal{O}_T(x_0)$, we have the following relations:

$$\begin{aligned}
D(T^2x,T^2y) &= \left|T^2x-T^2y\right|\\
&= \left|\frac{1}{3}\left[\left(\frac{x^2}{3}+\frac{x}{9}\right)^2-\left(\frac{y^2}{3}+\frac{y}{9}\right)^2\right]+\frac{1}{9}\left(\frac{x^2-y^2}{3}+\frac{x-y}{9}\right)\right|\\
&\le \frac{1}{3}\left|\frac{x^2}{3}+\frac{x}{9}-\frac{y^2}{3}-\frac{y}{9}\right|\left(\frac{x^2}{3}+\frac{x}{9}+\frac{y^2}{3}+\frac{y}{9}\right)\\
&\quad +\frac{1}{9}\left|x-y\right|\left(\frac{x+y}{3}+\frac{1}{9}\right)\\
&= \frac{1}{3}\left|Tx-Ty\right|\left(\frac{x^2}{3}+\frac{x}{9}+\frac{y^2}{3}+\frac{y}{9}\right)\\
&\quad +\frac{1}{9}\left|x-y\right|\left(\frac{x+y}{3}+\frac{1}{9}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
D(T^2x,T^2y) &\le \frac{1}{3}\frac{8}{9}\left|Tx-Ty\right|+\frac{1}{9}\frac{7}{9}\left|x-y\right|\\
&\le \frac{8}{27}D(Tx,Ty)+\frac{7}{81}D(x,y).
\end{aligned}$$

Note that $T\left(\left(0,\frac{1}{2}\right)\right)\subset\left(0,\frac{1}{2}\right)$, and observe that Theorem 3.1 and Theorem 3.2 are verified for any $x_0\in\left(0,\frac{1}{2}\right)$. In this case, $x=0$ is the fixed point of the operator T. However, Theorem 3.3 cannot be applied in this case, since the property (H) is missing in our case.

It has to be emphasized that known results in literature can be obtained by considering classes of metric spaces or b-metric spaces as subclasses of the JS-metrics used previously.

4. **Two sided type convex contractions**

Another series of results can be established following the notions introduced in [14], as the concept of a two sided generalized convex contraction, as follows.

Theorem 4.1. *Let (X,D) be a Jleli-Samet metric space, $T\colon X\to X$ be an operator and $\alpha\colon X\times X\to[0,\infty)$ be a given mapping. Presume that the following assertions are accomplished:*

i) *T is α-admissible;*

ii) *there is $x_0\in X$ with $\delta(D,T,x_0)<\infty$, such that $\alpha(x_0,Tx_0)\ge 1$;*

iii) *there are a_1, a_2, b_1, $b_2\in[0,1)$ with $a_1+a_2+b_1+b_2<1$, such that*

$$\begin{aligned}
\alpha(x,y)D(T^2x,T^2y)\le a_1D(x,Tx)+a_2D(Tx,T^2x)\\
+b_1D(y,Ty)+b_2D(Ty,T^2y),
\end{aligned}$$

for all x, $y\in\mathcal{O}_T(x_0)$;

Then the mapping T has the approximate fixed point property.

Proof. Let $x_0 \in X$ fulfill the above properties. Let us denote $p = a_1 + a_2 + b_1$ and $q = 1 - b_2$, and it is clear that $p < q$. Consider $v = \max\{D(Tx_0, x_0), D(T^2x_0, Tx_0)\}$. The contractive inequality implies:

$$\begin{aligned} D(T^2x_0, T^3x_0) &= D(T^2x_0, T^2Tx_0) \le \alpha(x_0, Tx_0)D(T^2x_0, T^2Tx_0) \\ &\le a_1D(x_0, Tx_0) + a_2D(Tx_0, T^2x_0) \\ &\quad +b_1D(Tx_0, T^2x_0) + b_2D(T^2x_0, T^3x_0) \\ &\le (a_1 + a_2 + b_1)v + b_2D(T^2x_0, T^3x_0) \end{aligned}$$

which lead to $D(T^2x_0, T^3x_0) \le \frac{p}{q}v$.

