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AMENABILITY MODULO AN IDEAL OF FRÉCHET ALGEBRAS

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Amenability modulo an ideal of a Banach algebra have been defined and studied. In this paper we introduce the concept of amenability modulo an ideal of a Fréchet algebra and investigate some known results about amenability modulo an ideal of a Fréchet algebra. Also we show that a Fréchet algebra $(\mathcal{A}, p_n)_{n\in\mathbb{N}}$ is amenable modulo an ideal if and only if $\mathcal{A}$ is isomorphic to a reduced inverse limit of amenable modulo an ideal of Banach algebras.

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1. Introduction

Some parts of theory of Banach algebras, have been introduced and studied for Fréchet algebras. For example the notion of amenability of a Fréchet algebra and their applications to harmonic analysis was introduced by Helemskii in [8] and [9], and was covered and studied by Pirkovskii [15]. Also in [11], approximate amenability and approximate contractibility of Fréchet algebras was introduced and investigated. Furthermore, in [1] we and Abtahi studied weak amenability of Fréchet algebras and generalized some results related to weak amenability of Banach algebras for Fréchet algebras. In [2] Amini and Rahimi introduced the notion of amenability modulo an ideal of Banach algebras. They showed that amenability of the semigroup algebra $\ell^1(S)$ modulo ideals by certain classes of group congruence of S is equivalent to the amenability of S. Then Rahimi and Tahmasebi [16] continued this vertification and studied basic properties of amenability modulo an ideal such as virtual diagonal modulo an ideal, approximate diagonal modulo an ideal and contractibility modulo an ideal for Banach algebras.

In the present work, we continue our study on amenability of Fréchet algebras. We generalize some basic definitions and results about the concept of amenability modulo an ideal in Banach algebra case for Fréchet algebras. According to the definition of amenability modulo an ideal of Banach algebras, we introduce the concept of amenability modulo an ideal of Fréchet algebras. Then we verified some concept in the Banach algebra case, for Fréchet algebras. The remainder of the paper

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is organized as follows. Section 2 presents some preliminaries and basic results and definitions about locally convex spaces and Fréchet algebras. In section 3 we define the notion of amenability modulo an ideal for a Fréchet algebra. As the main result of this section we investigate the relation of the amenability modulo an ideal of a Fréchet algebra with amenability modulo an ideal of Banach algebras which form this Fréchet algebra. More precisely we show that if $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$ and $I = \varprojlim \overline{I_n}$ be an Arens-Michael decomposition of I, then $\mathcal{A}$ is amenable modulo I if and only if for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is amenable modulo $\overline{I_n}$. Also we discuss about the relation between amenability modulo an ideal of a Fréchet algebra and the amenability of the quotient algebra of this Fréchet algebra. Moreover we provide some examples of amenable modulo an ideal of Fréchet algebras which are not amenable. Section 4 describes the notions of locally bounded approximate identity modulo an ideal for a Fréchet algebra. In this section we prove that all amenable modulo an ideal Fréchet algebras have locally bounded approximate identities modulo an ideal.

2. **Preliminaries**

In this section, we first exhibit basic definitions and results related to locally convex spaces and also Fréchet algebras, which will be used throughout the paper. We refer the reader to [5], [6], [7], [12] and [13] for these results.

By a locally convex space E we mean a topological vector space in which the origin has a local base of absolutely convex absorbing sets. We denote by (E, p_α), a locally convex space E with a fundamental system of seminorms $(p_\alpha)_\alpha$.

If $(E, p_\alpha)_{\alpha\in A}$ and $(F, q_\beta)_{\beta\in B}$ be locally convex spaces, by applying [13, Proposition 22.6] a linear mapping $T : E \to F$ is continuous if and only if for each $\beta \in B$ there exist an $\alpha \in A$ and $C > 0$, such that

$$q_\beta(T(x)) \leq C p_\alpha(x),$$

for all $x \in E$. Also by [6, page 24], for locally convex spaces (E, p_μ), (F, q_λ) and (G, r_ν), the bilinear map $\theta : E \times F \to G$ is jointly continuous if and only if for any ν_0 there exist μ_0 and λ_0 such that the bilinear map

$$\theta : (E, p_{\mu_0}) \times (F, q_{\lambda_0}) \longrightarrow (G, r_{\nu_0})$$

is jointly continuous. Separate continuity of a bilinear map also defined in [17]. In fact the bilinear map $f : E \times F \to G$ is said to be separately continuous if all partial maps $f_x : F \to G$ and $f_y : E \to G$ defined by $y \mapsto f(x, y)$ and $x \mapsto f(x, y)$, respectively, are continuous for each $x \in E$ and $y \in F$. By applying [17, chapter.III.5.1], we have the fact that separate continuity implies joint continuity for Fréchet spaces and in particular, Banach spaces.

