MATHEMATICAL MODELLING APPLICATION IN AERIAL COLLISION AVOIDANCE

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This paper presents a mathematical approach to a fundamental problem faced by the integration of unmanned aerial vehicles into airspace. The topic of collision avoidance is addressed here, formulated as a linear programming problem involving mixed integer constraints (MILP). The method proposed in this paper allows the optimization of the trajectory of an UAV without the risk of collision with fixed obstacles while minimizing the flight time of the mission. The mathematical details of the problem are discussed and several case studies are presented. The performance of the proposed approach is highlighted by progressively increasing the degree of difficulty of the obstacle avoidance problem.

Keywords: mathematical linear programming, trajectory planning, collision avoidance.

1. Introduction

Recently, unmanned aerial vehicle (UAV) has attracted the attention of researchers due to a wide range of applicability. Many unmanned aerial vehicle applications require the ability to navigate in urban or unknown areas where there are various obstacles (stationary or dynamic) of different types that may endanger the safety of vehicles and people. Because of the difficult missions in which aerial vehicles are involved, the control system design face multiple problems, such as the ability to plan trajectories in real time in order to avoid collisions with nearby objects.

The objective of this paper is to develop a mathematical approach of the obstacles and collisions avoidance problem in airspace. The trajectory generation in safe conditions involves defining the dynamic equations corresponding to the

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aerial vehicle motion and the positions of the existing obstacles. Thus, the model is mathematically obtained so as it satisfies the necessary Mixed-Integer Linear Programming requirements.

This paper focuses on the issue of collision avoidance between unmanned aerial vehicles and obstacles**.** Being a recent challenge of air traffic safety, a comprehensive analysis of the approaches is needed. Considering the wide range of typical missions of unmanned aerial vehicles, different types of methods have been developed. These approaches are distinguished by specific properties taking into account the limitations in the ability to be applied to a particular environment.

The conventional method of treating collision problems is based on geometrical ways that require certain information (location, velocity) about both obstacles and vehicles. Using the geometric approach, reference [1] deals with the collision problem determining the minimum distance between two aircraft. This is possible due to the ability to share data between aerial vehicles. If the risk is imminent, the vehicles follow the flight direction established to avoid the collision zone. Another approach which is analyzed in [2] is based on differential geometry to determine the maneuvers needed to avoid collision.

Other common methods involve the representation of the environment in a specific form: a set of cells or nodes, known as sampling based algorithms ([3], [4]). The feasible trajectory is generated by randomly searching within the space spanned by the optimization variables. Methods like Probabilistic Road Map (PRM), Rapidly Exploring Random Trees (RRT) and Voronoi are included in this category. The PRM method is a probabilistic approach of generating possible paths within a map based on free spaces and surfaces occupied by obstacles. As mentioned in [5], the algorithm is characterized by two phases: the learning phase and the query phase. In the preprocessing phase, the environment is stored in a suitable way for the following phase. This step involves assigning one node to the starting point and another node to the end point of the mission. The final trajectory represents the route between the initial position and the final one. The properties of Voronoi diagrams are detailed in [6]. These diagrams represent an important means that divides the geometric space so that the edge of each area is located at a maximum distance from all obstacles. RRT method aims to solve trajectory planning problems with the advantage of the possibility of considering multi-DOF cases. These approaches are presented in [3] and [4], describing the particular characteristics of each algorithm.

Another type of algorithms is called node-based methods which includes approaches like A-Star (A*) or Dijkstra's algorithm. The studies presented in [7] represent the fundamentals of developing the Dijkstra algorithm. It consists in determining the shortest trajectory using graphs in which the edges are considered known. A development of the previous method, the A-Star algorithm, detailed in

[8], [9], introduces a heuristic estimation for finding the path of the minimum expected cost.

