A NEW LOOK AT THE CLASSICAL SEQUENCE SPACES BY USING MULTIPLICATIVE CALCULUS

Yusuf Gurefe, Uğur Kadak, Emine Misirli and Alia Kurdi[1]

*The important point to be noted on the non-Newtonian calculus is a self-contained system independent of any other system of calculus. Therefore, the reader may be surprised to learn that there is a uniform relationship between the corresponding operators of this calculus and the classical calculus. In the present paper, some fundamental theorems and notions of the classical calculus are interpreted from the view point of multiplicative calculus and the analogies between them are given. We propose a concrete approach based on some topological properties with respect to the multiplicative calculus. Finally, we give *-completeness results on some sets of specific sequences.*

Keywords: Sequence spaces, multiplicative calculus, metric topology, complete metric space.

1. Introduction

In the period 1967-1972, Grossman and Katz [5] introduced the non-Newtonian calculus consisting of the branches of geometric, bigeometric, quadratic and biquadratic calculus etc. Also, Grossman extended this notion to the other fields in [6, 7]. All these calculi can be described simultaneously within the framework of a general theory. We prefer to use the name *non-Newtonian* to indicate any calculi other than the classical calculus. Every property in the classical calculus has an analogue in non-Newtonian calculus which is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example for wage-rate (in dollars, euro etc.) related problems, the use of bigeometric calculus which is a kind of non-Newtonian calculus is advocated instead of a traditional Newtonian one.

Bashirov *et al.* [1, 2] have recently concentrated on the multiplicative calculus and have given results with applications corresponding to the well-known properties of derivatives and integrals in the classical calculus. Also, Uzer [15] extended the non-Newtonian calculus to the complex valued functions and was interested in the statements of some fundamental theorems and concepts of multiplicative complex

[1] YG: Department of Econometrics, Faculty of Economics and Administrative Sciences, Usak University, Usak, Turkey; UK: Department of Mathematics, Faculty of Arts and Sciences, Bozok University, Yozgat, Turkey; EM: Department of Mathematics, Faculty of Sciences, Ege University, Bornova-Izmir, Turkey; AK: University Politehnica of Bucharest, Department of Mathematics & Informatics, 313 Splaiul Independenţei, 060042 Bucharest, Romania.

calculus, and proved some analogies between the multiplicative complex calculus and classical calculus by theoretical and numerical examples. Further Misirli and Gurefe introduced multiplicative Adams Bashforth-Moulton methods for differential equations in [13]. Some authors also worked on the classical sequence spaces and related topics by using non-Newtonian calculus: please, see Çakmak and Başar [3, 4], Tekin and Başar [14]. Further, Kadak [8] and Kadak *et al.* [9, 10, 11] have determined matrix transformations between certain sequence spaces over the non-Newtonian complex field and generalized Runge-Kutta method via non-Newtonian differentiation.

Following Cakmak and Başar [3], we construct the classical sequence spaces with respect to the multiplicative calculus. In Section 2, some required inequalities are presented in the sense of the multiplicative calculus, and the concepts of *metric and some related examples are given. Section 3 is devoted to introduce the corresponding results for the sequences concerning the convergent sequences of real numbers and to prove some basic topological properties. Additionally, by using the notion of *completeness, *limit and *convergence, other results are discussed in detail.

2. **Preliminaries and Basic Inequalities**

A *generator* is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$, the set of real numbers. Each generator generates exactly one arithmetic, and conversely each arithmetic is generated by exactly one generator. As a generator, we choose the function exp from $\mathbb{R}$ to the set $\mathbb{R}^+$ of positive reals, that is to say that

$$\begin{array}{rccc} \alpha: & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & x & \longmapsto & \alpha(x)=e^x=y \end{array} \quad \text{and} \quad \begin{array}{rccc} \alpha^{-1}: & \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & y & \longmapsto & \alpha^{-1}(y)=\ln y = x. \end{array}$$

If $I(x)=x$ for all $x\in\mathbb{R}$, then I is called *identity function* whose inverse is itself. In the special cases $\alpha=I$ and $\alpha=\exp$, α generates the classical and geometric arithmetics, respectively.