By a topological algebra we mean a linear associative algebra $\mathcal{A}$, whose underlying vector space is a topological vector space such that the multiplication

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}, \quad (a, b) \mapsto ab$$

is a separately continuous mapping; see [15]. An outstanding particular class of topological algebras is the class of Fréchet algebras. A Fréchet algebra, denoted by $(\mathcal{A}, p_n)$, is a complete topological algebra, whose topology is given by the countable family of increasing submultiplicative seminorms; see [5] and [7]. Also every closed subalgebra of a Fréchet algebra is clearly a Fréchet algebra.

For a Fréchet algebra $(\mathcal{A}, p_n)$, a locally convex $\mathcal{A}$-bimodule is a locally convex topological vector space X with an $\mathcal{A}$-bimodule structure such that the corresponding mappings are separately continuous. Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and X be a locally convex $\mathcal{A}$-bimodule. Following [7], a continuous derivation of $\mathcal{A}$ into X is a continuous mapping D from $\mathcal{A}$ into X such that

$$D(ab) = a.D(b) + D(a).b,$$

for all $a, b \in \mathcal{A}$. Furthermore for each $x \in X$ the mapping $\delta_x : \mathcal{A} \to X$ defined by

$$\delta_x(a) = a.x - x.a \quad (a \in \mathcal{A}),$$

is a continuous derivation and is called the inner derivation associated with x.

We recall definition of inverse limit from [4]. Let $(E_\alpha)_{\alpha\in\Lambda}$ be a family of algebras, where Λ is a directed set. Also suppose that $f_{\alpha\beta}$ is a family of homomorphisms defined from E_β into E_α for any $\alpha, \beta \in \Lambda$, with $\alpha \leq \beta$. A family $\{(E_\alpha, f_{\alpha\beta})\}$ is called a projective system of algebras, if it has the above relation and in addition satisfies the following condition

$$f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}, \quad (\alpha, \beta, \gamma \in \Lambda, \alpha \leq \beta \leq \gamma).$$

Now consider the cartesian product algebra $F = \prod_{\alpha\in\Lambda} E_\alpha$, and a subset of F,

$$E = \{x = (x_\alpha) \in F : x_\alpha = f_{\alpha\beta}(x_\beta), \ \alpha \leq \beta\}.$$

Then E is the projective (or inverse) limit of the projective system $\{(E_\alpha, f_{\alpha\beta})\}$ and we denote it by $E = \varprojlim\{(E_\alpha, f_{\alpha\beta})\}$ or simply $E = \varprojlim E_\alpha$.

Now let $(\mathcal{A}, p_\lambda)$ be a locally convex algebra. Obviously for each $\lambda \in \Lambda$, $N_\lambda = \ker p_\lambda$ is an ideal in $\mathcal{A}$ and $\frac{\mathcal{A}}{N_\lambda}$ is a normed algebra. Suppose that $\varphi_\lambda : \mathcal{A} \to \mathcal{A}_\lambda, \varphi_\lambda(x) = x_\lambda = x + N_\lambda$ be the corresponding quotient map. It is clear that φ_λ is a continuous surjective homomorphism. Now if $\lambda, \gamma \in \Lambda$ with $\lambda \leq \gamma$, one has $N_\lambda \subseteq N_\gamma$. So that the linking maps

$$\varphi_{\lambda\gamma} : \frac{\mathcal{A}}{N_\gamma} \to \frac{\mathcal{A}}{N_\lambda}, \quad \varphi_{\lambda\gamma}(x + N_\gamma) = x + N_\lambda,$$

are well defined continuous surjective homomorphisms such that $\varphi_{\lambda\gamma} \circ \varphi_\gamma = \varphi_\lambda$. Hence $\varphi_{\lambda\gamma}$'s have unique extentions to continuous homomorphisms between the Banach algebras $\mathcal{A}_\gamma$ and $\mathcal{A}_\lambda$, we use the symbol $\varphi_{\lambda\gamma}$ for the extentions too, which $\mathcal{A}_\gamma$ is the completion of $\frac{\mathcal{A}}{N_\gamma}$. The families $(\frac{\mathcal{A}}{N_\lambda}, \varphi_{\lambda\gamma})$ [respectively, $(\mathcal{A}_\lambda, \varphi_{\lambda\gamma})$], form inverse system of normed,[respectively, Banach] algebras. We denote the corresponding inverse limits by $\varprojlim \frac{\mathcal{A}}{N_\lambda}$ and $\varprojlim \mathcal{A}_\lambda$. Moreover if the initial algebra $\mathcal{A}$ is complete, one has

$$\mathcal{A} = \varprojlim \frac{\mathcal{A}}{N_\lambda} = \varprojlim \mathcal{A}_\lambda,$$

up to topological isomorphisms. Now let $\mathcal{A}$ be a Fréchet algebra with fundamental system of increasing submultiplicative seminorms $(p_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ let $\varphi_n : \mathcal{A} \to \frac{\mathcal{A}}{\ker p_n}$ be the quotient map. Then $\frac{\mathcal{A}}{\ker p_n}$ is naturally a normed algebra, normed by setting $\|\varphi_n(a)\|_n = p_n(a)$ for each $a \in \mathcal{A}$ and the completion $(\mathcal{A}_n, \|.\|_n)$ is a Banach algebra. We call the map φ_n from $\mathcal{A}$ into $\mathcal{A}_n$, the canonical map. It is important to note that $\varphi_n(\mathcal{A})$ is a dense subalgebra of $\mathcal{A}_n$ and in general $\mathcal{A}_n \neq \varphi_n(\mathcal{A})$.