Compared to the previous approaches, the mathematic model-based algorithms (linear algorithms or optimal control) allow modeling dynamic and kinematic constraints. These methods are based on obtaining the optimal solution of the cost function, which take into account the restrictions defined as equalities or inequalities. This kind of methods is commonly used in solving collision avoidance problems and path planning topics. For instance, the main objective of [10] is to analyze control strategies for UAV applications and paper [11] also uses optimal control for path planning problems. Linear planning algorithms include methods like MILP – Mixed Integer Linear Programming (treated in [12], [13], [14]) or BIP - Binary Linear Programming ([15]), which uses only binary variables.

The current paper is organized in several parts. The first section begins with a brief introduction of the problems faced by the integration of unmanned aerial vehicles into airspace. In the second part of the paper, the preliminaries and problem formulation are presented, describing the design approach. This section concerns on the mathematical linear programming including dynamics, constraints and cost function. In the last part, the proposed approach is illustrated by numerical examples. The topic of collision avoidance has been analyzed in other studies, but the case studies and mathematical implementation of the objectives are presented in this paper for the first time. The study ends with some concluding remarks.

2. Problem formulation

Considering the risks due to obstacles, the collision avoidance problem can be treated as a linear programming issue with mixed integer constraints called Mixed-Integer Linear Programming (MILP). This is an appropriate method for airspace collision avoidance problems. The main advantage of MILP is the efficiency in solving optimization problems. Compared to conventional approaches, the mathematical structure of MILP employs both real and integer variables. This enables the use of logical constraints such as obstacles and collisions or restricted flight areas avoidance rules, but also continuous constraints such as maximum speed.

The mathematical modelling of the collision avoidance problem is reduced to the linear definition of the objective function depending on the vector of variables, constants and constraints. Solving the trajectory planning problem involves minimizing a quadratic cost function whose variables satisfy the dynamic equations. The cost function has the following form:

$$\min_{x,u} J = \min_{x,u} \int_0^\infty (x^T Q x + u^T R u) dt \tag{1}$$

where $\dot{x} = Ax + Bu$, x - state vector, u - control inputs. For the general case, it is considered a number of aircraft N whose mission is carried out over a period of time ΔT. The position of the aircraft i at the time step t is described by (x_{ti}, y_{ti}) and the speed by $(v_{x_{ti}}, v_{y_{ti}})$ creating the state vector. It is assumed that each aircraft is operated by the forces of the two directions $(f_{x_{ti}}, f_{y_{ti}})$ which form the input vector.

The linear form of the discretized dynamics for the whole system in the case of a number N of air vehicles corresponding to a number T of time steps can be written as:

$$x_{(t+1)i} = A_i x_{ti} + B_i f_{ti}, \forall i \in [1 \dots N], \forall t \in [0 \dots T] \tag{2}$$

for which the initial conditions are known. In order to be able to approach the problem in the proposed way, it is necessary to define the constraints that influence the trajectory generation. To define the restrictions in terms of speeds and forces, a term $\boldsymbol{M}$ is introduced, a large enough value for a good approximation [13].

$$v_{x_{ti}} \sin\left(\frac{2\pi m}{M}\right) + v_{y_{ti}} \cos\left(\frac{2\pi m}{M}\right) \leq v_{max}, \forall i \in [1 \dots N], \forall t \in [0 \dots T-1], \forall m \in [1 \dots M] \tag{3}$$

$$f_{x_{ti}} \sin\left(\frac{2\pi m}{M}\right) + f_{y_{ti}} \cos\left(\frac{2\pi m}{M}\right) \leq f_{max}, \forall i \in [1 \dots N], \forall t \in [0 \dots T-1], \forall m \in [1 \dots M] \tag{4}$$

