DUALITY CONDITION FOR GABOR FRAMES $(\chi_{[0,c)}, a, b)$ AND $(\chi_{[0,d)}, a, b)$

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In this paper we find a sufficient and necessary condition for which two Gabor frames $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ form dual frames for $L_2(\mathbb{R})$, where a, b, c and d are positive numbers.

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1. Introduction

Frames were first introduced by Duffin and Schaeffer [6] in the study of nonharmonic Fourier series in 1952. Frames have very important and interesting properties which make them very useful in the characterization of function spaces, signal processing and many other fields. A frame is a family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [2]. Given a separable Hilbert space $\mathcal{H}$ with inner product $< .,. >$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |< f, f_k >|^2 \leq B\|f\|^2, \tag{1}$$

where A, B are the lower and upper frame bounds, respectively. The second inequality of the frame condition (1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. If $A = B$, then $\{f_k\}_{k=1}^{\infty}$ is called a tight frame. For more information concerning frames refer to [1, 4, 5, 8, 12].

For any $x, y \in \mathbb{R}$ the translation operator T_x and modulation operator E_y on $L_2(\mathbb{R})$ are defined by $(T_x g)(t) = g(t - x)$, $(E_y g)(t) = e^{2\pi i y t} g(t)$. A Gabor system (g, a, b) with window function $g \in L_2(\mathbb{R})$, time shift parameter $a > 0$ and frequency shift parameter $b > 0$ is the sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. A Gabor system (g, a, b) is called a Gabor frame if it is a frame for $L_2(\mathbb{R})$, i.e., if there exist constants $A > 0$, $B < \infty$ such that for all $f \in L_2(\mathbb{R})$,

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |< f, E_{mb}T_{na}g >|^2 \leq B\|f\|^2, \tag{2}$$

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where $\|.\|$ and $< ., . >$ denote the standard norm and inner product of $L_2(\mathbb{R})$.

It is a non-trivial problem that when is (g, a, b) a Gabor frame? It is well known tht if $ab \leq 1$, then (g, a, b) is Gabor frame. By the Ron-Shen theory [11] and [7], Sec. 1.2, the triple (g, a, b) is a Gabor frame with bounds $A > 0$, $B < \infty$ if and only if $bAI \leq M_g(t)M_g^*(t) \leq bBI$, $a.e.\ t \in \mathbb{R}$, where I denote the identity operator on $\ell_2(\mathbb{Z})$ and $M_g(t)$ is the bi-infinite matrix defined by $M_g(t) = (g(t+na-\frac{k}{b}))_{k,n\in\mathbb{Z}}$, $a.e.\ t \in \mathbb{R}$, where k is the row index and n is the column index. The case when g is a charactristic function $\chi_{[0,c)}$ of an interval $[0, c)$ for some $c > 0$ has been studied in [10]. As a result of the Ron-Shen theorem, $(\chi_{[0,c)}, a, b)$ is a Gabor frame with frame bounds $A > 0$, $B < \infty$ if and only if $(\chi_{[0,bc)}, ba, 1)$ is a Gabor frame with frame bounds bA and bB. For any $x \in \mathbb{R}$ the largest integer less than or equal to x denote by $\lfloor x \rfloor$ and the smallest integer greater than x denote by $\lceil x \rceil$. There are some results that has proved by Janssen in [10].

Theorem 1.1. *Let a, c be as above and $N = 2, 3, ...$*
a) $(\chi_{[0,c)}, a, 1)$ is not a Gabor frame when $a > c$,
b) $(\chi_{[0,c)}, a, 1)$ is a Gabor frame when $a \leq c \leq 1$,
c) $(\chi_{[0,c)}, 1, 1)$ is a Gabor frame if and only if $c = 1$,
d) When $1 \leq c < 2$ we have that $(\chi_{[0,c)}, N^{-1}, 1)$ is a Gabor frame $\Leftrightarrow c \in [1, 2-N^{-1}]$. When $c \geq 2$ we have that

$$(\chi_{[0,c)}, N^{-1}, 1) \text{ is a Gabor frame} \Leftrightarrow c - \lfloor c \rfloor \in [N^{-1}, 1 - N^{-1}],$$

e) $(\chi_{[0,c)}, a, 1)$ is not a Gabor frame when $c = 2, 3, ...,$
f) $(\chi_{[0,c)}, a, 1)$ is a Gabor frame when $a \leq min(c - \lfloor c \rfloor, 1 - (c - \lfloor c \rfloor))$,
g) $(\chi_{[0,c)}, a, 1)$ is a Gabor frame when $1 \leq c < 2$ and $a \leq 1 - (c - \lfloor c \rfloor)$.

Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{H}$ if

$$f = \sum_{k=1}^{\infty} < f, f_k > g_k = \sum_{k=1}^{\infty} < f, g_k > f_k,\ \forall f \in \mathcal{H}.$$

Dual frames are important to reconstruction of vectors (or signals) in terms of the frame elements. Two Gabor frames (g, a, b) and (h, a, b) form dual frames for $L_2(\mathbb{R})$ if for all $f \in L_2(\mathbb{R})$

$$f = \sum_{m,n\in\mathbb{Z}} < f, E_{mb}T_{na}g > E_{mb}T_{na}h.$$

In this paper we find a simple duality condition for the case that g and h are charactristic functions on intervals $[0, c)$ and $[0, d)$, respectively.

2. **Dual frames**

The duality condition for a pair of Gabor systems (g, a, b) and (h, a, b) is presented by Janssen as follows [9]:

Lemma 2.1. *Two Bessel sequences (g, a, b) and (h, a, b) form dual frames for $L_2(\mathbb{R})$ if and only if $\sum_{k\in\mathbb{Z}} \overline{g(x - ka - \frac{n}{b})}h(x - ka) = b\delta_{n,0}$, $a.e.\ \ x \in [0, a]$.*

Every Gabor system $(\chi_{[0,c)}, a, b)$ is a Bessel sequence [10]. In this section we are going to find duals of a Bessel sequnce $(\chi_{[0,c)}, a, b)$ having the form $(\chi_{[0,d)}, a, b)$.

Lemma 2.2. *If $c > \frac{1}{b}$ or $d > \frac{1}{b}$, then $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are not dual frames for $L_2(\mathbb{R})$.*

Proof. We first assume that $c > \frac{1}{b}$.
If $d > c - \frac{1}{b}$, then for all $x \in [0, e) \subseteq [0, a)$, where $e = min\{a, c - \frac{1}{b}\}$ we have $o \leq x < e \leq c - \frac{1}{b} < d$ and hence $\chi_{[0,d)}(x) = \chi_{[0,c)}(x + \frac{1}{b}) = 1$. Therefore for $n = -1$ we have $\sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x - ka + \frac{1}{b})\chi_{[0,d)}(x - ka) \geq \chi_{[0,c)}(x + \frac{1}{b})\chi_{[0,d)}(x) = 1$.
If $d \leq c - \frac{1}{b}$, then for all $x \in [0, e) \subseteq [0, a)$, where $e = min\{a, d\}$ we have $0 \leq x < e \leq d \leq c - \frac{1}{b}$ and hence $\chi_{[0,d)}(x) = \chi_{[0,c)}(x + \frac{1}{b}) = 1$. Therefore for $n = -1$ we have $\sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x - ka + \frac{1}{b})\chi_{[0,d)}(x - ka) \geq \chi_{[0,c)}(x + \frac{1}{b})\chi_{[0,d)}(x) = 1$. Thus $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are not dual frames for $L_2(\mathbb{R})$ by Lemma 2.1. The proof of the case $d > \frac{1}{b}$ is similar. □

For all $x \in [0, a)$ we have $0 \leq x - ka < d$ if and only if $\lceil \frac{x-d}{a} \rceil \leq k \leq \lfloor \frac{x}{a} \rfloor = 0$. Thus $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are dual frames for $L_2(\mathbb{R})$ if and only if $\sum_{k=\lceil \frac{x-d}{a} \rceil}^{\lfloor \frac{x}{a} \rfloor} \chi_{[0,c)}(x - ka - n/b) = b\delta_{n,0}, \quad a.e. \quad x \in [0, a]$. A sufficient and necessary condition for duality of two Bessel sequences $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ is given in the next theorem.