The above is the Arense-Michael decomposition of $\mathcal{A}$, which expresses Fréchet algebra as reduced inverse limit of Banach algebras. Now choose an Arens-Michael decomposition $\mathcal{A} = \varprojlim \mathcal{A}_n$ and let I be a closed ideal of $\mathcal{A}$. Then it is easy to see that $I = \varprojlim \overline{I_n}$ is an Arens-Michael decomposition of I, where $\varphi_n : I \to I_n$ is the canonical map.(see [15]).

According to [15] if $\mathcal{A}$ is a locally convex algebra and X is a left locally convex $\mathcal{A}$-module, then a continuous seminorm q on X is m-compatible if there exists a continuous submultiplicative seminorm p on $\mathcal{A}$ such that

$$q(a.x) \leq p(a)q(x), \quad (a \in \mathcal{A}, x \in X).$$

Also by [14, 3.4] if $\mathcal{A}$ is a Fréchet algebra and X is a complete left $\mathcal{A}$-module with a jointly continuous left module action, then the topology on X can be determined by a directed family of m-compatible seminorms.

3. Amenability modulo an ideal of a Fréchet algebra

Let $\mathcal{A}$ be a Banach algebra and I be a closed ideal of $\mathcal{A}$. According to [2], $\mathcal{A}$ is amenable modulo I, if for every Banach $\mathcal{A}$-bimodule E such that $I.E = E.I = 0$ and every derivation D from $\mathcal{A}$ into E^*, there exists $\varphi \in E^*$ such that

$$D(a) = a.\varphi - \varphi.a, \quad (a \in \mathcal{A} \setminus I).$$

We commence with the definition of amenability modulo an ideal of a Fréchet algebra. We extend some results of [16], for Fréchet algebras. Recall that for the algebra $\mathcal{A}$,

$$\mathcal{A}.\mathcal{A} = \{a.b : \quad a, b \in \mathcal{A}\}.$$

Also $\mathcal{A}^2$ is the linear span of $\mathcal{A}.\mathcal{A}$.

Definition 3.1. Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. We call $\mathcal{A}$ amenable modulo I, if for every Banach $\mathcal{A}$-bimodule E such that $E.I = I.E = 0$ each continuous derivation from $\mathcal{A}$ into E^* is inner on $\mathcal{A} \setminus I$.

Note that the concept of amenability modulo an ideal of a Fréchet algebra $\mathcal{A}$ coincides with the concept of amenability modulo an ideal, in the case where $\mathcal{A}$ is a Banach algebra. Also in view of [15, Theorem 9.6] each amenable Fréchet algebra is amenable modulo I for each closed ideal I. But at the end of this section we show that in general the converse of it, is not true. An easy computation shows that if a Fréchet algebra $\mathcal{A}$ is amenable modulo $I = \{0\}$, then $\mathcal{A}$ is amenable. Henceforth amenability modulo an ideal for a Fréchet algebra is a generalization of the concept of the amenability for a Fréchet algebra.

As for first result we extend [16, Theorem 8] for Fréchet algebras. The proof is similar to the Banach algebra case.

Proposition 3.1. *Let $\mathcal{A}$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$ and $\mathcal{A}$ be amenable modulo I. Suppose that $\mathcal{B}$ is a Fréchet algebra and J is a closed ideal of $\mathcal{B}$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism with dense range such that $\varphi(I) \subseteq J$. Then $\mathcal{B}$ is amenable modulo J.*

Proof. Suppose that E is a Banach $\mathcal{B}$-bimodule such that $J.E = E.J = 0$ and $D : \mathcal{B} \to E^*$ is a continuous derivation. Then E becomes a Banach $\mathcal{A}$-bimodule with the module actions defined by

$$a.x = \varphi(a)x \quad \text{and} \quad x.a = x\varphi(a) \quad (a \in \mathcal{A}, x \in E).$$

