INERTIAL EXTRAGRADIENT ALGORITHMS FOR SOLVING GENERALIZED EQUILIBRIA PROBLEMS

Lu-Chuan Ceng[1], Zhangsong Yao[2], Tzu-Chien Yin[3]

In this paper, we present an inertial extragradient algorithm for solving a generalized equilibrium problem with constraints of a split fixed point problem and a variational inequality problem, in which the process exploits the contractiveness of one operator at the upper-level problem and the pseudomonotonicity of another mapping at the lower level. Strong convergence result of the proposed process is established under some mild assumptions.

Keywords: inertial extragradient algorithm, equilibria problems, split problem, fixed point, pseudomonotone variational inequality.

MSC2020: 65Y05, 65K15, 68W10, 47H05, 47H10.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\Psi : C\times C \to \mathbf{R}$ be a bifunction. Recall that the equilibrium problem (EP) is to find $x^* \in C$ such that

$$\Psi(x^*, y) \geq 0,\ \forall y \in C. \tag{1}$$

The solution set of (1) is denoted by EP(Ψ). To solve (1), The following conditions need to be known in advance: (H1): $\Psi(y, y) = 0, \forall y \in C$; (H2): $\Psi(x, y) + \Psi(y, x) \leq 0, \forall x, y \in C$; (H3): $\lim_{\lambda\to 0^+} \Psi((1-\lambda)y + \lambda x, z) \leq \Psi(y, z), \forall x, y, z \in C$; (H4) $z \mapsto \Psi(y, z)$ is convex and lower semicontinuous (l.s.c.) for every $y \in C$.

In (1), if $\Psi(x, y) = \langle Ax, y - x\rangle, \forall x, y \in C$, then we have the well known variational inequality problem (VIP) which is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0,\ \forall y \in C. \tag{2}$$

The solution set of the VIP is denoted by VI(C, A). An important method to solve (1) and (2) is extragradient method introduced by Korpelevich [9]. Consequently, many algorithms and techniques were designed for finding the solution set of (1) and (2), see [13, 15, 16, 18–23].

Now, we consider the following a system of generalized equilibrium problems (SGEP) ([2]) which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \Psi_1(x^*, x) + \langle \varphi_1 y^*, x - x^*\rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^*\rangle \geq 0,\ \forall x \in C, \\ \Psi_2(y^*, y) + \langle \varphi_2 x^*, y - y^*\rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^*\rangle \geq 0,\ \forall y \in C, \end{cases} \tag{3}$$

where $\Psi_1, \Psi_2 : C \times C \to \mathbf{R}$ are two bifunctions, $\varphi_1, \varphi_2 : \mathcal{H} \to \mathcal{H}$ are two mappings and μ_1, μ_2 are two positive constants.

[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China, email: zenglc@shnu.edu.cn

[2]Corresponding author. School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China, email: yaozhsong@163.com

[3]Corresponding author. Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, e-mail: yintzuchien@hotmail.com

Setting $\Psi_1 = \Psi_2 = 0$, we have the following general system of variational inequalities (GSVI) ([3]) which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 \varphi_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \ \forall x \in C, \\ \langle \mu_2 \varphi_2 x^* + y^* - x^*, y - y^* \rangle \geq 0, \ \forall y \in C. \end{cases} \tag{4}$$

For any $x \in \mathcal{H}$, define $T_r^{\Psi}(x) := \{y \in C : \Psi(y,z) + \frac{1}{r}\langle z - y, y - x\rangle \geq 0, \forall z \in C\}$ and set $G = T_{\mu_1}^{\Psi_1}(I - \mu_1\varphi_1)T_{\mu_2}^{\Psi_2}(I - \mu_2\varphi_2)$. Let $\varphi_1, \varphi_2 : \mathcal{H} \to \mathcal{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. If (H1)-(H4) hold, then (x^*, y^*) is a solution of SGEP (3) ([4]) if and only if $x^* \in \mathrm{Fix}(G) := \{x \in \mathcal{H} : G(x) = x\}$ where $y^* = T_{\mu_2}^{\Psi_2}(I - \mu_2\varphi_2)x^*$.

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{W} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear mapping and $A, F : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear operators. Recall that the bilevel split variational inequality problem (BSVIP) ([2]) is to find $z^* \in \varOmega$ such that

$$\langle Fz^*, z - z^* \rangle \geq 0, \ \forall z \in \varOmega, \tag{5}$$

where $\varOmega := \{z \in \mathrm{VI}(C, A) : \mathcal{W}z \in \mathrm{VI}(Q, B)\}$ is the solution set of the split variational inequality problem (SVIP) ([5]) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \ \forall x \in C, \tag{6}$$

and $y^* = \mathcal{W}x^* \in Q$ such that

$$\langle By^*, y - y^* \rangle \geq 0, \ \forall y \in Q. \tag{7}$$

To solve SVIP, Censor et al. [5] proposed the following iterative algorithm: for any initial $x_1 \in \mathcal{H}_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I - \lambda A)(x_n + \gamma \mathcal{W}^*(P_Q(I - \lambda B) - I)\mathcal{W}x_n), \ \forall n \geq 1. \tag{8}$$

Consequently, the split problems have been investigated in the literature, see [7, 8, 10, 12, 17, 24, 25].

Very recently, Abuchu et al. [1] consider a bilevel split quasimonotone variational inequality problem (BSQVIP) ([1]): find $z^* \in \varOmega := \{z \in \mathrm{VI}(C, A) : \mathcal{W}z \in \mathrm{Fix}(S)\}$ such that

$$\langle Fz^*, z - z^* \rangle \geq 0, \ \forall z \in \varOmega \tag{9}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_1$ is quasimonotone and L-Lipschitz continuous, $F : \mathcal{H}_1 \to \mathcal{H}_1$ is κ-Lipschitzian and η-strongly monotone and $S : \mathcal{H}_2 \to \mathcal{H}_2$ is τ-demimetric mapping with $\tau \in (-\infty, 1)$. The authors [1] proposed a modified relaxed inertial subgradient extragradient iterative algorithm for solving the BSQVIP (9). Under suitable conditions, they proved the strong convergence of the proposed algorithm to a unique solution of the BSQVIP (9).

In this paper, we investigate the following SGEP with a bilevel split fixed point problem (BSFPP) and VIP constraint which is formulated as:

$$\text{find } z^* \in \varXi \text{ such that } P_{\varXi}(I - f)z^* = z^*, \tag{10}$$

where $\varXi := \mathrm{Fix}(G) \cap \varOmega \cap \mathrm{VI}(C, A)$ and $\varOmega = \{z \in \mathrm{Fix}(\overline{S}) : \mathcal{W}z \in \mathrm{Fix}(S)\}$. We propose hybrid inertial subgradient extragradient rule with line-search process for finding a solution of (10) in real Hilbert spaces, where the rule exploits the contractiveness of the operator f at the upper-level problem and the pseudomonotonicity of the mapping A at the lower level. The strong convergence result for the proposed algorithm is established under some mild restrictions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. For each $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|, \forall y \in C$. It is well known that P_C has the following properties: (i) $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle, \forall x, y \in \mathcal{H}$; (ii) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \le 0, \forall x \in \mathcal{H}, y \in C$; (iii) $\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in \mathcal{H}, y \in C$.