Consider any generator α with range A. By α-*arithmetic*, we mean the arithmetic whose domain is A and whose operations are defined as follows: for $x,y\in\mathbb{R}$ and any generator α,

$$\begin{array}{llcl} \alpha\text{-addition} & x\dot{+}y & = & \alpha\{\alpha^{-1}(x)+\alpha^{-1}(y)\} \\ \alpha\text{-subtraction} & x\dot{-}y & = & \alpha\{\alpha^{-1}(x)-\alpha^{-1}(y)\} \\ \alpha\text{-multiplication} & x\dot{\times}y & = & \alpha\{\alpha^{-1}(x)\times\alpha^{-1}(y)\} \\ \alpha\text{-division} & x\dot{/}y & = & \alpha\{\alpha^{-1}(x)\div\alpha^{-1}(y)\} \\ \alpha\text{-order} & x\dot{<}y & \Leftrightarrow & \alpha^{-1}(x)<\alpha^{-1}(y). \end{array}$$

Particularly, if we choose α-generator as exp function, $\alpha(z) = e^z$ for $z \in \mathbb{R}$ then $\alpha^{-1}(z) = \ln z$, α-arithmetic turns out to be the geometric arithmetic. Specifically,

$$\begin{array}{rclclcl} x \oplus y &=& \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} &=& e^{\{\ln x + \ln y\}} &=& x \cdot y \\ x \ominus y &=& \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\} &=& e^{\{\ln x - \ln y\}} &=& x \div y,\ y \neq 0 \\ x \odot y &=& \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\} &=& e^{\{\ln x \times \ln y\}} &=& x^{\ln y} = y^{\ln x} \\ x \oslash y &=& \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\} &=& e^{\{\ln x \div \ln y\}} &=& x^{\frac{1}{\ln y}},\ y \neq 1 \end{array}$$

that is

$$\begin{array}{lcl} \alpha\text{-addition} & \to & \text{geometric addition} \\ \alpha\text{-subtraction} & \to & \text{geometric subtraction} \\ \alpha\text{-multiplication} & \to & \text{geometric multiplication} \\ \alpha\text{-division} & \to & \text{geometric division} \end{array}$$

In this case, α-summation turns out to be geometric summation

$$\sideset{_*}{}\sum_{k=1}^{n} x_k = \alpha\left\{\sum_{k=1}^{n} \alpha^{-1}(x_k)\right\} = \alpha\{\alpha^{-1}(x_1) + \cdots + \alpha^{-1}(x_n)\} = e^{\{\ln x_1 + \cdots + \ln x_n\}} = \prod_{k=1}^{n} x_k.$$

Now, following Çakmak and Başar [3], we are able to give some inequalities by using exponential generator.

The α-square of a number x in $A \subset \mathbb{R}$ is denoted by $x \odot x$ which will be denoted by x^{2_*}. For each nonnegative number s, the symbol $\sqrt{x}^*$ will be used to denote $s = \alpha\{\sqrt{\alpha^{-1}(x)}\} = e^{\sqrt{\ln x}}$ which is the unique nonnegative number whose α-square is equal to x, which means $s^{2_*} = x$.

Through out this section we denote the p-th multiplicative exponent and the q-th multiplicative root of $x \in \mathbb{R}^+$ by x^{p_*} and $\sqrt[q]{x}^*$, respectively. Therefore, we have

$$\begin{array}{rcl} x^{2_*} = x \odot x &=& e^{\{\ln x \times \ln x\}} = e^{\ln^2 x} = x^{\ln x} \\ x^{3_*} = x^{2_*} \odot x &=& e^{\ln^3 x} = x^{\ln^2 x} \\ &\vdots& \\ x^{p_*} = x^{(p-1)_*} \odot x &=& e^{\ln^p x} = x^{\ln^{p-1} x} \\ &\vdots& \end{array}$$

The α-absolute value denoted by $|x|^*$ is defined $\alpha(|\alpha^{-1}(x)|) = e^{|\ln x|}$. For each number x in $A \subset \mathbb{R}^+$,

$$\sqrt{x^{2_*}}^* = |x|^* = e^{|\ln x|}. \tag{2.1}$$

Then we say,

$$|x|^* = \begin{cases} x & , \ x > 1 \\ 1 & , \ x = 1 \\ 1/x & , \ x < 1 \end{cases} = e^{|\ln x|}.$$

Definition 2.1 ([3])**.** Let X be a non-empty set and $d^*\colon X \times X \to \mathbb{R}^+$ be a function such that for all x, y, $z \in X$, the following axioms hold:

(M1) $d^*(x, y) = 1$ if and only if $x = y$,

(M2) $d^*(x, y) = d^*(y, x)$,

(M3) $d^*(x, y) \leq d^*(x, z) \oplus d^*(z, y)$.