Similarly, it can be seen that $D(T^3x_0, T^4x_0) \le \frac{p}{q}v$. By induction on k, we can prove that $D(T^nx_0, T^{n+1}x_0) \le \left(\frac{p}{q}\right)^k v$, for k the integer part of $\frac{n}{2}$ with $k \in \mathbb{N}^*$.

It follows that $\lim_{n\to\infty} D(T^nx_0, T^{n+1}x_0) = 0$, and the conclusion follows accordingly. □

Putting together adequate theoretical pieces, one can state the next theorem, which refers strictly to the existence of fixed points of two sided convex contractions.

Theorem 4.2. *Let (X, D) be a complete Jleli-Samet metric space, $T: X \to X$ be an operator and $\alpha: X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is a triangular α-admissible mapping;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \ge 1$;*

iii) *there are a_1, a_2, b_1, $b_2 \in [0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, such that*

$$\begin{aligned} \alpha(x, y)D(T^2x, T^2y) \le\ & a_1D(x, Tx) + a_2D(Tx, T^2x) \\ & +b_1D(y, Ty) + b_2D(Ty, T^2y), \end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) *T is continuous;*

v) *$\alpha(T^nx_0, T^nx_0) \ge 1$, for all $n \in \mathbb{N}$.*

Then the sequence $\{T^nx_0\}$ is convergent to x_*. Moreover, x_* is a fixed point of T.

Proof. We use the notation $\{x_n = T^nx_0\}$ for the Picard sequence associated to x_0. The property of triangular α-admissibility and condition v) ensure that $\alpha(x_m, x_n) \ge 1$, for all m, $n \in \mathbb{N}$ with $n \ge m$.

Considering $x = T^mx_0$ and $y = T^nx_0$, the third assertion gives, for all $n \ge m \ge k_0$ where $k_0 \in \mathbb{N}$:

$$\begin{aligned} D(x_{m+2}, x_{n+2}) \le\ & \alpha(x_m, x_n)D(T^2x_m, T^2x_n) \\ & \le a_1D(x_m, Tx_m) + a_2D(Tx_m, T^2x_m) \\ & +b_1D(x_n, Tx_n) + b_2D(Tx_n, T^2x_n). \end{aligned}$$

Taking into account the previous result, for every $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}^*$ such that $D(x_m, x_{m+1}) < \varepsilon$, $D(x_{m+1}, x_{m+2}) < \varepsilon$, $D(x_n, x_{n+1}) < \varepsilon$ and $D(x_{n+1}, x_{n+2}) < \varepsilon$, for all m, $n \ge n_\varepsilon$. It is clear that

$$D(x_{m+2}, x_{n+2}) \le (a_1 + b_1 + a_2 + b_2)\varepsilon < \varepsilon,$$

for all m, $n \ge n_\varepsilon$.

Using the definition of D-Cauchy sequences, we conclude that

$$\lim_{m,n\to\infty} D(x_m, x_n) = 0.$$

Note that (X,D) is D-complete, so we can find $x_* \in X$ such that

$$\lim_{n\to\infty} D(x_n, x_*) = 0.$$

Moreover, since T is continuous, it is clear that $\lim_{n\to\infty} D(Tx_n, Tx_*) = 0$, meaning that $Tx_* = x_*$. □

Separately, one can prove a uniqueness theorem without imposing the condition of continuity on the involved operator or additional hypotheses regarding α, but paying the price of using strong JS-spaces.

Theorem 4.3. *Let (X,D) be a strong Jleli-Samet metric space and $T\colon X \to X$ be a self-operator on X. Suppose that the following items hold true:*

i) *T is a triangular α-admissible mapping for some $\alpha\colon X \times X \to [0,\infty)$;*

ii) *for all $x \in X$, we have $\alpha(x,Tx) \geq 1$;*

iii) *there are a_1, a_2, b_1, $b_2 \in [0,1)$ with $a_1 + a_2 + b_1 + b_2 < 1$, such that*

$$\begin{aligned}\alpha(x,y)D(T^2x,T^2y) \leq a_1 D(x,Tx) + a_2 D(Tx,T^2x)\\ +b_1 D(y,Ty) + b_2 D(Ty,T^2y),\end{aligned}$$

for all x, $y \in X$;

iv) *T has the property (H);*

Then, T has at most one fixed point $x_* \in X$ which fulfills the property $D(x_*,x_*) = 0$.