Recall that a mapping $S : C \to \mathcal{H}$ is called

(1) L-Lipschitz continuous if $\exists L > 0$ such that $\|Sx - Sy\| \le L\|x - y\|, \forall x, y \in C$.
(2) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Sx - Sy, x - y\rangle \ge \alpha\|x - y\|^2, \forall x, y \in C$.
(3) monotone if $\langle Sx - Sy, x - y\rangle \ge 0, \forall x, y \in C$.
(4) pseudomonotone if $\langle Sx, y - x\rangle \ge 0 \Rightarrow \langle Sy, y - x\rangle \ge 0, \forall x, y \in C$.
(5) quasimonotone if $\langle Sx, y - x\rangle > 0 \Rightarrow \langle Sy, y - x\rangle \ge 0, \forall x, y \in C$.
(6) η-strictly pseudocontractive if $\exists\ \eta \in [0, 1)$ such that $\|Sx - Sy\|^2 \le \|x - y\|^2 + \eta\|(I - S)x - (I - S)y\|^2, \forall x, y \in C$.
(7) τ-demicontractive if $\exists \tau \in [0, 1)$ such that $\|Sx - y\|^2 \le \|x - y\|^2 + \tau\|x - Sx\|^2, \forall x \in C,\ y \in \mathrm{Fix}(S) \ne \emptyset$.
(8) τ-demimetric if $\exists \tau \in (-\infty, 1)$ such that $\langle x - Sx, x - y\rangle \ge \frac{1-\tau}{2}\|x - Sx\|^2, \forall x \in C,\ y \in \mathrm{Fix}(S) \ne \emptyset$.
(9) sequentially weakly continuous if $\forall\{x_n\} \subset C$, the relation holds: $x_n \rightharpoonup x \Rightarrow Sx_n \rightharpoonup Sx$.

If $S : C \to C$ is an η-strictly pseudocontractive mapping, then (i) for all $x, y \in C$, $\|\gamma(x - y) + \delta(Sx - Sy)\| \le (\gamma + \delta)\|x - y\|$, where $\gamma \ge 0, \delta \ge 0$ and $(\gamma + \delta)\eta \le \gamma$; (ii) $\|Sx - Sy\| \le \frac{1+\eta}{1-\eta}\|x - y\|, \forall x, y \in C$.

If $B : \mathcal{H} \to \mathcal{H}$ is a ζ-inverse-strongly monotone mapping, then $\|(I - \mu B)y - (I - \mu B)z\|^2 \le \|y - z\|^2 - \mu(2\zeta - \mu)\|By - Bz\|^2$. In particular, if $0 \le \mu \le 2\zeta$, then $I - \mu B$ is nonexpansive.

Lemma 2.1 ([6]). *Assume that $A : C \to \mathcal{H}$ is pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, v - u\rangle \ge 0, \forall v \in C$ if and only if $\langle Av, v - u\rangle \ge 0, \forall v \in C$.*

Lemma 2.2 ([14]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n\gamma_n, \forall n \ge 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$, and (ii) $\limsup_{n\to\infty} \gamma_n \le 0$ or $\sum_{n=1}^{\infty} |\lambda_n\gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3 ([11]). *Let $\{\Phi_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\Phi_{m_k}\} \subset \{\Phi_m\}$ such that $\Phi_{m_k} < \Phi_{m_k+1}, \forall k \ge 1$. Let the sequence $\{\psi(m)\}_{m\ge m_0}$ of integers be formulated as $\psi(m) = \max\{k \le m : \Phi_k < \Phi_{k+1}\}$ with integer $m_0 \ge 1$ satisfying $\{k \le m_0 : \Phi_k < \Phi_{k+1}\} \ne \emptyset$. Then, (i) $\psi(m_0) \le \psi(m_0 + 1) \le \cdots$ and $\psi(m) \to \infty$; (ii) $\Phi_{\psi(m)} \le \Phi_{\psi(m)+1}$ and $\Phi_m \le \Phi_{\psi(m)+1}, \forall m \ge m_0$.*

3. Main results

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and C be a nonempty, closed and convex subset of $\mathcal{H}_1$. Suppose that the following conditions hold:

(C1): $\Psi_1, \Psi_2 : C \times C \to \mathbf{R}$ are two bifunctions satisfying (H1)-(H4) and $\mathcal{W} : \mathcal{H}_1 \to \mathcal{H}_2$ is a non-zero bounded linear operator with the adjoint $\mathcal{W}^*$.
(C2): $f : \mathcal{H}_1 \to \mathcal{H}_1$ is a δ-contraction, $\overline{S} : \mathcal{H}_1 \to \mathcal{H}_1$ is an η-strictly pseudocontractive mapping and $S : \mathcal{H}_2 \to \mathcal{H}_2$ is a $\tau(\in (-\infty, 1))$-demimetric mapping such that $I - S$ is demiclosed at zero.
(C3): $A : \mathcal{H}_1 \to \mathcal{H}_1$ is a pseudomonotone and L-Lipschitz continuous mapping satisfying the condition: $\|Ax\| \le \liminf_{n\to\infty} \|Ax_n\|$ for each $\{x_n\} \subset C$ with $x_n \rightharpoonup x$, $\varphi_1, \varphi_2 : \mathcal{H}_1 \to \mathcal{H}_1$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

(C4): $\varXi := \mathrm{Fix}(G) \cap \varOmega \cap \mathrm{VI}(C, A) \neq \emptyset$, where $\varOmega := \{z \in \mathrm{Fix}(\overline{S}) : \mathcal{W}z \in \mathrm{Fix}(S)\}$ and $G := T_{\mu_1}^{\Psi_1}(I - \mu_1\varphi_1)T_{\mu_2}^{\Psi_2}(I - \mu_2\varphi_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$.

Let $\{\varepsilon_n\} \subset [0, 1]$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\} \subset (0, 1)$ satisfying

(i) $\sup_{n\geq 1} \frac{\varepsilon_n}{\alpha_n} < \infty$, $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)\eta \leq \gamma_n, \forall n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \delta_n > 0$.

Algorithm 3.1. *Let $\lambda > 0$, $\ell \in (0, 1)$, $\sigma \geq 0$, $\mu \in (0, 1)$ and $x_1, x_0 \in \mathcal{H}_1$ be arbitrary. Calculate x_{n+1} as follows:*

Step 1. Set $w_n = x_n + \varepsilon_n(x_n - x_{n-1})$ and calculate $y_n = P_C(w_n - \xi_n Aw_n)$, where ξ_n is chosen to be the largest $\xi \in \{\lambda, \lambda\ell, \lambda\ell^2, ...\}$ satisfying

$$\xi\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|. \tag{11}$$

Step 2. Construct the half-space $C_n := \{y \in \mathcal{H}_1 : \langle w_n - \xi_n Aw_n - y_n, y_n - y\rangle \geq 0\}$, and compute $v_n = P_{C_n}(w_n - \xi_n Ay_n)$.

Step 3. Calculate $z_n = \alpha_n f(x_n) + (1 - \alpha_n)[v_n - \sigma_n\mathcal{W}^*(I - S)\mathcal{W}v_n]$, where for any fixed $\epsilon > 0$, σ_n is chosen to be the bounded sequence satisfying

$$0 < \epsilon \leq \sigma_n \leq \frac{(1 - \tau)\|\mathcal{W}v_n - S\mathcal{W}v_n\|^2}{\|\mathcal{W}^*(\mathcal{W}v_n - S\mathcal{W}v_n)\|^2} - \epsilon \quad \text{if } \mathcal{W}v_n \neq S\mathcal{W}v_n, \tag{12}$$

otherwise set $\sigma_n = \sigma \geq 0$.

Step 4. Calculate

$$x_{n+1} = \beta_n x_n + \gamma_n Gz_n + \delta_n\overline{S}Gz_n. \tag{13}$$

Set $n := n + 1$ and return to Step 1.