Obviously $I.E = E.I = 0$ and also $D \circ \varphi : \mathcal{A} \to E^*$ is a continuous derivation. On the other hand $\mathcal{A}$ is amenable modulo I, so there exists $\eta \in E^*$ such that $(D \circ \varphi)(a) = a.\eta - \eta.a$ on $\mathcal{A} \setminus I$. Now if $b \in \mathcal{B} \setminus J$, then there exists a net $(a_\alpha)_\alpha \subseteq \mathcal{A}$ such that $b = \lim_\alpha \varphi(a_\alpha)$. Since $\varphi(I) \subseteq J$, we may assume that $(a_\alpha)_\alpha \subseteq \mathcal{A} \setminus I$. Now we have

$$\begin{aligned} D(b) &= \lim_\alpha (D \circ \varphi)(a_\alpha) \\ &= \lim_\alpha (a_\alpha.\eta - \eta.a_\alpha) \\ &= \lim_\alpha (\varphi(a_\alpha)\eta - \eta\varphi(a_\alpha)) \\ &= b\eta - \eta b. \end{aligned}$$

Thus $\mathcal{B}$ is amenable modulo J. □

In [15, Theorem 9.5], Pirkovskii asserts that a Fréchet algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is isomorphic to a reduced inverse limit of amenable Banach algebras. In the following theorem, we extend this result as the main result of this section, for amenability modulo an ideal of Fréchet algebras.

During this section, for simplicity's sake we use the notation $I_n = \varphi_n(I)$, where $\varphi_n : \mathcal{A} \to \mathcal{A}_n$ is the canonical map for each $n \in \mathbb{N}$.

Theorem 3.1. *Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. Then the following assertions are equivalent;*

(i) *$\mathcal{A}$ is amenable modulo I.*
(ii) *For each Arens-Michael decomposition of $\mathcal{A} = \varprojlim \mathcal{A}_n$ all $\mathcal{A}_n$'s are amenable Banach algebras modulo $\overline{I_n}$'s, where $I = \varprojlim \overline{I_n}$ is an Arens-Michael decomposition of I.*

Proof. $(i) \Rightarrow (ii)$. Let $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$ and $\mathcal{A}$ be amenable modulo I. Since $\varphi_n : \mathcal{A} \to \mathcal{A}_n$ is a continuous homomorphism with dense range, by Proposition 3.1, $\mathcal{A}_n$ is amenable modulo $\overline{I_n}$ for each $n \in \mathbb{N}$.
$(ii) \Rightarrow (i)$. Let $\mathcal{A}_n$ be amenable modulo $\overline{I_n}$ for each $n \in \mathbb{N}$ and let E be a Banach $\mathcal{A}$-bimodule such that $E.I = I.E = 0$ and $D : \mathcal{A} \to E^*$ be a continuous derivation.

Since $(E^*, \|\cdot\|)$ is a Banach space, continuity of D implies the existence of a constant $C > 0$ and an $n_0 \in \mathbb{N}$ such that

$$\|D(a)\| \leq Cp_{n_0}(a), \qquad (a \in \mathcal{A}).$$

Since $\{p_n\}_n$ is a fundamental system of increasing seminorms, we have $\|D(a)\| \leq Cp_m(a)$, for each $m \geq n_0$. Thus $\ker p_m \subseteq \ker D$, for each $m \geq n_0$, so the function

$$D_m : \frac{\mathcal{A}}{\ker p_m} \to E^*, \quad D_m(a + \ker p_m) = D_m(a_m) = D(a), \quad (a \in \mathcal{A}),$$

is well defined and is a continuous derivation on $\frac{\mathcal{A}}{\ker p_m}$, for each $m \geq n_0$. The unique extention of D_m to the Banach algebra $\mathcal{A}_m$ is a continuous derivation is denoted by D_m. On the other hand since $(E, \|\cdot\|)$ is a Banach $\mathcal{A}$-bimodule, the norm on E is m-compatible, so there exists $n_1 \in \mathbb{N}$ such that $\|a.x\| \leq p_{n_1}(a)\|x\|$, for all $a \in \mathcal{A}$ and $x \in E$. (see [15, Page 7]). Thus

$$\|a.x\| \leq p_m(a)\|x\|, \qquad (a \in \mathcal{A},\ x \in E,\ m \geq n_1).$$

Then E is a Banach $\mathcal{A}_m$-bimodule in a natural way for each $m \geq n_1$.(see [15, Page 7]). In conclusion, since $E.I = I.E = 0$ we have the following

$$E.I_m = I_m.E = 0, \ \ (m \geq n_1),$$

and so

$$E.\overline{I_m} = \overline{I_m}.E = 0, \quad (m \geq n_1).$$

Now set $n = \max\{n_1, n_0\}$. In view of the above arguments

$$D_n : \frac{\mathcal{A}}{\ker p_n} \to E^*, \quad D_n(a + \ker p_n) = D(a),$$

is a continuous derivation and $\overline{I_n}.E = E.\overline{I_n} = 0$ and the unique extention of D_n to the Banach algebra $\mathcal{A}_n$ is a continuous derivation also denoted by D_n. Since $\mathcal{A}_n$ is amenable modulo $\overline{I_n}$, there exists $\varphi \in E^*$ such that

$$D_n(a_n) = a_n.\varphi - \varphi.a_n \quad (a_n \in \mathcal{A}_n \setminus \overline{I_n}).$$