Then, the pair (X, d^*) and d^* are called a *multiplicative metric space* and a *multiplicative metric* (shortly, **metric*) on X, respectively.

Proposition 2.1. *Let (X, d^*) be a $*$-metric space. Then, the inequality*

$$\left|\frac{d^*(x,z)}{d^*(y,z)}\right|^* \leq d^*(x,y)$$

holds for all $x, y, z \in X$.

Proof. By using (M3), if $d^*(x, z) \leq d^*(x, y)\ d^*(y, z)$, then $d^*(x, z)/d^*(y, z) \leq d^*(x, y)$, and if $d^*(y, z) \leq d^*(y, x)\ d^*(x, z)$, then $d^*(y, z)/d^*(x, z) \leq d^*(y, x) = d^*(x, y)$. Thus

$$\frac{1}{d^*(x,y)} \leq \frac{d^*(x,z)}{d^*(y,z)} \leq d^*(x,y). \tag{2.2}$$

From (2.2), we obtain

$$\left|\frac{d^*(x,z)}{d^*(y,z)}\right|^* \leq d^*(x,y),$$

hence the proof. □

Proposition 2.2. *The following relations hold:*

(i) $|xy|^* \leq |x|^*|y|^*$

(ii) $|\ln x| = \ln |x|^*$

Proof. We can show that $1/|x|^* \leq x \leq |x|^*$, $x \in \mathbb{R}^*$. Indeed, $1/|x|^* = x < 1 < |x|^*$, and if $x \geq 1$, then $1/|x|^* \leq x = |x|^*$. Therefore we have

$$\frac{1}{|x|^*} \leq x \leq |x|^*. \tag{2.3}$$

(i) By using $1/|x|^* \leq x \leq |x|^*$ and $1/|y|^* \leq y \leq |y|^*$, we immediately get $1/(|x|^*|y|^*) \leq xy \leq |x|^*|y|^*$ and $|xy|^* \leq |x|^*|y|^*$.

(ii) If $x \geq 1$, then $|x|^* = x$ and $|\ln x| = \ln x = \ln |x|^*$, otherwise if $0 < x < 1$, then $|x|^* = 1/x$, $\ln x < 0$ and $|\ln x| = -\ln x = \ln(1/x) = \ln |x|^*$.

The proof is complete. □

Lemma 2.1. [Minkowski's inequality] *Let $p > 1$ and x_k, $y_k \in \mathbb{R}^+$ for $k \in \{1, 2, \dots, n\}$. Then,*

$$\sqrt[p]{{}_*\sum_{k=1}^{n} (x_k \oplus y_k)^{p_*}}^{\,*} \leq \sqrt[p]{{}_*\sum_{k=1}^{n} {x_k}^{p_*}}^{\,*} \oplus \sqrt[p]{{}_*\sum_{k=1}^{n} {y_k}^{p_*}}^{\,*}.$$

Proof. Taking the values $x_k \leftarrow \ln x_k$ and $y_k \leftarrow \ln y_k$ in the classic Minkowski inequality, we have

$$\left(\sum_k |\ln(x_k y_k)|^p\right)^{\frac{1}{p}} \leq \left(\sum_k |\ln x_k|^p\right)^{\frac{1}{p}} + \left(\sum_k |\ln y_k|^p\right)^{\frac{1}{p}}$$

for all $(x_k), (y_k) \in \mathbb{R}^+$. From relation $|\ln x| = \ln |x|^*$ in Proposition 2.2(ii), then

$$\exp\left\{\left[\sum_k \left(\ln |x_k y_k|^*\right)^p\right]^{\frac{1}{p}}\right\} \leq \exp\left\{\left[\sum_k \left(\ln |x_k|^*\right)^p\right]^{\frac{1}{p}}\right\} \exp\left\{\left[\sum_k \left(\ln |y_k|^*\right)^p\right]^{\frac{1}{p}}\right\} \tag{2.4}$$

holds for all $1 \leq p < \infty$ and $k \in \mathbb{N}$. The rest can be obtained by taking into account the notions of geometric summation and p-th non-Newtonian exponent together. □

3. Topological Properties, Convergence and Completeness

We know that sequences of real numbers play an important role in calculus, and it is the metric on $\mathbb{R}$ which enables us to define the basic concept of convergence of such a sequence. The same holds for sequences; in this case we have to use the *metric on the set of real numbers, $\mathbb{R}$. In an arbitrary metric space $X = (X, d^*)$ the situation is quite similar, that is, we may consider a sequence $(x_n) \in \mathbb{R}^+$ of elements $x_1, x_2, \ldots$ of X and use d^* to define convergence in multiplicative calculus. We define the *completeness of *metric space.