Remark 3.1. *The line-search process* (11) *is well defined and* $\min\{\lambda, \frac{\mu\ell}{L}\} \leq \xi_n \leq \lambda$.

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, the stepsize σ_n formulated in* (12) *is well-defined.*

Proof. It is sufficient to show $\|\mathcal{W}^*(\mathcal{W}v_n - S\mathcal{W}v_n)\|^2 \neq 0$. Pick a $q \in \varXi$ arbitrarily. Since S is τ-demimetric mapping, one gets

$$\|v_n - q\|\|\mathcal{W}^*(\mathcal{W}v_n - S\mathcal{W}v_n)\| \geq \langle v_n - q, \mathcal{W}^*(\mathcal{W}v_n - S\mathcal{W}v_n)\rangle \geq \frac{1 - \tau}{2}\|\mathcal{W}v_n - S\mathcal{W}v_n\|^2. \tag{14}$$

If $\mathcal{W}v_n \neq S\mathcal{W}v_n$, one has $\|\mathcal{W}v_n - S\mathcal{W}v_n\|^2 > 0$. Therefore, $\|\mathcal{W}^*(\mathcal{W}v_n - S\mathcal{W}v_n)\|^2 > 0$. □

Lemma 3.2. *Let $\{w_n\}$, $\{y_n\}$, $\{v_n\}$ be the sequences constructed in Algorithm 3.1. Then,*

$$\|v_n - q\|^2 \leq \|w_n - q\|^2 - (1 - \mu)\|y_n - w_n\|^2 - (1 - \mu)\|y_n - v_n\|^2, \forall q \in \varXi. \tag{15}$$

Proof. First, for each $q \in \varXi \subset C \subset C_n$, one has

$$\|v_n - q\|^2 \leq \frac{1}{2}\|v_n - q\|^2 + \frac{1}{2}\|w_n - q\|^2 - \frac{1}{2}\|v_n - w_n\|^2 - \langle v_n - q, \xi_n Ay_n\rangle.$$

So, it follows that $\|v_n - q\|^2 \leq \|w_n - q\|^2 - \|v_n - w_n\|^2 - 2\langle v_n - q, \xi_n Ay_n\rangle$, which together with (11) and the pseudomonotonicity of A, implies that $\langle Ay_n, y_n - q\rangle \geq 0$ and

$$\begin{aligned}\|v_n - q\|^2 &\leq \|w_n - q\|^2 - \|v_n - w_n\|^2 + 2\xi_n(\langle Ay_n, q - y_n\rangle + \langle Ay_n, y_n - v_n\rangle)\\ &\leq \|w_n - q\|^2 - \|v_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \xi_n Ay_n - y_n, v_n - y_n\rangle.\end{aligned}$$

Since $v_n = P_{C_n}(w_n - \xi_n Ay_n)$ with $C_n := \{y \in \mathcal{H}_1 : \langle w_n - \xi_n Aw_n - y_n, y_n - y\rangle \geq 0\}$, we have $\langle w_n - \xi_n Aw_n - y_n, y_n - v_n\rangle \geq 0$, which together with (11), implies that

$$\begin{aligned}&2\langle w_n - \xi_n Ay_n - y_n, v_n - y_n\rangle = 2\langle w_n - \xi_n Aw_n - y_n, v_n - y_n\rangle\\ &+ 2\xi_n\langle Aw_n - Ay_n, v_n - y_n\rangle \leq 2\mu\|w_n - y_n\|\|v_n - y_n\| \leq \mu(\|y_n - w_n\|^2 + \|y_n - v_n\|^2).\end{aligned} \tag{16}$$

Consequently, we obtain the desired result. $\square$

Lemma 3.3. *Let* $\{x_n\}$ *be the sequence constructed in Algorithm 3.1. Then,* $\{x_n\}$ *is bounded provided* $\sup_{n\geq 1}\|x_n - x_{n-1}\| < \infty$.

Proof. Set $z_n = \alpha_n f(x_n) + (1-\alpha_n)u_n$ where $u_n := v_n - \sigma_n \mathcal{W}^*(I-S)\mathcal{W}v_n, \forall n \geq 1$. Since $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [a,b] \subset (0,1)$. It is clear that $P_{\Xi} \circ f$ is a contraction with the unique fixed point $z^* \in \mathcal{H}_1$. So, there exists the unique solution $z^* \in \Xi = \mathrm{Fix}(G) \cap \Omega \cap \mathrm{VI}(C,A)$ to the VIP

$$\langle (I-f)z^*, y - z^*\rangle \geq 0, \forall y \in \Xi. \tag{17}$$

Hence, there exists the unique solution $z^* \in \Xi$ to the SGEP (10) with the BSFPP and VIP constraint. Note that $\|w_n - z^*\| \leq \|x_n - z^*\| + \alpha_n \cdot \frac{\varepsilon_n}{\alpha_n}\|x_n - x_{n-1}\|$. From $\sup_{n\geq 1}\frac{\varepsilon_n}{\alpha_n} < \infty$ and $\sup_{n\geq 1}\|x_n - x_{n-1}\| < \infty$, it follows that $\{\frac{\varepsilon_n}{\alpha_n}\|x_n - x_{n-1}\|\}$ is bounded. Thus, $\exists M_1 > 0$ s.t. $\frac{\varepsilon_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1, \forall n \geq 1$. Hence,

$$\|w_n - z^*\| \leq \|x_n - z^*\| + \alpha_n M_1, \forall n \geq 1. \tag{18}$$

Moreover,

$$\|z_n - z^*\| \leq \alpha_n\delta\|x_n - z^*\| + (1-\alpha_n)\|u_n - z^*\| + \alpha_n\|f(z^*) - z^*\|. \tag{19}$$

Observe that $\|u_n - z^*\|^2 = \|v_n - z^*\|^2 - 2\sigma_n\langle \mathcal{W}(v_n - z^*), (I-S)\mathcal{W}v_n\rangle + \sigma_n^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2$. Since the operator S is τ-demimetric, we have

$$\|u_n - z^*\|^2 \leq \|v_n - z^*\|^2 + \sigma_n[\sigma_n\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 - (1-\tau)\|(I-S)\mathcal{W}v_n\|^2]. \tag{20}$$

Taking into account (12), we get $\sigma_n + \epsilon \leq \frac{(1-\tau)\|\mathcal{W}v_n - S\mathcal{W}v_n\|^2}{\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2}$ which implies that

$$\sigma_n(\sigma_n\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 - (1-\tau)\|\mathcal{W}v_n - S\mathcal{W}v_n\|^2) \leq -\sigma_n\epsilon\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2. \tag{21}$$

Using $0 < \epsilon \leq \sigma_n$ in (12), we have that $-\epsilon^2 \geq -\sigma_n\epsilon$ and hence

$$-\sigma_n\epsilon\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 \leq -\epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2. \tag{22}$$

Combining (20), (21) and (22), we obtain

$$\|u_n - z^*\|^2 \leq \|v_n - z^*\|^2 - \sigma_n\epsilon\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 \leq \|v_n - z^*\|^2. \tag{23}$$

In addition, by Lemma 3.2, we get

$$\|v_n - z^*\|^2 \leq \|w_n - z^*\|^2 - (1-\mu)\|y_n - w_n\|^2 - (1-\mu)\|y_n - v_n\|^2 \leq \|w_n - z^*\|^2. \tag{24}$$