Therefore $D(a) = a.\varphi - \varphi.a$ for each $a \in \mathcal{A} \setminus I$. Note that if $a = (a_n) \notin I = \varprojlim \overline{I_n}$, then there is an $n_2 \in \mathbb{N}$ such that $a_{n_2} \notin \overline{I_{n_2}}$. Since the mappings φ_{mn_2} are defined by $\varphi_{mn_2}(a_m) = a_{n_2}$, then $a_m \notin \overline{I_m}$, for each $m \geq n_2$. So we can assume that $a_n \in \mathcal{A}_n \setminus \overline{I_n}$ for each $n \geq n_2$. Therefore similar to the definition of D_n for $n = \max\{n_1, n_0\}$ we can define D_n for $n = \max\{n_2, n_1, n_0\}$. Since E is a Banach $\mathcal{A}_n$-bimodule in natural way we have

$$D(a) = D_n(a_n) = a_n.\varphi - \varphi.a_n = a.\varphi - \varphi.a \quad (a \in \mathcal{A} \setminus I).$$

So $\mathcal{A}$ is amenable modulo I. □

The following theorem is a generalization of [2, Theorem 1]. The proof is completely different from the Banach algebra case.

Theorem 3.2. *Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. Then the following statements hold;*

(i) *If* $\mathcal{A}$ *is amenable modulo* I*, then* $\frac{\mathcal{A}}{I}$ *is amenable.*
(ii) *If* $\mathcal{A}$ *is amenable modulo* I *and* I *is amenable, then* $\mathcal{A}$ *is amenable.*
(iii) *If* $\frac{\mathcal{A}}{I}$ *is amenable and* $I^2 = I$*, then* $\mathcal{A}$ *is amenable modulo* I*.*

Proof. (i). Let $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$ and $I = \varprojlim \overline{I_n}$ be an Arens-Michael decomposition of I. By the hypothesis $\mathcal{A}$ is amenable modulo I, so by using Theorem 3.1, $\mathcal{A}_n$ is amenable modulo $\overline{I_n}$, for each $n \in \mathbb{N}$. Moreover $\frac{\mathcal{A}}{I} = \varprojlim \frac{\mathcal{A}_n}{\overline{I_n}}$ is an Arens-Michael decomposition of $\frac{\mathcal{A}}{I}$, by [4, Theorem 3.14]. The assertion now follows from [15, Theorem 9.5] and [2, Theorem 1].

(ii). Let $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$. By [15, Theorem 9.5] it is sufficient to show that $\mathcal{A}_n$ is amenable for each $n \in \mathbb{N}$, for amenability of $\mathcal{A}$. Suppose that $I = \varprojlim \overline{I_n}$ is an Arens-Michael decomposition of I. Since $\mathcal{A}$ is amenable modulo I, Theorem 3.1, implies that $\mathcal{A}_n$ is amenable modulo $\overline{I_n}$, for each $n \in \mathbb{N}$ and by [15, Theorem 9.5], $\overline{I_n}$, is amenable for each $n \in \mathbb{N}$. Consequently by [2, Theorem 1], $\mathcal{A}_n$ is amenable for each $n \in \mathbb{N}$.
(iii). Let E be a Banach $\mathcal{A}$-bimodule such that $I.E = E.I = 0$ and $D : \mathcal{A} \to E^*$ be a continuous derivation. Suppose that $\mathcal{A} = \varprojlim \mathcal{A}_n$ is an Arens-Michael decomposition of $\mathcal{A}$. Since $(E^*, \|\cdot\|)$ is a Banach space, by similar arguments to the proof of Theorem 3.1, there exists $n \in \mathbb{N}$ and a continuous derivation on $\mathcal{A}_n$ defined by;

$$D_n : \mathcal{A}_n \to E^*, \quad D_n(a_n) = D_n(a + \ker p_n) = D(a), \quad \overline{I_n}.E = E.\overline{I_n} = 0.$$

On the other hand since φ_n is a continuous homomorphism we have;

$$\begin{aligned} I_n = \varphi_n(I) = \varphi_n(I^2) &= \varphi_n(span\{ab : \quad a, b \in I\}) \\ &= span\{\varphi_n(ab) : \quad a, b \in I\} \\ &= span\{\varphi_n(a)\varphi_n(b) : \quad a, b \in I\} \\ &= span\{a_n b_n : \quad a_n, b_n \in I_n\} = I_n^2. \end{aligned}$$