Proof. Start by considering x_*, $y_* \in X$ two fixed points of T with $D(x_*,x_*) = D(y_*,y_*) = 0$. The property (H) ensures that there is $w \in X$ such that $\alpha(x_*,w) \geq 1$ and $\alpha(y_*,w) \geq 1$. Taking advantage from the triangular α-admissibility, we get that $\alpha(x_*,T^n w) \geq 1$ and $\alpha(y_*,T^n w) \geq 1$ for all $n \in \mathbb{N}^*$. Moreover, combining the hypotheses, one can prove by induction that $\alpha(T^n w, T^m w) \geq 1$, for all m, $n \in \mathbb{N}$. Keeping the notations from Theorem 3.3, we are going to work again with the quantities $p = a_1 + a_2 + b_1$, $q = 1 - b_2$ and $v = \max\{D(T^2w,Tw), D(Tw,w)\}$. Let us study the contractive inequality when $x = w$ and $y = Tw$.

$$\begin{aligned}D(T^2w,T^3w) &= D(T^2w,T^2Tw) \leq \alpha(w,Tw)D(T^2w,T^2Tw)\\ &\leq a_1 D(w,Tw) + a_2 D(Tw,T^2w)\\ &\quad +b_1 D(Tw,T^2w) + b_2 D(T^2w,T^3w)\\ &\leq (a_1+a_2+b_1)v + b_2 D(T^2w,T^3w)\end{aligned}$$

leading to $D(T^2w,T^3w) \leq \frac{p}{q}v$.

Continuing this procedure, it can be proved that

$$D(T^n w, T^{n+1}w) \leq \left(\frac{p}{q}\right)^k v,$$

for all $n = 2k+1$ or $n = 2k$, when $k \in \mathbb{N}$. Now, it is clear that

$$\begin{aligned}D(x_*,T^{n+2}w) &= D(T^2x_*,T^2T^n w) \leq \alpha(x_*,T^n w)D(T^2x_*,T^{n+2}w)\\ &\leq a_1 D(x_*,x_*) + a_2 D(x_*,x_*)\\ &\quad +b_1 D(T^n w,T^{n+1}w) + b_2 D(T^{n+1}w,T^{n+2}w)\\ &\leq b_1\left(\frac{p}{q}\right)^k v + b_2\left(\frac{p}{q}\right)^{k+1} v\end{aligned}$$

for $n = 2k$ or $n = 2k+1$, so the relation remains true for every $n \in \mathbb{N}^*$.

It follows that $\lim_{n\to\infty} D(x_*, T^n w) = 0$. Doing the same thing for y_*, we obtain that $\lim_{n\to\infty} D(y_*, T^n w) = 0$, and the uniqueness of the limit in this framework compels that $x_* = y_*$. □

For $a_2 = b_2 = 0$, the next corollary follows.

Corollary 4.1. Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:

i) T is a triangular α-admissible mapping;

ii) there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;

iii) there are a_1, $b_1 \in [0, 1)$ with $a_1 + b_1 < 1$, such that

$$\alpha(x, y)D(T^2x, T^2y) \leq a_1 D(x, Tx) + b_1 D(y, Ty),$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) T is continuous;

v) $\alpha(T^n x_0, T^n x_0) \geq 1$, for all $n \in \mathbb{N}$.

Then the sequence $\{T^n x_0\}$ is convergent to x_*. Moreover, x_* is a fixed point of T.

Other interesting corollaries can be obtained by considering various combinations of quadruples (a_1, a_2, b_1, b_2), having one or more components being null, or choosing subclasses of the JS-metrics family as frameworks.