Combining (18), (23) and (24), we obtain

$$\|u_n - z^*\| \leq \|v_n - z^*\| \leq \|w_n - z^*\| \leq \|x_n - z^*\| + \alpha_n M_1, \forall n \geq 1. \tag{25}$$

From (19) and (25), we have

$$\begin{aligned}
\|x_{n+1} - z^*\| &= \|\beta_n(x_n - z^*) + \gamma_n(Gz_n - z^*) + \delta_n\overline{S}(Gz_n - z^*)\| \\
&\leq \beta_n\|x_n - z^*\| + (1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(Gz_n - z^*) + \delta_n\overline{S}(Gz_n - z^*)]\| \\
&\leq [1 - \alpha_n(1-\beta_n)(1-\delta)]\|x_n - z^*\| + \alpha_n(1-\beta_n)[M_1 + \|f(z^*) - z^*\|] \\
&\leq \max\{\|x_n - z^*\|, \frac{M_1 + \|f(z^*) - z^*\|}{1-\delta}\}.
\end{aligned} \tag{26}$$

By induction, we obtain $\|x_n - z^*\| \leq \max\{\|x_1 - z^*\|, \frac{M_1 + \|f(z^*) - z^*\|}{1-\delta}\}, \forall n \geq 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Gz_n\}$, $\{\overline{S}Gz_n\}$. $\square$

Lemma 3.4. *Let $\{v_n\}, \{x_n\}, \{z_n\}$ be the sequences generated by Algorithm 3.1. Suppose that $x_n - x_{n+1} \to 0$, $v_n - z_n \to 0$ and $z_n - Gz_n \to 0$. Then $\omega_w(\{x_n\}) \subset \Xi$ provided $\sup_{n\geq 1} \|x_n - x_{n-1}\| < \infty$ where $\omega_w(\{x_n\}) = \{z \in \mathcal{H}_1 : x_{n_k} \rightharpoonup z$ for some $\{x_{n_k}\} \subset \{x_n\}\}$.*

Proof. Observe that $\|w_n - x_n\| = \varepsilon_n \|x_n - x_{n-1}\| \leq \alpha_n M_1 \to 0 (n \to \infty)$. Take a fixed $z \in \omega_w(\{x_n\})$ arbitrarily. Then, $\exists\{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z \in \mathcal{H}_1$. Thanks to $w_n - x_n \to 0$, we know that $\exists\{w_{n_k}\} \subset \{w_n\}$ s.t. $w_{n_k} \rightharpoonup z \in \mathcal{H}_1$. Next we show that

$$\begin{aligned}\|z_n - z^*\|^2 \leq \alpha_n\delta\|x_n - z^*\|^2 + (1-\alpha_n)\|w_n - z^*\|^2 - (1-\alpha_n)(1-\mu)[\|y_n - w_n\|^2 \\ + \|y_n - v_n\|^2] + 2\alpha_n\langle (f-I)z^*, z_n - z^*\rangle.\end{aligned} \tag{27}$$

Indeed, from Algorithm 3.1, we have $z_n - z^* = \alpha_n(f(x_n) - f(z^*)) + (1-\alpha_n)(u_n - z^*) + \alpha_n(f(z^*) - z^*)$. From (15) and (25), we have

$$\begin{aligned}\|z_n - z^*\|^2 &\leq \|\alpha_n(f(x_n) - f(z^*)) + (1-\alpha_n)(u_n - z^*)\|^2 + 2\alpha_n\langle (f-I)z^*, z_n - z^*\rangle \\ &\leq \alpha_n\delta\|x_n - z^*\|^2 + (1-\alpha_n)\|w_n - z^*\|^2 - (1-\alpha_n)(1-\mu)[\|y_n - w_n\|^2 \\ &\quad + \|y_n - v_n\|^2] + 2\alpha_n\langle (f-I)z^*, z_n - z^*\rangle.\end{aligned}$$

By (25) and (27), we have

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\leq \beta_n\|x_n - z^*\|^2 + (1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(Gz_n - z^*) + \delta_n(\overline{S}Gz_n - z^*)]\|^2 \\ &\leq (\|x_n - z^*\| + \alpha_n M_1)^2 - (1-\beta_n)(1-\alpha_n)(1-\mu)[\|y_n - w_n\|^2 \\ &\quad + \|y_n - v_n\|^2] + \alpha_n M_2,\end{aligned} \tag{28}$$

where $\sup_{n\geq 1} 2\|(f-I)z^*\|\|z_n - z^*\| \leq M_2$ for some $M_2 > 0$. This immediately implies that

$$\begin{aligned}&(1-\beta_n)(1-\alpha_n)(1-\mu)[\|y_n - w_n\|^2 + \|y_n - v_n\|^2] \\ &\leq (\alpha_n M_1 + \|x_n - x_{n+1}\|)(\alpha_n M_1 + \|x_n - z^*\| + \|x_{n+1} - z^*\|) + \alpha_n M_2.\end{aligned}$$

It follows that $\lim_{n\to\infty} \|y_n - w_n\| = \lim_{n\to\infty} \|y_n - v_n\| = 0$ which together with $w_n - x_n \to 0$ and $v_n - z_n \to 0$, leads to

$$\|x_n - z_n\| \leq \|x_n - w_n\| + \|w_n - y_n\| + \|y_n - v_n\| + \|v_n - z_n\| \to 0 (n \to \infty). \tag{29}$$

Consequently, this yields $\|v_n - x_n\| \leq \|v_n - z_n\| + \|z_n - x_n\| \to 0 (n \to \infty)$.

In what follows, we claim that $z \in \Xi$. In fact, from $y_n = P_C(w_n - \xi_n A w_n)$, we have $\langle w_n - \xi_n A w_n - y_n, y_n - y\rangle \geq 0, \forall y \in C$, and hence

$$\frac{1}{\xi_n}\langle w_n - y_n, y - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \leq \langle Aw_n, y - w_n\rangle, \forall y \in C. \tag{30}$$

Note that $\xi_n \geq \min\{\lambda, \frac{\mu\ell}{L}\}$. So, from (30) we get $\liminf_{k\to\infty}\langle Aw_{n_k}, y - w_{n_k}\rangle \geq 0, \forall y \in C$. Meantime, observe that $\langle Ay_n, y - y_n\rangle = \langle Ay_n - Aw_n, y - w_n\rangle + \langle Aw_n, y - w_n\rangle + \langle Ay_n, w_n - y_n\rangle$. Since $w_n - y_n \to 0$, from L-Lipschitz continuity of A we obtain $Aw_n - Ay_n \to 0$, which together with (30) arrives at $\liminf_{k\to\infty}\langle Ay_{n_k}, y - y_{n_k}\rangle \geq 0, \forall y \in C$.