According to [12], we will define basic concepts *neighborhood (*open and *closed ball),*open sets, *closed sets, *interior, *closure and *limit point.

Definition 3.1. Given a point $x_0 \in X$. Then, for a real number $r > 0$,

$$B(x_0; r) = \{x \in X |\ d^*(x, x_0) < r\}$$

is a **neighborhood*, or **open ball*, of centre x_0 and radius r and

$$B[x_0; r] = \{x \in X |\ d^*(x, x_0) \leq r\}$$

is a **closed ball* of centre x_0 and radius r.

We see that an *open ball of radius r is the set all of points in X whose multiplicative distance from the center of the ball is less than r and we say directly from the definition that every *neighborhood of x_0 contains x_0; in other words, x_0 is a point of each of its *neighborhoods.

Definition 3.2. Let (X, d^*) be a *metric space. Then $G \subset X$ is called **open set* if and only if every point of G has a *neighborhood contained in G. Also $G \subset X$ is called **closed* set if and only if its complement is *open.

Proposition 3.1. *Every *open ball is an *open set.*

Proof. Let $x \in X$ and $r > 0$, we will show that the *open ball $B(x; r)$ is an *open set in a *metric space. Now we take an element $y \in B(x; r)$ so the condition $d^*(x, y) < r$

is satisfied. By taking a radius $s = \frac{r}{d^*(x,y)}$, we must show that $B(y;s) \subset B(x;r)$. Let $z \in B(y;s)$ for $d^*(y,z) < s$ and by the using *triangle inequalities we have

$$d^*(x,z) \leq d^*(x,y)\ d^*(y,z) < d^*(x,y)\ s = d^*(x,y)\frac{r}{d^*(x,y)} = r$$

$d^*(x,z) < r$ and $z \in B(x;r)$. Then the *open ball $B(y;s)$ is a subset the *open ball $B(x;r)$. This completes the proof. □

Proposition 3.2. *Every *closed ball is *closed set.*

Proof. Let $x \in X$ and $r > 0$, we will show that the *closed ball $B[x;r]$ is a *closed set in *metric space. We know that $G \subset X$ is called *closed set if and only if its complement is *open.

Since by definition the complement is

$$B[x;r]^C = \{y \in X|\ d^*(x,y) > r\}$$

we take an element $y \in B[x;r]^C$ for which the condition $d^*(x,y) > r$ is satisfied. By taking a radius $s = \frac{d^*(x,y)}{r}$, we must show that $B(y;s) \subset B[x;r]^C$. Let $z \in B(y;s)$ for $d^*(y,z) < s$ and by the using multiplicative triangle inequalities

$$d^*(x,y) \leq d^*(x,z)\ d^*(z,y) < d^*(x,z)\ s = \frac{d^*(x,y)}{r} d^*(x,z)$$

we have $d^*(x,z) > r$ and $z \in B[x;r]^C$. Then the *open ball $B(y;s)$ is a subset of the complement of the *closed ball $B[x;r]$. This completes the proof. □

The notion of a *closed set is closely connected with the idea of *limit point. Let G be a set in (X,d^*) and x be a point of G. Then x is called a *limit point of G if and only if every *neighborhood of x contains a point of G different from x. The set of *limit points is denoted G'.

Definition 3.3. Let G be any subset of (X,d^*). Then the following definitions can be given.

(i) The **interior* G^o is the largest *open set contained in G. The other words, the *interior of G is the union of all *open sets contained in G.
(ii) The **closure* $\overline{G}$ is the smallest *closed set containing G. The other words, the *closure of G is the intersection of all *closed sets containing G.

Definition 3.4 ([3])**.** The sequence (x_n) is said to be *multiplicative convergent* (shortly, **convergent*) to x in $X = (X,d^*)$, if for every $\epsilon > 1$ and there exists $n_0 \in \mathbb{N}$ such that $d^*(x_n,x) < \epsilon$ for every $n > n_0$. In other words; $\lim_{n\to\infty} d^*(x_n,x) = 1$, and x is called the *multiplicative limit* of (x_n) and we write $^*\lim_{n\to\infty} x_n = x$ or, simply, $x_n \longrightarrow^* x$. We say that (x_n) **converges* to x or has the *limit x. If (x_n) is not *convergent, it is said to be **divergent*.