Another set of theorems can be formulated in the spirit of [13]. The technical difficulties come from the improvement of the last contractive inequality by adding another two terms.

Theorem 4.4. *Let (X, D) be a Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0, \infty)$ a given mapping. Presume that the following requirements are accomplished:*

i) T is α-admissible;

ii) there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;

iii) there are a_i, b_i, $c_i \in [0, 1)$, for $i = \overline{1, 2}$ with $\sum_{i=1}^{2}(a_i + b_i + c_i) < 1$, such that

$$\begin{aligned}\alpha(x, y)D(T^2x, T^2y) \quad \leq \quad & a_1 D(x, y) + a_2 D(Tx, Ty) \\ & + b_1 D(x, Tx) + b_2 D(Tx, T^2x) \\ & + c_1 D(y, Ty) + c_2 D(Ty, T^2y),\end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

Then the mapping T has the approximate fixed point property.

Proof. Firstly, let us consider $x_0 \in X$ such that $\delta(D, T, x_0) < \infty$ and $\alpha(x_0, Tx_0) \geq 1$. We are going to keep the previous notation for the Picard sequence. Now, since T is α-admissible, it follows that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. We can point out the terms $v = D(x_0, x_1) + D(x_1, x_2)$, $p = \sum_{i=1}^{2}(a_i + b_i) + c_1$ and $q = 1 - c_2$. In order to find out more information, we apply the contractive inequality:

$$\begin{aligned}\alpha(x_0, x_1)D(x_2, x_3) \quad \leq \quad & a_1 D(x_0, x_1) + a_2 D(x_1, x_2) + b_1 D(x_0, x_1) \\ & + b_2 D(x_1, x_2) + c_1 D(x_1, x_2) + c_2 D(x_2, x_3) \\ \leq \quad & (a_1 + a_2 + b_1 + b_2 + c_1)v + c_2 D(x_2, x_3) \\ \leq \quad & pv + c_2 D(x_2, x_3).\end{aligned}$$

As a consequence, it follows that $D(x_2, x_3) \leq \frac{p}{q} v$. We proceed further:

$$\begin{aligned}\alpha(x_1, x_2)D(x_3, x_4) &\leq a_1 D(x_1, x_2) + a_2 D(x_2, x_3) + b_1 D(x_1, x_2)\\ &\quad + b_2 D(x_2, x_3) + c_1 D(x_2, x_3) + c_2 D(x_3, x_4)\\ &= (a_1 + b_1)D(x_1, x_2) + (a_2 + b_2 + c_1)D(x_2, x_3)\\ &\quad + c_2 D(x_3, x_4)\\ &\leq pv + c_2 D(x_3, x_4).\end{aligned}$$

which gives us $D(x_3, x_4) \leq \frac{p}{q}$.

It can be proved that $D(x_n, x_{n+1}) \leq \left(\frac{p}{q}\right)^k v$, where k is the integer part of $\frac{n}{2}$, $k \in \mathbb{N}$, using induction on k.

Since $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$, the conclusion follows. □

Theorem 4.5. *Let (X, D) be a complete Jleli-Samet metric space, $T : X \to X$ be an operator and $\alpha : X \times X \to [0, \infty)$ a given mapping. Suppose that the following conditions are fulfilled:*

i) *T is a triangular α-admissible mapping;*

ii) *there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;*

iii) *there are a_i, b_i, $c_i \in [0, 1)$, for $i = \overline{1, 2}$ with $\sum_{i=1}^{2}(a_i + b_i + c_i) < 1$, such that*

$$\begin{aligned}\alpha(x, y)D(T^2x, T^2y) &\leq a_1 D(x, y) + a_2 D(Tx, Ty)\\ &\quad + b_1 D(x, Tx) + b_2 D(Tx, T^2x)\\ &\quad + c_1 D(y, Ty) + c_2 D(Ty, T^2y),\end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) *T is continuous;*

v) *$\alpha(T^n x_0, T^n x_0) \geq 1$, for all $n \in \mathbb{N}$.*

Then the sequence $\{T^n x_0\}$ is convergent to x_* which is a fixed point of T. Moreover, if $y_* \in X$ is another fixed point of T with $\alpha(x_*, y_*) \geq 1$, $D(x_*, y_*) < \infty$, and $D(y_*, y_*) = 0$, so that iii) *is checked by (x_*, y_*), we can conclude that $x_* = y_*$.*

Proof. As we used to do in previous cases, we recognize $\{x_n\}$ as the Picard sequence associated to the point x_0.