Take a sequence $\{\kappa_k\} \subset (0,1)$ satisfying $\kappa_k \downarrow 0$ as $k \to \infty$. For all $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\langle Ay_{n_i}, y - y_{n_i}\rangle + \kappa_k \geq 0, \forall i \geq m_k. \tag{31}$$

Since $\{\kappa_k\}$ is decreasing, it is clear that $\{m_k\}$ is increasing. From the assumption on A, we know that $\liminf_{k\to\infty}\|Ay_{n_k}\| \geq \|Az\|$. If $Az = 0$, then z is a solution, i.e., $z \in \mathrm{VI}(C, A)$. Let $Az \neq 0$. Then we have $0 < \|Az\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|$. Without loss of generality, we may assume that $Ay_{n_k} \neq 0, \forall k \geq 1$. Noticing that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $Ay_{n_k} \neq 0, \forall k \geq 1$, we set $\hbar_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we get $\langle Ay_{m_k}, \hbar_{m_k}\rangle = 1, \forall k \geq 1$. So, from (31) we get $\langle Ay_{m_k}, y+\kappa_k\hbar_{m_k}-y_{m_k}\rangle \ge 0, \forall k\ge 1$. Again from the pseudomonotonicity of A we have $\langle A(y+\kappa_k\hbar_{m_k}), y+\kappa_k\hbar_{m_k}-y_{m_k}\rangle \ge 0, \forall k\ge 1$. This immediately yields

$$\langle Ay, y-y_{m_k}\rangle \ge \langle Ay-A(y+\kappa_k\hbar_{m_k}), y+\kappa_k\hbar_{m_k}-y_{m_k}\rangle - \kappa_k\langle Ay, \hbar_{m_k}\rangle, \forall k\ge 1. \quad (32)$$

We claim that $\lim_{k\to\infty}\kappa_k\hbar_{m_k}=0$. Indeed, from $y_{n_k}\rightharpoonup z$ (due to $x_n-y_n\to 0$), $\{y_n\}\subset C$ guarantees $z\in C$. Note that $\{y_{m_k}\}\subset\{y_{n_k}\}$ and $\kappa_k\downarrow 0$ as $k\to\infty$. So it follows that $0\le \limsup_{k\to\infty}\|\kappa_k\hbar_{m_k}\| = \limsup_{k\to\infty}\frac{\kappa_k}{\|Ay_{m_k}\|}\le \frac{\limsup_{k\to\infty}\kappa_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|}=0$. Hence we get $\kappa_k\hbar_{m_k}\to 0$.

Next we show that $z\in\Xi$. Indeed, letting $k\to\infty$, we deduce that the right-hand side of (32) tends to zero by the uniform continuity of A, the boundedness of $\{w_{m_k}\}, \{\hbar_{m_k}\}$ and the limit $\lim_{k\to\infty}\kappa_k\hbar_{m_k}=0$. Thus, we get $\langle Ay, y-z\rangle = \liminf_{k\to\infty}\langle Ay, y-y_{m_k}\rangle \ge 0, \forall y\in C$. By Lemma 2.1 we have $z\in \mathrm{VI}(C,A)$. Furthermore, we claim $Tz\in \mathrm{Fix}(S)$. In fact, since $z_n=\alpha_n f(x_n)+(1-\alpha_n)u_n$ where $u_n:=v_n-\sigma_n\mathcal{W}^*(I-S)\mathcal{W}v_n$, using $0<\epsilon\le\sigma_n$ and $v_n-z_n\to 0$, we obtain that $\|u_n-v_n\|\le\|z_n-v_n\|+\alpha_n\|u_n-v_n\|+\alpha_n\|f(x_n)-v_n\|\to 0 (n\to\infty)$ and hence

$$\epsilon\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|\le\sigma_n\|\mathcal{W}^*(I-S)\mathcal{W}v_n\| = \|v_n-u_n\|\to 0 \quad (n\to\infty),$$

which together with the τ-demimetricness of S, leads to

$$\frac{1-\tau}{2}\|(I-S)\mathcal{W}v_n\|^2\le\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|\|v_n-z^*\|\to 0\ (n\to\infty). \quad (33)$$

It follows that

$$\begin{aligned}\|\overline{S}Gz_n-x_n\| &= \frac{1}{\delta_n}\|x_{n+1}-x_n-\gamma_n(Gz_n-x_n)\|\\ &\le \frac{1}{\delta_n}(\|x_{n+1}-x_n\|+\|Gz_n-z_n\|+\|z_n-x_n\|)\to 0\ (n\to\infty),\end{aligned}$$

and

$$\begin{aligned}\|\overline{S}x_n-x_n\| &\le \|\overline{S}x_n-\overline{S}Gz_n\|+\|\overline{S}Gz_n-x_n\|\\ &\le \frac{1+\eta}{1-\eta}(\|x_n-z_n\|+\|z_n-Gz_n\|)+\|\overline{S}Gz_n-x_n\|\to 0\ (n\to\infty).\end{aligned}$$

Since $I-\overline{S}$ is demiclosed at zero, $x_n-\overline{S}x_n\to 0$ and $x_{n_k}\rightharpoonup z$, we have $z\in\mathrm{Fix}(\overline{S})$. Also, noticing $v_n-x_n\to 0$ and $x_{n_k}\rightharpoonup z$, we get $v_{n_k}\rightharpoonup z$. Since $\mathcal{W}$ is bounded linear operator, it is easy to see that $\mathcal{W}$ is weakly continuous on $\mathcal{H}_1$. So, we obtain that $\mathcal{W}v_{n_k}\rightharpoonup\mathcal{W}z$. By the assumption on S, we know that $I-S$ is demiclosed at zero. Hence, from (33) we derive $\mathcal{W}z\in\mathrm{Fix}(S)$, which immediately yields $z\in\Omega$. In addition, noticing $x_n-z_n\to 0$ and $x_{n_k}\rightharpoonup z$, we get $z_{n_k}\rightharpoonup z$. Therefore, $z\in\mathrm{VI}(C,A)\cap\Omega=\Xi$. □

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ converges strongly to the unique solution $z^*\in\Xi$ of the SGEP* (10) *with the BSFPP and VIP constraint, provided* $\sup_{n\ge 1}\|x_n-x_{n-1}\|<\infty$.

Proof. In terms of Lemma 3.3 we obtain that $\{x_n\}$ is bounded. Note that there exists the unique solution $z^*\in\Xi$ of the SGEP (10) with the BSFPP and VIP constraint, that is, the VIP (17) has the unique solution $z^*\in\Xi$. For convenience, we write $y^*:=T_{\mu_2}^{\Psi_2}(I-\mu_2\varphi_2)z^*$, $q_n:=T_{\mu_2}^{\Psi_2}(I-\mu_2\varphi_2)z_n$ and $p_n:=T_{\mu_1}^{\Psi_1}(I-\mu_1\varphi_1)q_n$. Then $z^*=Gz^*=T_{\mu_1}^{\Psi_1}(I-\mu_1\varphi_1)y^*$ and $p_n=Gz_n$.

Note that $\|q_n-y^*\|^2\le\|z_n-z^*\|^2-\mu_2(2\beta-\mu_2)\|\varphi_2z_n-\varphi_2z^*\|^2$ and $\|p_n-z^*\|^2\le \|q_n-y^*\|^2-\mu_1(2\alpha-\mu_1)\|\varphi_1q_n-\varphi_1y^*\|^2$. Combining these two inequalities, we obtain

$$\|p_n-z^*\|^2\le\|z_n-z^*\|^2-\mu_2(2\beta-\mu_2)\|\varphi_2z_n-\varphi_2z^*\|^2-\mu_1(2\alpha-\mu_1)\|\varphi_1q_n-\varphi_1y^*\|^2. \quad (34)$$