Let $X = (X, d^*)$ be any multiplicative metric space and M, N subsets of X. We can define

$$\begin{aligned} \delta^*(x, M) &= \inf \{d^*(x, a) \mid a \in M\}, \\ \delta^*(M, N) &= \inf \{d^*(a, y) \mid a \in M,\ y \in N\}, \\ \delta^*(M) &= \sup \{d^*(x, y) \mid x, y \in M\}, \end{aligned}$$

which represents: $\delta^*(x, M)$ the multiplicative *distance between the point* x *and the set* M, $\delta^*(M, N)$ the **distance between the sets* M *and* N, $\delta^*(M)$ the *diameter of the set* M.

Definition 3.5. Let $M \subset X$ a nonempty set and $x, y \in M$. If its diameter $\delta^*(M) = \sup\ d^*(x, y)$ is finite, then M is called *multiplicative bounded* or **bounded.*

Definition 3.6. A sequence (x_n) in a multiplicative metric space $X = (X, d^*)$ is said to be *multiplicative bounded* (**bounded*) if and only if there exists the constant $M \geq 1$, $|x_n|^* \leq M$, for every $n \in \mathbb{N}$.

Proposition 3.3. *Let* $X = (X, d^*)$ *be a multiplicative metric space. Then*
(i) *Any *convergent sequence in* X *is *bounded and its *limit is unique.*
(ii) *If* $x_n \longrightarrow^* x$ *and* $y_n \longrightarrow^* y$*, then* $d^*(x_n, y_n) \longrightarrow^* d^*(x, y)$*.*
(iii) *For all* (x_{n_k})*, called subsequences of* (x_n)*, if* $x_n \longrightarrow^* x_0$*, then* $x_{n_k} \longrightarrow^* x_0$*.*

Proof. (i) Suppose that $x_n \longrightarrow^* x$. Then, taking $\epsilon = 2$, we can find a $n_0 \in \mathbb{N}$ such that $d^*(x_n, x) < 2$ for every $n > n_0$. Hence by the *triangle inequality for all n, we have $d^*(x_n, x) < a$ where

$$a = \max\{d^*(x_1, x), \ldots, d^*(x_{n_0}, x), 2\}.$$

This shows that (x_n) is *bounded.
Assuming that $x_n \longrightarrow^* x$ and $x_n \longrightarrow^* z$, we obtain (M3)

$$1 \leq d^*(x, z) \leq d^*(x, x_n)\ d^*(x_n, z) \longrightarrow^* 1$$

and the uniqueness $x = z$ of the limit follows from (M1).

(ii) For $x_n \longrightarrow^* x$ and $y_n \longrightarrow^* y$, from the inverse *triangle inequality

$$\begin{aligned} \left|\frac{d^*(x_n, y_n)}{d^*(x, y)}\right|^* &= \left|\frac{d^*(x_n, y_n)}{d^*(x, y_n)}\ \frac{d^*(x, y_n)}{d^*(x, y)}\right|^* \\ &\leq \left|\frac{d^*(x_n, y_n)}{d^*(x, y_n)}\right|^* \left|\frac{d^*(x, y_n)}{d^*(x, y)}\right|^* \\ &\leq d^*(x_n, x)\ d^*(y_n, y) \longrightarrow^* 1. \end{aligned}$$

(iii) Let (x_{n_k}) be any subsequence of (x_n) which converges to x_0 in multiplicative mean, for every $\epsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $d^*(x_n, x_0) < \epsilon$ for every $n > n_0$. Then, $\lim_{k\to\infty} n_k = \infty$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, we get

$$d^*(x_{n_k}, x_0) < \epsilon \Rightarrow {}^*\lim_{k\to\infty} x_{n_k} = x_0.$$

The proof is complete. □

Definition 3.7. A sequence (x_n) in a multiplicative metric space $X = (X, d^*)$ is said to be a *Cauchy sequence* if for every $\epsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $d^*(x_m, x_n) < \epsilon$ for every $m, n > n_0$.

The space is said to be *complete if every Cauchy sequence in X *converges.