Also $I_n.E^* = E^*.I_n = 0$. Therefore $I_n \subseteq \ker D_n$, but $\ker D_n$ is a closed subspace of $\mathcal{A}_n$, so $\overline{I_n} \subseteq \ker D_n$. Thus we can define a continuous derivation $\widetilde{D_n} : \frac{\mathcal{A}_n}{\overline{I_n}} \to E^*$, by $\widetilde{D_n}(a_n + \overline{I_n}) = D_n(a_n)$. On the other hand $\frac{\mathcal{A}}{I} = \varprojlim \frac{\mathcal{A}_n}{\overline{I_n}}$ is an Arens-Michael decomposition of $\frac{\mathcal{A}}{I}$, by [4, Theorem 3.14]. Since $\frac{\mathcal{A}}{I}$ is amenable, so $\frac{\mathcal{A}_n}{\overline{I_n}}$ is an amenable Banach algebra for each $n \in \mathbb{N}$ by [15, Theorem 9.5]. Therefore there exists $\eta_n \in E^*$ such that $\widetilde{D_n}(a_n + \overline{I_n}) = (a_n + \overline{I_n}).\eta_n - \eta_n.(a_n + \overline{I_n})$ for each $a_n \in \mathcal{A}_n$. So for each $a \in \mathcal{A} \setminus I$ we have

$$\begin{aligned} D(a) = D_n(a + \ker p_n) = D_n(a_n) &= \widetilde{D_n}(a_n + \overline{I_n}) \\ &= (a_n + \overline{I_n}).\eta_n - \eta_n.(a_n + \overline{I_n}) \\ &= a_n.\eta_n - \eta_n.a_n \\ &= a.\eta_n - \eta_n.a. \end{aligned}$$

Thus $\mathcal{A}$ is amenable modulo I. □

Some examples of amenable modulo an ideal of Banach algebras which are not amenable can be found in [2]. Before we proceed to examples of amenability modulo an ideal for Fréchet algebras we give some necessary background material. We shall give the definitions and some basic properties of semigroup algebras.

Let S be a semigroup and $s \in S$, and let δ_s denote the function on S which is 1 at s, and 0 elsewhere. A generic element of $\ell^1(S)$ is of the form

$$f = \sum_{s\in S} \alpha_s \delta_s, \quad \sum_{s\in S} |\alpha_s| < \infty.$$

Now consider $f = \sum_{r\in S} \alpha_r \delta_r$ and $g = \sum_{s\in S} \beta_s \delta_s \in \ell^1(S)$. Set

$$f \star g = \sum_{r\in S} \alpha_r \delta_r \star \sum_{s\in S} \beta_s \delta_s = \sum_{t\in S} (\sum_{rs=t} \alpha_r \beta_s) \delta_t,$$

where $\sum_{rs=t} \alpha_r \beta_s = 0$ when there are no elements r and s in S with $rs = t$. Then $(\ell^1(S), \star)$ is called the semigroup algebra of S. Take

$$\|f\|_1 = \sum_{s\in S} |\alpha_s|.$$

Then $(\ell^1(S), \star, \|\cdot\|_1)$ is a Banach algebra. If $\theta : S \to T$ is an ephimorphism of semigroups, then by [3] there exists a contractive ephimorphism $\theta_* : \ell^1(S) \to \ell^1(T)$ determined by

$$\theta_*(\delta_s) = \delta_{\theta(s)}, \quad (s \in S).$$

Let S be a semigroup. A relation R on the set S is called left[respectively, right] compatible if $s, t, a \in S$ and $(s,t) \in R$ implies that $(as, at) \in R$)[respectively, $(sa, ta) \in R$] and it is called compatible if $s, t, s', t' \in S$ and $(s,t) \in R$ and $(s', t') \in R$ implies $(ss', tt') \in R$. A compatible equivalence relation is called congruence. By [10, Theorem 1.5.2] if ρ is a congruence on the semigroup S, then the quotient set $\frac{S}{\rho}$ is a semigroup with respect to the operation defined by

$$(a\rho)(b\rho) = (ab)\rho, \quad (a, b \in S).$$

A congruence ρ on S is called a group congruence on S if $\frac{S}{\rho}$ is a group. We denote the least group congruence on S by σ. Also we denote the set of idempotent elements of S by $E(S)$. A semigroup S is called an E-semigroup if $E(S)$ forms a subsemigroup of S and E-inverse if for all $s \in S$ there exists $t \in S$ such that $st \in E(S)$. An inverse semigroup S is called E-unitary if for each $s \in S$ and $e \in E(S)$, $es \in E(S)$ implies $s \in E(S)$. According to [2], if R is a ring and S is a semigroup, the semigroup ring $R[S]$ is the ring whose elements are of the form $\sum_{s\in S} r_s s$, where $r_s \in R$ and all but infinitely many of the cofficient are zero. If $R = K$ is a field, then $K[S]$ is called a semigroup algebra. Now let S be a semigroup, ρ a congruence on S and $\pi : S \to \frac{S}{\rho}$ be the quotient map. Then one can extends π to an algebra ephimorphim

$$\pi_* : K[S] \to K[\frac{S}{\rho}],$$

whose kernel I_ρ is the ideal in $K[S]$, generated by the set

$$\{s - t, \quad s, t \in S \quad with \;\; (s,t) \in \rho\}.$$

Hence $K[\frac{S}{\rho}] \cong \frac{K[S]}{I_\rho}$.