It is desired that the aircraft reach the final position in a shorter time than $\boldsymbol{T}$. As it is suggested in [13], the binary variables are used to define the corresponding constraints. The $\boldsymbol{b_{ti}}$ variables have the value equal to 1 if the aircraft $\boldsymbol{i}$ reaches its destination on time step $\boldsymbol{t}$ and a null value otherwise. Thus, the set of restrictions for formulating the mathematical model is expressed as follows:

$$\forall \boldsymbol{i} \in [\boldsymbol{1} \dots \boldsymbol{N}], \forall \boldsymbol{t} \in [\boldsymbol{1} \dots \boldsymbol{T}] \quad \begin{aligned} &\boldsymbol{x_{ti} - x_{Fi} \leq R(1 - b_{ti})} \\ &\textbf{and } \boldsymbol{x_{ti} - x_{Fi} \geq -R(1 - b_{ti})} \\ &\textbf{and } \boldsymbol{y_{ti} - y_{Fi} \leq R(1 - b_{ti})} \\ &\textbf{and } \boldsymbol{y_{ti} - y_{Fi} \geq -R(1 - b_{ti})} \end{aligned} \tag{5}$$

$$\forall i \in [1 \dots N] \quad \sum_{t=1}^{T} b_{ti} = 1 \tag{6}$$

where R is a positive value higher than any of the state variables.

It can be noticed that if $b_{ti} = 1$, equation (5) forces the aircraft to be in the final desired position. Equation (6) assumes that each aircraft reaches the final position at a certain time step. The obstacles are defined by minimum and maximum coordinates, more precisely an obstacle is characterized by the points $(x_{\min}, y_{\min})$ and $(x_{\max}, y_{\max})$. As explained in reference [16], at each time step, the position of the vehicle has to be outside the perimeter delimited by the above coordinates.

$$\forall t \in [1 \ldots T], \quad \begin{array}{c} x_t \leq x_{min} \\ or\ x_t \geq x_{max} \\ or\ y_t \leq y_{min} \\ or\ y_t \geq y_{max} \end{array} \tag{7}$$

In order to be able to solve the problem caused by the presence of obstacles, the required conditions must be simultaneously satisfied, which implies the need of type "and" restrictions. To convert the type "or" restrictions into "and" restrictions, another binary constraints c_{tk} are introduced:

$$\forall t \in [1 \ldots T] \quad \begin{array}{c} x_t \leq x_{min} + Mc_{t1} \\ and\ x_t \geq x_{max} - Mc_{t2} \\ and\ y_t \leq y_{min} + Mc_{t3} \\ and\ y_t \geq y_{max} - Mc_{t4} \\ and\ \sum_{k=1}^{4} c_{tk} \leq 3 \end{array} \tag{8}$$

The solution of minimizing the flight time is determined by minimizing the sum of the final times for each vehicle.

$$\min_{x,u,b,c} J = \sum_{t=1}^{T} \sum_{i=1}^{N} t b_{ti} \tag{9}$$

This form of the cost function is not sufficient because there can be multiple solutions determined at the same time step, fact caused by the discretization of the time span. Furthermore, the states and commands for the time steps do not influence the cost function, but they have a significant impact on the time solution [13]. To solve this problem, reference [13] proposes the introduction of a low value term, defining the cost function as follows:

$$\min_{x,u,b,c} J = \sum_{i=1}^{N} \left(\sum_{t=1}^{T} t b_{ti} + \varepsilon \sum_{t=0}^{T-1} \left(|f_{x_{ti}}| + |f_{y_{ti}}| \right) \right) \tag{10}$$

where ε is a small positive value. The introduction of the penalty in the cost function allows searching for the solution only in the relevant regions. Thereby, the minimization algorithm requires a shorter time. In this way, the efficiency of the algorithm is highlighted by the unique optimal solution.

3. Numerical simulation

To define the mathematical model, it is necessary to establish certain characteristics. Both the initial and the final position of the vehicle trajectory are established. The positions, dimensions and number of obstacles in the airspace are also known. The aerial vehicle starts from the initial position $\boldsymbol{A}(\boldsymbol{x_0}, \boldsymbol{y_0})$ and reaches the desired destination $\boldsymbol{B}(\boldsymbol{x_F}, \boldsymbol{y_F})$ by following the optimal path while avoiding obstacles. The final trajectory generated by the algorithm minimizes the necessary flight time providing the minimum distance between the two desired positions.