Taking into account the triangular α-admissibility and hypothesis v), we get that $\alpha(x_m, x_n) \geq 1$, for all m, $n \in \mathbb{N}$ with $n \geq m$.

The contractive relation imposes, for all $n \geq m \geq k$, the next relations:

$$\begin{aligned}D(x_{m+2}, x_{n+2}) &\leq \alpha(x_m, x_n)D(T^2x_m, T^2x_n)\\ &\leq a_1 D(x_m, x_n) + a_2 D(x_{m+1}, x_{n+1})\\ &\quad + b_1 D(x_m, x_{m+1}) + b_2 D(x_{m+1}, x_{m+2})\\ &\quad + c_1 D(x_n, x_{n+1}) + c_2 D(x_{n+1}, x_{n+2}).\end{aligned}$$

In other words, we have

$$\begin{aligned}D(x_{m+2}, x_{n+2}) &\leq a_1 \delta_k(D, T, x_0) + (a_2 + b_1 + c_1)\delta_{k+1}(D, T, x_0)\\ &\quad + (b_2 + c_2)\delta_{k+2}(D, T, x_0).\end{aligned}$$

After passing through the supremum in the last relation, it follows:

$$\begin{aligned}\delta_{k+2}(D, T, x_0) &\leq a_1 \delta_k(D, T, x_0) + (a_2 + b_1 + c_1)\delta_{k+1}(D, T, x_0)\\ &\quad + (b_2 + c_2)\delta_{k+2}(D, T, x_0).\end{aligned} \tag{2}$$

Next, $\{\delta_k(D,T,x_0)\}$ is a non-increasing sequence and all its terms are positive numbers. Therefore, the sequence is convergent. Denote by $L \in [0,\infty)$ its limit. Let us suppose that $L \neq 0$. Passing through the limit over k in relation (2), we obtain

$$L \leq \sum_{i=1}^{2}(a_i+b_i+c_i)L < L,$$

which is a contradiction.

It follows that $\lim_{m,n\to\infty} D(x_n,x_m)=0$, and the Picard sequence is D-Cauchy. As X is presumed to be D-complete, there exists $x_* \in X$ such that $\lim_{n\to\infty} D(x_n,x_*)=0$.

Taking into consideration the continuity of T, we conclude that

$$\lim_{n\to\infty} D(Tx_n,Tx_*)=0.$$

Since the limit of a sequence is unique in our spaces, it follows that $Tx_*=x_*$ and the first part of the theorem is proved.

For the uniqueness part, we presume that $y_* \in X$ is another fixed point of T and x_*, y_* verify the contractive inequality:

$$\begin{aligned} D(x_*,y_*) \leq\ & a_1D(x_*,y_*)+a_2D(x_*,y_*)+b_1D(x_*,x_*) \\ & +b_2D(x_*,x_*)+c_1D(y_*,y_*)+c_2D(y_*,y_*). \end{aligned}$$

We obtain that

$$(1-a_1-a_2)D(x_*,y_*) \leq 0$$

which is a contradiction, so $x_*=y_*$. □

It is important to point out that each item required in the above theorem must be achieved in order to obtain the conclusion. In the next example we are going to see the importance of the α-admissibility in the problems dealing with this type of contraction.