Now we provide some examples of Fréchet algebras which are amenable modulo an ideal I, but are not amenable.

Example 3.1. *Let $\mathcal{S} = (S_n, \theta_n^m)_{n,m\in\mathbb{N}}$ be an inverse sequence of semigroups such that the linking maps θ_n^m are onto. Set $\mathcal{L}^1(\mathcal{S}) = \varprojlim(\ell^1(S_n), (\theta_n^m)_*)$. Clearly $\mathcal{L}^1(\mathcal{S})$ is a Fréchet algebra.*

(1) *If S_n is an amenable E-unitary inverse semigroup with $E(S_n)$ infinite, then $\ell^1(S_n)$ is not amenable but is amenable modulo I_{σ_n}, by* [2] *. So $\mathcal{L}^1(\mathcal{S})$ is not amenable by* [15, Theorem 9.5]*, but by applying Theorem 3.1, $\mathcal{L}^1(\mathcal{S})$ is amenable modulo $I_\sigma = \varprojlim I_{\sigma_n}$.*

(2) *For each $n \in \mathbb{N}$, let G_n be an amenable group with identity 1_n and T be an abelian semigroup with infinitely many idempotents which is not an inverse semigroup. Also set $S_n = G_n \times T$. Then by applying* [2, Example (iii)]*, for each $n \in \mathbb{N}$, $\ell^1(S_n)$ is amenable modulo I_{σ_n} but is not amenable Banach algebra. Therefore $\mathcal{L}^1(\mathcal{S})$ is not amenable by* [15, Theorem 9.5]*, but by applying Theorem3.1, $\mathcal{L}^1(\mathcal{S})$ is amenable modulo $I_\sigma = \varprojlim I_{\sigma_n}$.*

All amenable Fréchet algebras are amenable modulo I for each closed ideal I. In the next example we give a Fréchet algebra which is amenable by [15, Corollary 9.8].

(3) *All nuclear $\sigma-$ C^*-algebras are amenable and so amenable modulo an ideal I for each closed ideal I. (see* [15, Corollary 9.8] *for more details)*

In the next example we give a Fréchet algebra which is not amenable and so is not amenable modulo $I = \{0\}$.

(4) *Let $C^\infty([0,1])$ be the space of infinitely many differentiable functions on* $[0,1]$ *with pointwise multiplication.Then $C^\infty([0,1])$ is a Fréchet algebra with respect to the system of seminorms p_n given by*

$$p_n(f) = 2^{n-1} \sup\{|f^{(k)}(x)| : \quad x \in [0,1], k = 0, ..., n-1\}.$$

$C^\infty([0,1])$ *is not weakly amenable by* [1, Theorem 1.3] *and so is not amenable by* [15, Theorem 9.6]*. Therefore $C^\infty([0,1])$ is not amenable modulo $I = \{0\}$.*

4. Locally bounded approximate identity modulo an ideal of a Fréchet algebra

We recall from [16] the concept of bounded approximate identity modulo an ideal for a Banach algebra. A Banach algebra $\mathcal{A}$ has a bounded approximte identity modulo I if there exists a bounded net $(u_\alpha)_\alpha$ in $\mathcal{A}$ such that

$$\lim_\alpha u_\alpha a = \lim_\alpha a u_\alpha = a, \quad (a \in \mathcal{A} \setminus I).$$

Also in [15] Pirkovskii asserts the concept of locally bounded approximate identity for a locally convex algebra. Similar to these definitions we define bounded and locally bounded approximate identities modulo an ideal for a Fréchet algebra.

Definition 4.1. Let $(\mathcal{A}, p_\lambda)$ be a locally convex algebra and suppose that I is a closed ideal of $\mathcal{A}$. A bounded net $(e_\alpha)_\alpha$ is a bounded approximate identity modulo I for $\mathcal{A}$, if

$$\lim_\alpha p_\lambda(ae_\alpha - a) = \lim_\alpha p_\lambda(e_\alpha a - a) = 0, \quad (a \in \mathcal{A} \setminus I, \ \ \lambda \in \Lambda).$$

Furthermore we say that $\mathcal{A}$ has a locally bounded approximate identity modulo I, if there exists a family $\{C_\lambda : \quad \lambda \in \Lambda\}$ of positive real numbers such that for each finite set $F \subseteq \mathcal{A} \setminus I$, each $\lambda \in \Lambda$, and each $\varepsilon > 0$ there exists $b \in \mathcal{A}$ with $p_\lambda(b) \leq C_\lambda$ and $p_\lambda(ab - a) < \varepsilon$ and $p_\lambda(ba - a) < \varepsilon$, for all $a \in F$.