Theorem 3.1. *Let (x_n) be a sequence in a multiplicative metric space $X = (X, d^*)$. Then the following holds:*

(i) *Every *convergent sequence in a multiplicative metric space is a Cauchy sequence.*

(ii) *Every Cauchy sequence is *bounded.*

(iii) *If the Cauchy sequence (x_n) have a subsequence (x_{n_k}) which converges to x_0, then $x_n \longrightarrow^* x_0$.*

Proof. Let (X, d^*) be a multiplicative metric space. Then

(i) if $x_n \longrightarrow^* x$, then for every $\epsilon > 1$, there exists $n_0 \in \mathbb{N}$ such that $d^*(x_n, x) < \sqrt{\epsilon}$ for all $n > n_0$. Hence by using the condition (M3) we obtain for $m, n > n_0$

$$d^*(x_m, x_n) \leq d^*(x_m, x)\ d^*(x, x_n) < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

This proves that (x_n) is a Cauchy sequence.

(ii) Let (x_n) be any Cauchy sequence in X. Then, for every $\epsilon > 1$ there exists $n_0 \in \mathbb{N}$ such that $d^*(x_n, x_m) < \epsilon$ for every $m, n > n_0$. By denoting

$$M = \max\{d^*(x_1, x_{n_0}), \ldots, d^*(x_{n_0-1}, x_{n_0}), \epsilon\},$$

we get $d^*(x_n, x_{n_0}) \leq M$, for all $n \in \mathbb{N}$. For all $m, n \in \mathbb{N}$ and by using (M3) we have

$$d^*(x_n, x_m) \leq d^*(x_n, x_{n_0})\ d^*(x_m, x_{n_0}) \leq M^2.$$

Hence (x_n) is *bounded.

(iii) Let (x_n) be any Cauchy sequence which have a subsequence (x_{n_k}) *converges to x_0. If $\epsilon > 1$ there exists $n_0 \in \mathbb{N}$ such that $d^*(x_n, x_m) < \sqrt{\epsilon}$ for every $m, n > n_0$. Similarly, for $\epsilon > 1$ there exists $k_1 \in \mathbb{N}$ such that $d^*(x_{n_k}, x_0) < \sqrt{\epsilon}$ for every $k > k_1$. Using these results for all $k > k_0 = \max\{n_0, k_1\}$, we have

$$d^*(x_k, x_0) \leq d^*(x_k, x_{n_k})\ d^*(x_{n_k}, x_0) < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

Hence $^*\lim_{k\to\infty} x_k = x_0$.

The proof is complete. □

Now, consider $\omega = \{(x_n) \,|\, x_n \in M,\ x_n > 0\}$. We define the classical sets $\ell_\infty^*(M)$, $c^*(M)$, $c_0^*(M)$ and $\ell_p^*(M)$ consisting of the multiplicative bounded, convergent, null and absolutely p-summable sequence, as follows:

$$\begin{aligned}
\ell_\infty^*(M) &:= \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} d^*(x_k, 1) < \infty\right\},\\
c^*(M) &:= \left\{x = (x_k) \in \omega : \exists l \in \mathbb{R},\ {}^*\lim_{k\to\infty} d^*(x_k, l) = 1\right\},\\
c_0^*(M) &:= \left\{x = (x_k) \in \omega :\ {}^*\lim_{k\to\infty} d^*(x_k, 1) = 1\right\},\\
\ell_p^*(M) &:= \left\{x = (x_k) \in \omega : \exp\left(\sum_k (\ln |x_k|^*)^p\right) < \infty\right\},\ (1 \le p < \infty).
\end{aligned}$$

Theorem 3.2. *Define the distance function d_∞^* by*

$$d_\infty^* : \gamma(M) \times \gamma(M) \longrightarrow \mathbb{R}^+, \quad d_\infty^*(x, y) = \sup\left\{|x_k \ominus y_k|^* : k \in \mathbb{N}\right\},$$

*where γ denotes any of the spaces ℓ_∞^*, c^* and c_0^*, and $x = (x_k)$, $y = (y_k) \in \gamma(M)$. Then, $(\gamma(M), d_\infty^*)$ is a *complete metric space.*

Proof. Since the proof is similar for $c^*(M)$ and $c_0^*(M)$, we prove the theorem only for $\ell_\infty^*(M)$. One can easily see that the axioms (M1) and (M2) are trivial. We will prove only the axiom (M3) by taking into account multiplicative triangular inequality as follows:

(M3) Let $x = (x_k), y = (y_k), z = (z_k) \in \ell_\infty^*(M)$. Then,

$$\begin{aligned}
d_\infty^*(x, y) &= \sup\left\{|x_k \ominus y_k|^* : k \in \mathbb{N}\right\} = \sup\left\{|(x_k \ominus z_k) \oplus (z_k \ominus y_k)|^* : k \in \mathbb{N}\right\}\\
&\le \sup\left\{|x_k \ominus z_k|^* : k \in \mathbb{N}\right\} \oplus \sup\left\{|z_k \ominus y_k|^* : k \in \mathbb{N}\right\}\\
&\le d_\infty^*(x, z) \oplus d_\infty^*(z, y).
\end{aligned}$$

It remains to prove the *completeness of $\ell_\infty^*(M)$. Let $x_m = (x_1^{(m)}, x_2^{(m)}, \ldots)$ be a Cauchy sequence on $\ell_\infty^*(M)$. Then for every $\epsilon > 1$, there exists $m_0 \in \mathbb{N}$ for all $m, r > m_0$ such that

$$d_\infty^*(x_m, x_r) = \sup\left\{\left|x_k^{(m)} \ominus x_k^{(r)}\right|^* : k \in \mathbb{N}\right\} < \epsilon. \tag{3.1}$$

For all $m, r > m_0$ and $k \in \mathbb{N}$, by the using *completeness of $\mathbb{R}$ (see [3]), we say $(x_k^{(1)}, x_k^{(2)}, \ldots)$ is a Cauchy sequence and *converges.

Let $x = (x_1, x_2, \ldots)$ and ${}^*\lim_{m\to\infty} x_k^{(m)} = x_k$. We have to show that

$${}^*\lim_{m\to\infty} d_\infty^*(x_m, x) = 1 \text{ and } x \in \ell_\infty^*(M).$$

Taking the *limit for $r \to \infty$ in (3.1), we get

$$\left|x_k^{(m)} \ominus x_k\right|^* \le \epsilon, \quad \text{for } m > m_0. \tag{3.2}$$

We know that $x_m = (x_1^{(m)}, x_2^{(m)}, \ldots)$ is a Cauchy sequence on $\ell_\infty^*(M)$, and that there exists a constant M such that $|x_k^{(m)}|^* \le M$, for all $k \in \mathbb{N}$. Therefore, by the using multiplicative triangle inequality

$$|x_k|^* \leq \left|x_k \ominus x_k^{(m)}\right|^* \oplus |x_k^{(m)}|^* \leq \epsilon \oplus M, \quad k \in \mathbb{N},\ m > m_0.$$

So, $x = (x_1, x_2, \ldots)$ is *bounded and $x \in \ell_\infty^*(M)$. According to (3.2)

$$d_\infty^*(x_m, x) = \sup\left\{\left|x_k^{(m)} \ominus x_k\right|^* : k \in \mathbb{N}\right\} \leq \epsilon.$$

Therefore, the space $(\ell_\infty^*(M), d_\infty^*)$ is *complete. □

Theorem 3.3. *Let $x = (x_k), y = (y_k) \in \ell_p^*(M)$ be sequences, and*

$$d_p^*\colon \ell_p^*(M) \times \ell_p^*(M) \longrightarrow \mathbb{R}^+ \quad d_p^*(x,y) = \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k}{y_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\}.$$

*$(\ell_p^*(M), d_p^*)$ is a *complete metric space.*

Proof. (M1) Let $x = (x_k), y = (y_k) \in \ell_p^*(M)$. It is trivial that if $x_k = y_k$ for all $k \in \mathbb{N}$ then $d_p^*(x,y) = 1$. Conversely if $d_p^*(x,y) = 1$, then $\ln\left|\frac{x_k}{y_k}\right|^* = 0 \Rightarrow \left|\frac{x_k}{y_k}\right|^* = 1$. Thus $x_k = y_k$ for all $k \in \mathbb{N}$ and $x = y$.

(M2) It is obvious that

$$\begin{aligned} d_p^*(x,y) &= \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k}{y_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} \\ &= \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{y_k}{x_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} = d_p^*(y,x). \end{aligned}$$

(M3) Let $x = (x_k), y = (y_k), z = (z_k) \in \ell_p^*(M)$ and by using the multiplicative Minkowski inequality we obtain

$$\begin{aligned} d_p^*(x,y) &= \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k}{z_k}\frac{z_k}{y_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} \\ &\leq \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k}{z_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\}\exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{z_k}{y_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} \\ &\leq d_p^*(x,z) \oplus d_p^*(z,y). \end{aligned}$$

Therefore, $(\ell_p^*(M), d_p^*)$ is a *metric space.