According to (23), (25) and (28), we have

$$\begin{aligned}\|z_n - z^*\|^2 &\le \alpha_n\delta\|x_n - z^*\|^2 + (1-\alpha_n)\|u_n - z^*\|^2 + 2\alpha_n\langle (f-I)z^*, z_n - z^*\rangle \\ &\le \alpha_n\delta\|x_n - z^*\|^2 + (1-\alpha_n)[\|w_n - z^*\|^2 - \epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2] \\ &\quad + 2\alpha_n\langle (f-I)z^*, z_n - z^*\rangle,\end{aligned}$$

which together with (34), arrives at

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\le \beta_n\|x_n - z^*\|^2 + (1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(p_n - z^*) + \delta_n(\overline{S}p_n - z^*)]\|^2 \\ &\le \beta_n\|x_n - z^*\|^2 + (1-\beta_n)[\alpha_n\delta\|x_n - z^*\|^2 + (1-\alpha_n)(\|w_n - z^*\|^2 \\ &\quad - \epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2) + \alpha_n M_2 - \mu_2(2\beta - \mu_2)\|\varphi_2 z_n - \varphi_2 z^*\|^2 \\ &\quad - \mu_1(2\alpha - \mu_1)\|\varphi_1 q_n - \varphi_1 y^*\|^2],\end{aligned} \tag{35}$$

where $\sup_{n\ge1} 2\|(f-I)z^*\|\|z_n - z^*\| \le M_2$ for some $M_2 > 0$. Moreover, from (25) we have

$$\|w_n - z^*\|^2 \le \|x_n - z^*\|^2 + \alpha_n M_3, \tag{36}$$

where $\sup_{n\ge1}(2M_1\|x_n - z^*\| + \alpha_n M_1^2) \le M_3$ for some $M_3 > 0$. Combining (35) and (36), we obtain

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\le \|x_n - z^*\|^2 - (1-\beta_n)[(1-\alpha_n)\epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 \\ &\quad + \mu_2(2\beta-\mu_2)\|\varphi_2 z_n - \varphi_2 z^*\|^2 + \mu_1(2\alpha-\mu_1)\|\varphi_1 q_n - \varphi_1 y^*\|^2] + \alpha_n M_4,\end{aligned}$$

where $M_4 := M_2 + M_3$. This immediately implies that

$$\begin{aligned}&(1-\beta_n)[(1-\alpha_n)\epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 + \mu_2(2\beta-\mu_2)\|\varphi_2 z_n - \varphi_2 z^*\|^2 \\ &\quad + \mu_1(2\alpha-\mu_1)\|\varphi_1 q_n - \varphi_1 y^*\|^2] \le \|x_n - z^*\|^2 - \|x_{n+1} - z^*\|^2 + \alpha_n M_4.\end{aligned} \tag{37}$$

Observe that

$$\|w_n - z^*\|^2 \le \|x_n - z^*\|^2 + \varepsilon_n\|x_n - x_{n-1}\|(2\|x_n - z^*\| + \varepsilon_n\|x_n - x_{n-1}\|). \tag{38}$$

By (25), (35) and (38), we have

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\le \beta_n\|x_n - z^*\|^2 + (1-\beta_n)[\alpha_n\delta\|x_n - z^*\|^2 + (1-\alpha_n)\|w_n - z^*\|^2 \\ &\quad + 2\alpha_n\langle (f-I)z^*, z_n - z^*\rangle] \\ &\le [1 - \alpha_n(1-\beta_n)(1-\delta)]\|x_n - z^*\|^2 + \alpha_n(1-\beta_n)(1-\delta) \\ &\quad \times \{\frac{3M}{1-\delta}\frac{\varepsilon_n}{\alpha_n}\cdot\|x_n - x_{n-1}\| + \frac{2\langle (f-I)z^*, z_n - z^*\rangle}{1-\delta}\},\end{aligned} \tag{39}$$

where $\sup_{n\ge1}\{\|x_n - z^*\|, \varepsilon_n\|x_n - x_{n-1}\|\} \le M$ for some $M > 0$.

Set $\Phi_n = \|x_n - z^*\|^2$. Now, we show the convergence of $\{\Phi_n\}$ to zero by two cases.

Case 1. Suppose that there exists an integer $n_0 \ge 1$ such that $\{\Phi_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\Phi_n = d < +\infty$ and $\lim_{n\to\infty}(\Phi_n - \Phi_{n+1}) = 0$. From (37) we obtain

$$\begin{aligned}&(1-\beta_n)[(1-\alpha_n)\epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_n\|^2 + \mu_2(2\beta-\mu_2)\|\varphi_2 z_n - \varphi_2 z^*\|^2 \\ &\quad + \mu_1(2\alpha-\mu_1)\|\varphi_1 q_n - \varphi_1 y^*\|^2] \le \Phi_n - \Phi_{n+1} + \alpha_n M_4.\end{aligned}$$

Since $\alpha_n \to 0$, $\Phi_n - \Phi_{n+1} \to 0$, $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$, one has

$$\lim_{n\to\infty}\|\varphi_2 z_n - \varphi_2 z^*\| = \lim_{n\to\infty}\|\varphi_1 q_n - \varphi_1 y^*\| = \lim_{n\to\infty}\|\mathcal{W}^*(I-S)\mathcal{W}v_n\| = 0. \tag{40}$$

Furthermore, by the firm nonexpansivity of $T_{\mu_1}^{\Theta_1}$ we obtain that

$$\begin{aligned}\|p_n - z^*\|^2 &\le \langle q_n - y^*, p_n - z^*\rangle + \mu_1\langle \varphi_1 y^* - \varphi_1 q_n, p_n - z^*\rangle \\ &\le \frac{1}{2}[\|q_n - y^*\|^2 + \|p_n - z^*\|^2 - \|q_n - p_n + z^* - y^*\|^2] \\ &\quad + \mu_1\|\varphi_1 y^* - \varphi_1 q_n\|\|p_n - z^*\|,\end{aligned}$$

which hence arrives at

$$\|p_n - z^*\|^2 \le \|q_n - y^*\|^2 - \|q_n - p_n + z^* - y^*\|^2 + 2\mu_1\|\varphi_1 y^* - \varphi_1 q_n\|\|p_n - z^*\|.$$

In a similar way, one gets

$$\|q_n - y^*\|^2 \le \|z_n - z^*\|^2 - \|z_n - q_n + y^* - z^*\|^2 + 2\mu_2\|\varphi_2 z^* - \varphi_2 z_n\|\|q_n - y^*\|.$$

Combining the last two inequalities, one deduces that

$$\begin{aligned}\|p_n - z^*\|^2 \le{}& \|z_n - z^*\|^2 - \|z_n - q_n + y^* - z^*\|^2 - \|q_n - p_n + z^* - y^*\|^2 \\ &+ 2\mu_2\|\varphi_2 z^* - \varphi_2 z_n\|\|q_n - y^*\| + 2\mu_1\|\varphi_1 y^* - \varphi_1 q_n\|\|p_n - z^*\|,\end{aligned}$$

which together with (25) and (35), leads to

$$\begin{aligned}\|x_{n+1} - z^*\|^2 \le{}& (\|x_n - z^*\| + \alpha_n M_1)^2 - (1-\beta_n)[\|z_n - q_n + y^* - z^*\|^2 \\ &+ \|q_n - p_n + z^* - y^*\|^2] + 2\mu_2\|\varphi_2 z^* - \varphi_2 z_n\|\|q_n - y^*\| \\ &+ 2\mu_1\|\varphi_1 y^* - \varphi_1 q_n\|\|p_n - z^*\|.\end{aligned}$$

This immediately ensures that

$$\begin{aligned}&(1-\beta_n)[\|z_n - q_n + y^* - z^*\|^2 + \|q_n - p_n + z^* - y^*\|^2] \\ &\le (\sqrt{\Phi_n} + \alpha_n M_1)^2 - \Phi_{n+1} + 2\mu_2\|\varphi_2 z^* - \varphi_2 z_n\|\|q_n - y^*\| + 2\mu_1\|\varphi_1 y^* - \varphi_1 q_n\|\|p_n - z^*\|.\end{aligned}$$