Example 4.1. One can take $X=\{1,2\}$ endowed with the metric

$$D(x,y)=\begin{cases} 0, & \text{if } x=y, \\ 2, & \text{otherwise.} \end{cases}$$

One can show that (X,D) is a classic metric space, meaning that it is a Jleli-Samet space as well. Moreover, the only sequences that are convergent in this space are those which are stationary from a given index. Therefor, the D-Cauchy sequences coincide with those of the form $\{x_n\} \subset X$ such that $x_n=g$, for all $n \geq n_0$, where $n_0 \in \mathbb{N}$ and $g \in \{1,2\}$. Hence, (X,D) is a complete Jleli-Samet metric space.

Define now $T\colon X \to X$ by $T1=2$ and $T2=1$, as well as $\alpha\colon X\times X \to [0,\infty)$ given by

$$\alpha(x,y)=\begin{cases} \frac{1}{2}, & \text{if } x\neq y, \\ 1, & \text{otherwise.} \end{cases}$$

Taking $a_i=b_i=c_i=\frac{1}{12}$, for all $i=\overline{1,2}$, one can prove that

$$\begin{aligned} \alpha(x,y)D(T^2x,T^2y) \leq\ & a_1D(x,y)+a_2D(Tx,Ty) \\ & +b_1D(x,Tx)+b_2D(Tx,T^2x) \\ & +c_1D(y,Ty)+c_2D(Ty,T^2y), \end{aligned}$$

for all x, $y \in X$.

If $x = y$, the above inequality is verified. Now, if $x = 1$ and $y = 2$ we have:

$$\begin{aligned} 1 = \frac{1}{2}2 = \alpha(1,2)D(T^2 1, T^2 2) &\leq a_1 D(1,2) + a_2 D(1,2) \\ &\quad + b_1 D(1,2) + b_2 D(1,2) \\ &\quad + c_1 D(1,2) + c_2 D(1,2) \\ &\leq \frac{6}{12} D(1,2) = 1. \end{aligned}$$

Despite the fact that the contractive inequality is true for all x, $y \in X$, there is no $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. As consequence the last three results cannot be applied in this case.

Important results already known in the field can be retrieved from these theorems. In this respect, we mention here two corollaries to Theorem 4.5.

Corollary 4.2. Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0,\infty)$ a given mapping. Suppose that the following conditions are fulfilled:

i) T is a triangular α-admissible mapping;

ii) there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;

iii) there are a_1, a_2, b_1 and $c_1 \in [0, 1)$ with $a_1 + a_2 + b_1 + c_1 < 1$, such that

$$\begin{aligned} \alpha(x,y)D(T^2x, T^2y) &\leq a_1 D(x,y) + a_2 D(Tx,Ty) \\ &\quad + b_1 D(x,Tx) + c_1 D(y,Ty), \end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) T is continuous;

v) $\alpha(T^n x_0, T^n x_0) \geq 1$, for all $n \in \mathbb{N}$.

Then the sequence $\{T^n x_0\}$ is convergent to x_* which is a fixed point of T. Moreover, if $y_* \in X$ is another fixed point of T with $\alpha(x_*, y_*) \geq 1$, $D(x_*, y_*) < \infty$, and $D(y_*, y_*) = 0$, so that iii) is checked by (x_*, y_*), we can conclude that $x_* = y_*$.

Corollary 4.3. Let (X, D) be a complete Jleli-Samet metric space, $T\colon X \to X$ be an operator and $\alpha\colon X \times X \to [0,\infty)$ a given mapping. Suppose that the following conditions are fulfilled:

i) T is a triangular α-admissible mapping;

ii) there is $x_0 \in X$ with $\delta(D, T, x_0) < \infty$, such that $\alpha(x_0, Tx_0) \geq 1$;

iii) there are a_1, a_2, b_2 and $c_2 \in [0, 1)$ with $a_1 + a_2 + b_2 + c_2 < 1$, such that

$$\begin{aligned} \alpha(x,y)D(T^2x, T^2y) &\leq a_1 D(x,y) + a_2 D(Tx,Ty) \\ &\quad + b_2 D(Tx,T^2x) + c_2 D(Ty,T^2y), \end{aligned}$$

for all x, $y \in \mathcal{O}_T(x_0)$;

iv) T is continuous;

v) $\alpha(T^n x_0, T^n x_0) \geq 1$, for all $n \in \mathbb{N}$.