For the case study presented in this paper, the performances of the proposed optimization approach are highlighted by progressively increasing the degree of difficulty of the obstacle avoidance problem. Each numerical simulation analyzes different airspace configurations.

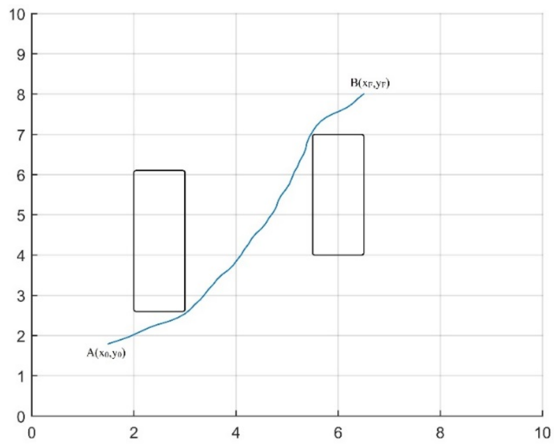

Fig. 1. The optimal trajectory in the presence of two obstacles

The first case considers the presence of two obstacles into the airspace. The proposed approach allows the generation of an optimal trajectory of the aerial vehicle so that there is no risk of collisions, as seen in Fig. 1.

A different arrangement of the two obstacles is represented in Fig. 2. Furthermore, the change of the initial and the final position highlights the capacity of the algorithm to generate the trajectory in safe conditions.

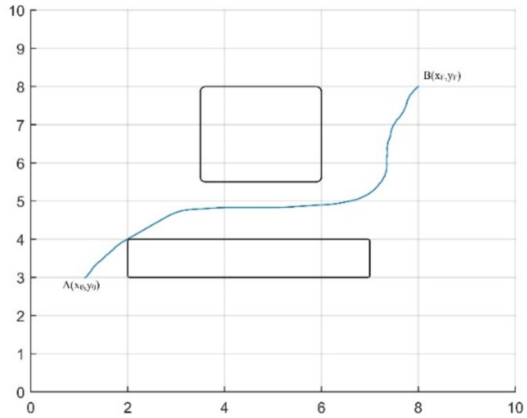

Fig. 2. The optimal trajectory for different arrangement of two obstacles

The second case involves the extension of the number of present obstacles (Fig. 3). The numerical simulations in the following figures show the determined trajectory of the aerial vehicle using the proposed approach, changing the position of the obstacles for Fig. 4. It can be seen that the position of the obstacles does not represent a risk in achieving the desired performance.

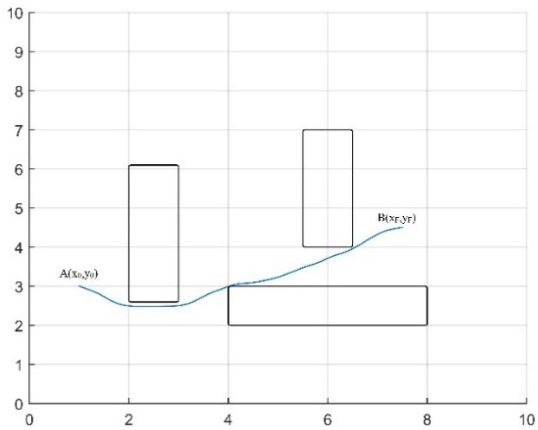

Fig. 3. The optimal trajectory in the presence of three obstacles

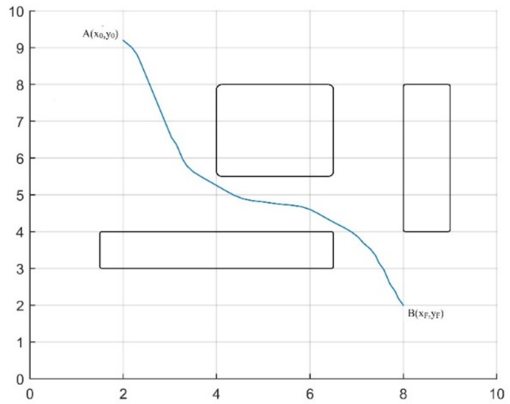

Fig. 4. The optimal trajectory in a different environment consisting of three obstacles