From (40) and the boundedness of $\{p_n\}, \{q_n\}$, one has

$$\lim_{n\to\infty}\|z_n - q_n + y^* - z^*\| = \lim_{n\to\infty}\|q_n - p_n + z^* - y^*\| = 0,$$

which hence yields

$$\|z_n - Gz_n\| = \|z_n - p_n\| \le \|z_n - q_n + y^* - z^*\| + \|q_n - p_n + z^* - y^*\| \to 0\ (n\to\infty). \quad (41)$$

Observe that $\|w_n - x_n\| = \alpha_n \cdot \frac{\varepsilon_n}{\alpha_n}\|x_n - x_{n-1}\| \le \alpha_n M_1 \to 0 (n\to\infty)$. By (40), we get

$$\begin{aligned}\|z_n - v_n\| &\le \alpha_n\|f(x_n) - v_n\| + (1-\alpha_n)\sigma_n\|\mathcal{W}^*(I-S)\mathcal{W}v_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|v_n\|) + \sigma_n\|\mathcal{W}^*(I-S)\mathcal{W}v_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

Using (28) one has that

$$\begin{aligned}&(1-\beta_n)(1-\alpha_n)(1-\mu)[\|y_n - w_n\|^2 + \|y_n - v_n\|^2] \\ &\le (\sqrt{\Phi_n} + \alpha_n M_1)^2 - \|x_{n+1} - z^*\|^2 + \alpha_n M_2.\end{aligned}$$

Hence, $\lim_{n\to\infty}\|y_n - w_n\| = \lim_{n\to\infty}\|y_n - v_n\| = 0$ which together with $w_n - x_n \to 0$ and $z_n - v_n \to 0$, leads to $\|x_n - z_n\| \le \|x_n - w_n\| + \|w_n - y_n\| + \|y_n - v_n\| + \|v_n - z_n\| \to 0\ (n\to\infty)$. From (41) it follows that

$$\|Gz_n - x_n\| \le \|Gz_n - z_n\| + \|z_n - x_n\| \to 0\ (n\to\infty). \quad (42)$$

On the other hand, using (28) we deduce that

$$\begin{aligned}\|x_{n+1}-z^*\|^2 &= \beta_n\|x_n-z^*\|^2+(1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(p_n-z^*)+\delta_n(\overline{S}p_n-z^*)]\|^2\\&\quad-\beta_n(1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(x_n-p_n)+\delta_n(x_n-\overline{S}p_n]\|^2\\&\le(\|x_n-z^*\|+\alpha_nM_1)^2+\alpha_nM_2\\&\quad-\beta_n(1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(x_n-p_n)+\delta_n(x_n-\overline{S}p_n]\|^2,\end{aligned}$$

which hence yields

$$\beta_n(1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(x_n-p_n)+\delta_n(x_n-\overline{S}p_n]\|^2\le(\sqrt{\Phi_n}+\alpha_nM_1)^2-\Phi_{n+1}+\alpha_nM_2.$$

It follows that

$$\lim_{n\to\infty}\|\frac{1}{1-\beta_n}[\gamma_n(x_n-p_n)+\delta_n(x_n-\overline{S}p_n]\|=0. \tag{43}$$

Note that

$$\begin{aligned}\delta_n\|x_n-\overline{S}Gz_n\| &\le \frac{\delta_n}{1-\beta_n}\|x_n-\overline{S}Gz_n\|\\&\le\|\frac{1}{1-\beta_n}[\gamma_n(x_n-p_n)+\delta_n(x_n-\overline{S}p_n]\|+\frac{1}{1-\beta_n}\|x_n-Gz_n\|.\end{aligned}$$

From (42) and (43) we obtain that $\lim_{n\to\infty}\|x_n-\overline{S}Gz_n\|=0$. So,

$$\|x_{n+1}-x_n\|\le\|Gz_n-x_n\|+\|\overline{S}Gz_n-x_n\|\to0\ (n\to\infty).$$

In addition, from the boundedness of $\{x_n\}$ it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(f-I)z^*,x_n-z^*\rangle=\lim_{k\to\infty}\langle(f-I)z^*,x_{n_k}-z^*\rangle. \tag{44}$$

Since $\mathcal{H}_1$ is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k}\rightharpoonup\widetilde{z}$. Thus, from (44) one gets

$$\begin{aligned}\limsup_{n\to\infty}\langle(f-I)z^*,x_n-z^*\rangle &= \lim_{k\to\infty}\langle(f-I)z^*,x_{n_k}-z^*\rangle\\&=\langle(f-I)z^*,\widetilde{z}-z^*\rangle.\end{aligned} \tag{45}$$

Applying Lemma 3.4, we deduce that $\widetilde{z}\in\omega_w(\{x_n\})\subset\varXi$. From (17) and (45) one gets

$$\limsup_{n\to\infty}\langle(f-I)z^*,x_n-z^*\rangle=\langle(f-I)z^*,\widetilde{z}-z^*\rangle\le0,$$

which together with $x_n-z_n\to0$, leads to

$$\limsup_{n\to\infty}\langle(f-I)z^*,z_n-z^*\rangle\le\limsup_{n\to\infty}[\|(f-I)z^*\|\|z_n-x_n\|+\langle(f-I)z^*,x_n-z^*\rangle]\le0.$$

Note that $\limsup_{n\to\infty}\{\frac{3M}{1-\delta}\cdot\frac{\varepsilon_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\frac{2\langle(f-I)z^*,z_n-z^*\rangle}{1-\delta}\}\le0$. Consequently, applying Lemma 2.2 to (39), one has $\lim_{n\to\infty}\|x_n-z^*\|^2=0$.

Case 2. Suppose that $\exists\{\Phi_{n_k}\}\subset\{\Phi_n\}$ s.t. $\Phi_{n_k}<\Phi_{n_k+1},\forall k\in\mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\psi:\mathcal{N}\to\mathcal{N}$ by $\psi(n):=\max\{k\le n:\Phi_k<\Phi_{k+1}\}$. By Lemma 2.3, we get $\Phi_{\psi(n)}\le\Phi_{\psi(n)+1}$ and $\Phi_n\le\Phi_{\psi(n)+1}$. From (37), we have

$$\begin{aligned}&(1-\beta_{\psi(n)})[(1-\alpha_{\psi(n)})\epsilon^2\|\mathcal{W}^*(I-S)\mathcal{W}v_{\psi(n)}\|^2+\mu_2(2\beta-\mu_2)\|\varphi_2z_{\psi(n)}-\varphi_2z^*\|^2\\&\quad+\mu_1(2\alpha-\mu_1)\|\varphi_1q_{\psi(n)}-\varphi_1y^*\|^2]\le\Phi_{\psi(n)}-\Phi_{\psi(n)+1}+\alpha_{\psi(n)}M_4,\end{aligned}$$

which immediately yields

$$\lim_{n\to\infty}\|\varphi_2 z_{\psi(n)}-\varphi_2 z^*\|=\lim_{n\to\infty}\|\varphi_1 q_{\psi(n)}-\varphi_1 y^*\|=\lim_{n\to\infty}\|\mathcal{W}^*(I-S)\mathcal{W}v_{\psi(n)}\|=0.$$

So it follows that

$$\|z_{\psi(n)}-v_{\psi(n)}\|\le\alpha_{\psi(n)}(\|f(x_{\psi(n)})\|+\|v_{\psi(n)}\|)+\sigma_{\psi(n)}\|\mathcal{W}^*(I-S)\mathcal{W}v_{\psi(n)}\|\to 0\ (n\to\infty).$$