Theorem 2.1. *Let a, b, c and d be positive numbers. Then two Bessel sequences $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are dual frames for $L_2(\mathbb{R})$ if and only if $b \in \mathbb{N}$, $c \leq \frac{1}{b}$, $d \leq \frac{1}{b}$ and $ab = min\{c, d\}$.*

Proof. We first assume that $d \leq c$ and we show that $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are dual frames for $L_2(\mathbb{R})$ if and only if $b \in \mathbb{N}$, $c \leq \frac{1}{b}$, $d \leq \frac{1}{b}$ and $ab = d$.
Let $b \in \mathbb{N}$, $c \leq \frac{1}{b}$, $d \leq \frac{1}{b}$, $ab = d$ and $x \in [0, a)$. Thus $0 \leq \frac{x}{a} < 1$ and hence $-b \leq \frac{x-d}{a} < -b + 1$. Thus $\lfloor \frac{x}{a} \rfloor = 0$ and $\lceil \frac{x-d}{a} \rceil = -b + 1$. Since $d \leq c$ we have

$$\sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x - ka)\chi_{[0,d)}(x - ka) = \sum_{k=\lceil \frac{x-d}{a} \rceil}^{\lfloor \frac{x}{a} \rfloor} \chi_{[0,c)}(x - ka)\chi_{[0,d)}(x - ka)$$

$$= \lfloor \frac{x}{a} \rfloor - \lceil \frac{x - d}{a} \rceil + 1 = b.$$

Also $d \leq c \leq \frac{1}{b}$ implies that if $0 \leq x - ka < d$, then $x - ka - \frac{n}{b} \leq x - ka - \frac{1}{b} \leq x - ka - d < o$ for $n > 0$ and $x - ka - \frac{n}{b} \geq x - ka + \frac{1}{b} \geq \frac{1}{b} \geq c$, for $n < 0$ and hence $x - ka - \frac{n}{b} \notin [0, c)$ for all $n \in \mathbb{Z} - \{0\}$. Thus for all $n \in \mathbb{Z} - \{0\}$

$$\sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x - ka - \frac{n}{b})\chi_{[0,d)}(x - ka) = 0.$$

So $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are dual frames for $L_2(\mathbb{R})$ by Lemma 2.1.
Convesely let $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are dual frames for $L_2(\mathbb{R})$. Now $b = \sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x - ka)\chi_{[0,d)}(x - ka) \in \mathbb{N}$ by Lemma 2.1 and $c, d \leq \frac{1}{b}$ by Lemma 2.2. If $ab < d$, then for all $x \in [0, e) \subseteq [0, a)$, where $e = min\{a, d - ab\}$ we have $\lfloor \frac{x}{a} \rfloor = 0$

and $\lceil \frac{x-d}{a} \rceil \leq -b$. Thus

$$\sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x-ka)\chi_{[0,d)}(x-ka) = \sum_{k=\lceil \frac{x-d}{a} \rceil}^{\lfloor \frac{x}{a} \rfloor} \chi_{[0,c)}(x-ka)\chi_{[0,d)}(x-ka)$$
$$= \lfloor \frac{x}{a} \rfloor - \lceil \frac{x-d}{a} \rceil + 1 \geq b+1 > b.$$

If $ab > d$, then for all $x \in [e,a) \subseteq [0,a)$, where $e = max\{0, d-ab+a\}$ we have $\lfloor \frac{x}{a} \rfloor = 0$ and $\lceil \frac{x-d}{a} \rceil \geq -b+2$. Thus

$$\sum_{k\in\mathbb{Z}} \chi_{[0,c)}(x-ka)\chi_{[0,d)}(x-ka) = \sum_{k=\lceil \frac{x-d}{a} \rceil}^{\lfloor \frac{x}{a} \rfloor} \chi_{[0,c)}(x-ka)\chi_{[0,d)}(x-ka)$$
$$= \lfloor \frac{x}{a} \rfloor - \lceil \frac{x-d}{a} \rceil + 1 \leq b-1 < b.$$

Hence $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are not dual frames for $L_2(\mathbb{R})$ by Lemma 2.1 and so $ab = d$.
Now assume that $c \leq d$. A similar argument shows that $(\chi_{[0,c)}, a, b)$ and $(\chi_{[0,d)}, a, b)$ are dual frames for $L_2(\mathbb{R})$ if and only if $b \in \mathbb{N}$, $c \leq \frac{1}{b}$, $d \leq \frac{1}{b}$ and $ab = c$. □

Corollary 2.1. *Let a, c and d be positive numbers. Then two Bessel sequences $(\chi_{[0,c)}, a, 1)$ and $(\chi_{[0,d)}, a, 1)$ are dual frames for $L_2(\mathbb{R})$ if and only if $c \leq 1$, $d \leq 1$ and $a = min\{c, d\}$.*

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