It remains to prove the *completeness of $(\ell_p^*(M), d_p^*)$.

Let $x_m = (x_1^{(m)}, x_2^{(m)}, \ldots)$ be a Cauchy sequence in $\ell_p^*(M)$. Then for every $\epsilon > 1$, there exists $m_0 \in \mathbb{N}$, so that for all $m, r > m_0$

$$d_p^*(x_m, x_r) = \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k^{(m)}}{x_k^{(r)}}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} < \epsilon. \tag{3.3}$$

Then, the following inequalities hold

$$\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k^{(m)}}{x_k^{(r)}}\right|^*\right)^p < \ln^p \epsilon$$

therefore, it follows that

$$\left|\frac{x_k^{(m)}}{x_k^{(r)}}\right|^* < \epsilon.$$

For all $m, r > m_0$ and $k \in \mathbb{N}$, by using the *completeness of $\mathbb{R}$, the sequence $(x_k^{(1)}, x_k^{(2)}, \ldots)$ is a Cauchy sequence and *converges.

Let us consider $x = (x_1, x_2, \ldots)$ and ${}^*\lim_{m\to\infty} x_k^{(m)} = x_k$. We have to prove that ${}^*\lim_{m\to\infty} d_p^*(x_m, x) = 1$ and $x \in \ell_p^*(M)$. Taking the limit for $r \to \infty$ in (3.3), we get

$$\exp\left[\sum_{k=1}^{j}\left(\ln\left|\frac{x_k^{(m)}}{x_k}\right|^*\right)^p\right] \leq \exp\left(\ln^p \epsilon\right), \quad j \in \mathbb{N},\ m > m_0. \tag{3.4}$$

Taking the limit for $j \to \infty$ in (3.4), we have

$$\exp\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k^{(m)}}{x_k}\right|^*\right)^p\right] \leq \exp\left(\ln^p \epsilon\right) < \infty. \tag{3.5}$$

Thus the sequence $\left(\frac{x_m}{x}\right) = \left(\frac{x_1^{(m)}}{x_1}, \frac{x_2^{(m)}}{x_2}, \ldots\right) \in \ell_p^*(M)$. Using $\left|\frac{x_k^{(m)}}{x_k}\right|^* = \left|\frac{x_k}{x_k^{(m)}}\right|^*$ and taking $x_k = x_k^{(m)}\frac{x_k}{x_k^{(m)}}$ we consider the reformulation of the multiplicative Minkowski inequality

$$\exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln|x_k|^*\right)^p\right]^{\frac{1}{p}}\right\} \leq \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln|x_k^{(m)}|^*\right)^p\right]^{\frac{1}{p}}\right\}\exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k}{x_k^{(m)}}\right|^*\right)^p\right]^{\frac{1}{p}}\right\}.$$

Since there exist $\delta > 0$ such that

$$\exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln|x_k^{(m)}|^*\right)^p\right]^{\frac{1}{p}}\right\} < \delta,$$

and from (3.5) we have

$$\exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k}{x_k^{(m)}}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} < \epsilon,$$

we obtain

$$\exp\left[\sum_{k=1}^{\infty}\left(\ln|x_k|^*\right)^p\right] < \epsilon_1 < \infty$$

where $\epsilon_1 = \exp\left(\ln^p(\delta\epsilon)\right)$. So, $x = (x_1, x_2, \ldots) \in \ell_p^*(M)$. According to (3.4)

$$d_p^*(x_m, x) = \exp\left\{\left[\sum_{k=1}^{\infty}\left(\ln\left|\frac{x_k^{(m)}}{x_k}\right|^*\right)^p\right]^{\frac{1}{p}}\right\} \leq \epsilon.$$

Hence ${}^*\lim_m d_p^*(x_m, x) = 1$, the space $(\ell_p^*(M), d_p^*)$ is *complete. □

4. **Conclusion**

In this paper, some fundamental theorems and notions of the classical calculus are interpreted from the view point of multiplicative calculus and the analogies between them are given. We proposed a concrete approach based on some topological properties with respect to the multiplicative calculus. Finally, we have introduced *-completeness results on some sets of specific sequences.

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