Using the same inferences as in the proof of Case 1, we deduce that $\lim_{n\to\infty}\|z_{\psi(n)}-Gz_{\psi(n)}\|=\lim_{n\to\infty}\|x_{\psi(n)+1}-x_{\psi(n)}\|=0$ and $\limsup_{n\to\infty}\langle(f-I)z^*,z_{\psi(n)}-z^*\rangle\le 0$. On the other hand, from (39) we obtain

$$\begin{aligned}\alpha_{\psi(n)}(1-\beta_{\psi(n)})(1-\delta)\Phi_{\psi(n)}\le\Phi_{\psi(n)}-\Phi_{\psi(n)+1}+\alpha_{\psi(n)}(1-\beta_{\psi(n)})(1-\delta)[\frac{3M}{1-\delta}\cdot\frac{\varepsilon_{\psi(n)}}{\alpha_{\psi(n)}}\\ \times\|x_{\psi(n)}-x_{\psi(n)-1}\|+\frac{2\langle(f-I)z^*,z_{\psi(n)}-z^*\rangle}{1-\delta}]\\ \le\alpha_{\psi(n)}(1-\beta_{\psi(n)})(1-\delta)[\frac{3M}{1-\delta}\cdot\frac{\varepsilon_{\psi(n)}}{\alpha_{\psi(n)}}\cdot\|x_{\psi(n)}-x_{\psi(n)-1}\|\\ +\frac{2\langle(f-I)z^*,z_{\psi(n)}-z^*\rangle}{1-\delta}],\end{aligned}$$

which hence arrives at $\limsup_{n\to\infty}\Phi_{\psi(n)}\le 0$. Thus, $\lim_{n\to\infty}\|x_{\psi(n)}-z^*\|^2=0$. Also, note that

$$\|x_{\psi(n)+1}-z^*\|^2-\|x_{\psi(n)}-z^*\|^2\le 2\|x_{\psi(n)+1}-x_{\psi(n)}\|\|x_{\psi(n)}-z^*\|+\|x_{\psi(n)+1}-x_{\psi(n)}\|^2.$$

Owing to $\Phi_n\le\Phi_{\psi(n)+1}$, we get

$$\|x_n-z^*\|^2\le\|x_{\psi(n)}-z^*\|^2+2\|x_{\psi(n)+1}-x_{\psi(n)}\|\|x_{\psi(n)}-z^*\|+\|x_{\psi(n)+1}-x_{\psi(n)}\|^2\to 0.$$

That is, $x_n\to z^*$ as $n\to\infty$. This completes the proof. □

4. **Conclusion**

In this paper, we introduce hybrid inertial subgradient extragradient rules with linesearch process for solving a system of generalized equilibrium problems with a bilevel split fixed point problem and a variational inequality constraint, where the rule exploits the contractiveness of one operator at the upper-level problem and the pseudomonotonicity of another mapping at the lower level. The bounded linear operator in the bilevel split fixed point problem involves a fixed-point problem of a strict pseudocontraction mapping in its domain space and a demimetric mapping in its range space. The strong convergence result for the proposed algorithm is established under some additional conditions.

REFERENCES

[1] *J. A. Abuchu, G. C. Ugwunnadi and O. K. Narain*, An inertial projection and contraction method for solving bilevel quasimonotone variational inequality problems, J. Anal., **31**(2023), 2915-2942.

[2] *T. V. Anh*, Linesearch methods for bilevel split pseudomonotone variational inequality problems, Numer. Algorithms, **81**(2019), 1067-1087.

[3] *L. C. Ceng, C. Y. Wang and J. C. Yao*, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Methods Oper. Res., **67**(2008), 375-390.

[4] *L. C. Ceng and J. C. Yao*, A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem, Nonlinear Anal., **72**(2010), 1922-1937.

[5] *Y. Censor, A. Gibali and S. Reich*, Algorithms for the split variational inequality problem, Numer. Algorithms **59**(2012), 301-323.

[6] *R. W. Cottle and J. C. Yao*, Pseudomonotone complementarity problems in Hilbert space, J. Optim. Theory Appl., **75**(1992), 281-295.

[7] *V. Dadashi and M. Postolache*, Forward-backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators, Arab. J. Math., **9**(2020), 89-99.

[8] *N. Gupta, M. Postolache, A. Nandal and R. Chugh*, A cyclic iterative algorithm for multiple-sets split common fixed point problem of demicontractive mappings without prior knowledge of operator norm, Mathematics, **9**(2021), Article number 372.

[9] *G. M. Korpelevich*, The extragradient method for finding saddle points and other problems, Ekonomikai Matematicheskie Metody, **12**(1976), 747-756.

[10] *A. Kumar, B. S. Thakur and M. Postolache*, Dynamic stepsize iteration process for solving split common fixed point problems with applications, Math. Comput. Simulat., **218**(2024), 498-511.

[11] *P. E. Maingé*, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal., **16**(2008), 899-912.

[12] *D. R. Sahu, A. Pitea and M. Verma*, A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems, Numer. Algor., **83(2)**(2020), 421-449.

[13] *B. S. Thakur and M. Postolache*, Existence and approximation of solutions for generalized extended nonlinear variational inequalities, J. Inequal. Appl., **2013**(2013), Art. No. 590.

[14] *H. K. Xu and T. H. Kim*, Convergence of hybrid steepest-descent methods for variational inequalities, J. Optim. Theory Appl., **119**(2003), 185-201.

[15] *Y. Yao, A. Adamu and Y. Shehu*, Strongly convergent inertial forward-backward-forward algorithm without on-line rule for variational inequalities, Acta Math. Sci., **44**(2024), 551-566.

[16] *Y. Yao, H. Li and M. Postolache*, Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions, Optim., **71(9)**(2022), 2451-2469.

[17] *Y. Yao, M. Postolache and J. C. Yao*, An approximation algorithm for solving a split problem of fixed point and variational inclusion, Optim., DOI: 10.1080/02331934.2023.2256769.

[18] *Y. Yao, N. Shahzad, M. Postolache and J. C. Yao*, Convergence of self-adaptive Tseng-type algorithms for split variational inequalities and fixed point problems, Carpathian J. Math., **40**(2024), 755-770.

[19] *Y. Yu and T. C. Yin*, Weak convergence of a self-adaptive Tseng-type algorithm for solving variational inclusion problems, U.P.B. Sci. Bull., Series A, **85**(2023), 51-58.

[20] *Y. Yu and T.C. Yin*, Strong convergence theorems for a nonmonotone equilibrium problem and a quasi-variational inclusion problem, J. Nonlinear Convex Anal., **25**(2024), 503-512.

[21] *Y. Yu and T.C. Yin*, Strong convergence of an algorithm for solving split variational inclusions, J. Nonlinear Convex Anal., in press.

[22] *Y. Yu, Y. Zhao and T. C. Yin*, Convergence of extragradient-type methods for fixed point problems and quasimonotone variational inequalities, J. Nonlinear Convex Anal., **24**(2023), 2225-2237.

[23] *X. Zhao, J. C. Yao and Y. Yao*, A nonmonotone gradient method for constrained multiobjective optimization problems, J. Nonlinear Var. Anal., **6(6)**(2022), 693-706.

[24] *L. J. Zhu, J. C. Yao and Y. Yao*, Approximating solutions of a split fixed point problem of demicontractive operators, Carpathian J. Math., **40**(2024), 195-206.

[25] *L. J. Zhu and Y. Yao*, Algorithms for approximating solutions of split variational inclusion and fixed point problems, Mathematics, **11** (2023), Article number 641.