Then the sequence $\{T^n x_0\}$ is convergent to x_* which is a fixed point of T. Moreover, if $y_* \in X$ is another fixed point of T with $\alpha(x_*, y_*) \geq 1$, $D(x_*, y_*) < \infty$, and $D(y_*, y_*) = 0$, so that iii) is checked by (x_*, y_*), we can conclude that $x_* = y_*$.

Other important corollaries can be obtained by considering particular cases of Jleli-Samet spaces.

REFERENCES

[1] *M.U. Ali, T. Kamran, M. Postolache*, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, Nonlinear Anal. Modelling Control, **22**(2017), 389-400.

[2] *H.H. Alsulami, S. Chandok, M.A. Taoudi, I.M. Erhan*, Some fixed point theorems for (α, ψ)-rational type contractive mappings, Fixed Point Theory Appl., **97**(2015).

[3] *I. Altun, B. Samet*, Pseudo Picard operators on generalized metric spaces, Appl. Anal. Discrete Math., **12**(2018), 389-400.

[4] *M. A. Alghamdi, S. H. Alnafei, S. Radenović, N. Shahzad*, Fixed point theorems for convex contraction mappings on cone metric spaces, Math. Comput. Model., **54**(2011), 2020-2026.

[5] *I.A. Bakhtin*, The contraction mapping principle in almost metric spaces, Funct. Anal. Gos. Ped. Inst. Ulianowsk, **30**(1989), 26-37.

[6] *M. Berinde*, Approximate fixed point theorems, Stud. Univ. Babeş-Bolyai, Math., **51**(2006), 11-25.

[7] *S. Czerwik*, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav., 1(1993), 5-11.

[8] *V.I. Istrăţescu*, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters, I. Ann. Mat. Pura Appl., **130**(1982), 89-104.

[9] *M. Jleli, B. Samet*, A generalized metric space and related fixed point theorems, Fixed Point Theory Appl., **61**(2015).

[10] *T. Kamran, M. Samreen, Q. Ul. Ain*, A generalization of b-metric space and some fixed point theorems, Mathematics, **5**(2017).

[11] *E. Karapınar, D. O'Regan, A. Roldan, N. Shahzad*, Fixed point theorems in new generalized metric spaces, J. Fixed Point Theory Appl., **18**(2016), 645-671.

[12] *E. Karapınar, B. Samet, D. Zhang*, Meir-Keeler type contractions on JS-metric spaces and related fixed point theorems, J. Fixed Point Theory Appl., **20**(2018).

[13] *M. S. Khan, Y. M. Singh, G. Maniu, M. Postolache*, On generalized convex contractions of 2-type in b-metric and 2-metric spaces, J. Nonlinear Sci. Appl., **10**(2017), 2902-2913.

[14] *M. A. Miandaragh, M. Postolache, S. Rezapour*, Approximate fixed points of generalized convex contractions, Fixed Point Theory Appl., Art.No 2013:255(2013).

[15] *H. Nawab, E. Karapınar, P. Salimi, F. Akbar*, α-admissible mappings and related fixed point theorems, J. Inequal. Appl., **114**(2013).

[16] *M. Samreen, T. Kamran, M. Postolache*, Extended b-metric space, extended b-comparison function and nonlinear contractions, U. Politech. Buch. Ser. A, **80**(2016), 21-28.

[17] *T. Senapati, L.K. Dey, D. Dolićanin Dekić*, Extentions of Ćirić and Wardowski type fixed point theorems in D-generalized metric spaces, Fixed Point Theory Appl., Art. No. 2016:133(2016).

[18] *W. Shatanawi*, Fixed and common fixed point for mappings satisfying some nonlinear contractions in b-metric spaces, J. Math. Anal., **7**(2016), 1-12.

[19] *X. Wu, L. Zhao*, Fixed point theorems for generalized alpha-psi type contractive mappings in b-metric spaces and applications, J. Math. Comput. Sci., **18**(2018), 49-62.