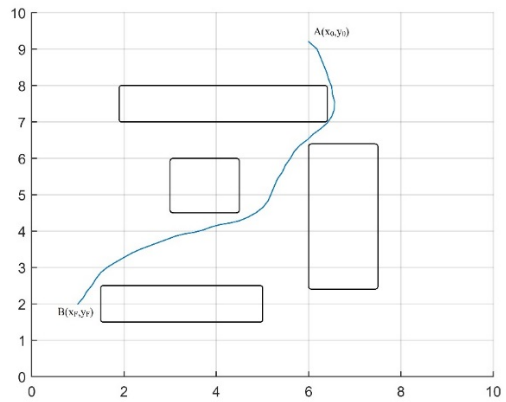

Fig. 5. The optimal trajectory in the presence of four obstacles

Compared to previous cases, the generation of the flight path faces the avoidance of a greater number of obstacles. The presence of a new obstacle in the airspace does not affect the capacity of the proposed method to obtain the route of the desired mission. Fig. 5 illustrates that the obtained trajectory takes into account the position of the obstacles and successfully avoids any collision. For the simulation in Fig. 6, both the initial and the final position are modified. As for the previous cases, the proposed approach proves the ability to determine a safe flight path. In this configuration, it is shown that the obstacles presented in the airspace do not endanger the performances of the mission.

The purpose of the case studies presented here was to illustrate the proposed method on several practical situations. As it is currently implemented, the algorithm is more suitable for offline computation of optimal trajectories since it employs a high number of points along the trajectory and the obstacles are known a priori. However, in many real time applications, the position and size of the obstacles must be detected online, creating the requirement for the vehicle to be able to rapidly and efficiently adapt its trajectory to avoid the collision. For this reason, the online implementation of the algorithm must be very efficient in terms of computational time. Implementation of the proposed method for real time applications will be the purpose of future work.

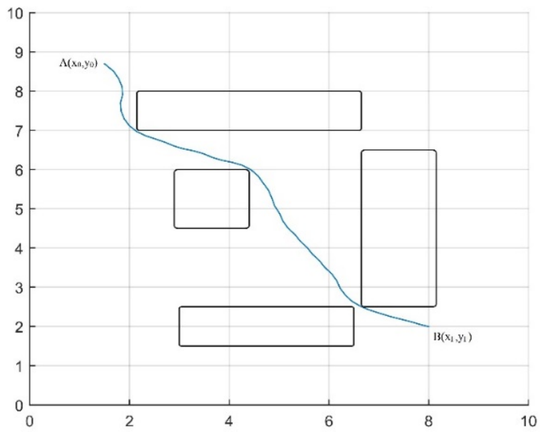

Fig. 6. The optimal trajectory for the case of changing the two desired positions

4. Conclusions

The present work focuses on collision avoidance problem between unmanned aerial vehicles and obstacles, which represents a major challenge for the integration of UAVs in civil airspace. Based on the analysis of current progress in the field of aerial vehicles, this study uses Mixed-Integer Linear Programming for planning and optimizing the flight path.

In order to be able to solve the problem caused by the presence of obstacles, the dynamic equations and the conditions imposed to avoid collisions are translated into a form suitable for the structure of MILP. This is realized by discretizing the continuous constraints and equations of motion, resulting in a parameter optimization problem.

This paper includes the mathematical modelling of the approach, which allowed obtaining numerical simulations in which different situations were analyzed. The case study highlights the generation of flight trajectories in the presence of a different number and types of obstacles in the airspace. In each analyzed situation, it can be seen that the trajectory of the aerial vehicle successfully avoids obstacles without generating other conflicts.

Although the MILP approach applied to an aerial vehicle motion in an obstacle environment is optimal, due to the generation of a conflict-free trajectory, it requires an increase in computing power with the choice of several points along the trajectory and the introduction of a larger number of obstacles. The analysis of situations that include several vehicles represents a topic for further research.

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