We commence with the following proposition which gives us necessary and sufficient conditions for the exitence of a bounded approximate identity modulo an ideal for a Fréchet algebra.

Proposition 4.1. *Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. $\mathcal{A}$ has a bounded approximate identity modulo I if and only if there exists a bounded set $B \subseteq \mathcal{A}$ such that for each finite set $F \subseteq \mathcal{A} \setminus I$, each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there exists $b \in B$ such that*

$$p_n(ab - a) < \varepsilon \quad \textit{and} \quad p_n(ba - a) < \varepsilon,$$

for each $a \in F$.

Proof. First suppose that $(e_\alpha)_\alpha \subseteq \mathcal{A}$ is a bounded approximate identity modulo I for $\mathcal{A}$. So for each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there exists α_0 such that for each $\alpha \geq \alpha_0$ we have

$$p_n(ae_\alpha - a) < \varepsilon \ \ \textit{and} \ \ p_n(e_\alpha a - a) < \varepsilon, \quad (a \in \mathcal{A} \setminus I).$$

Now set $B = (e_\alpha)_\alpha \subseteq \mathcal{A}$ and let $n \in \mathbb{N}$, $\varepsilon > 0$ and $F \subseteq \mathcal{A} \setminus I$ be a finite set. Therefore there exists $b = m_\alpha$ which $\alpha \geq \alpha_0$ such that

$$p_n(ab - a) < \varepsilon \quad \textit{and} \ \ p_n(ba - a) < \varepsilon, \quad (a \in F).$$

Conversely, suppose that there exists a bounded set $B \subseteq \mathcal{A}$ such that for each finite set $F \subseteq \mathcal{A} \setminus I$, each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there exists $b \in B$ with

$$p_n(ab - a) < \varepsilon \ \ \textit{and} \ \ p_n(ba - a) < \varepsilon, \quad (a \in F).$$

Now take

$$S = \{(n, \varepsilon, F) : \ \ n \in \mathbb{N}, \ \ \varepsilon > 0, \ \ F \subseteq \mathcal{A} \setminus I \ \ is \ \ a \ \ finite \ \ set\}.$$

So S is a directed set as follows:

$$(n_1, \varepsilon_1, F_1) \leq (n_2, \varepsilon_2, F_2) \ \ \Leftrightarrow \ \ n_1 \leq n_2, \ \ \varepsilon_2 \leq \varepsilon_1, \ \ F_1 \subseteq F_2.$$

For each $\alpha = (n, \varepsilon, F)$, there exists $b = e_\alpha \in B$ and so we have a bounded net $(e_\alpha) \subseteq \mathcal{A}$. Furthermore let $k \in \mathbb{N}$ and $\varepsilon > 0$ and the finite set $F \subseteq \mathcal{A} \setminus I$ be arbitrary. Therefore for each $(n, \delta, C) \geq (k, \varepsilon, F)$ we have

$$p_k(ae_\alpha - a) \leq p_n(ae_\alpha - a) < \delta < \varepsilon,$$
$$p_k(e_\alpha a - a) \leq p_n(e_\alpha a - a) < \delta < \varepsilon, \quad (a \in \mathcal{A} \setminus I).$$

So $\mathcal{A}$ has a bounded approximate identity modulo I. $\square$

Remark 4.1. *In view of Proposition 4.1, if $\mathcal{A}$ is normable, then the notions of bounded and locally bounded approximate identity modulo I are equivalent.*

By [16, Theorem 4] it is known that if a Banach algebra $\mathcal{A}$ is amenable modulo an ideal I, it has a bounded approximate identity modulo I. The following results are interesting in theirs own right. In fact we use them to extend [16, Theorem 4] for Fréchet algebrs.

Proposition 4.2. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism of Fréchet algebras with dense range and let I be a closed ideal of $\mathcal{A}$ and J be a closed ideal of $\mathcal{B}$ such that $\varphi(I) \subseteq J$. Suppose that $\mathcal{A}$ has a locally bounded approximate identity modulo I. Then $\mathcal{B}$ has a locally bounded approximate identity modulo J.*

Proof. Let $\{p_n : n \in \mathbb{N}\}$ be a family of fundamental system of seminorms generating the topology of $\mathcal{A}$ and $\{q_m : m \in \mathbb{N}\}$ generating the topology of $\mathcal{B}$. Continuity of φ implies for each $m \in \mathbb{N}$ the existence of $k_m > 0$ and $n_0 \in \mathbb{N}$ such that

$$q_m(\varphi(a)) \leq k_m p_{n_0}(a), \quad (a \in \mathcal{A}).$$

Since $\mathcal{A}$ has a locally bounded approximate identity modulo I one concludes from Definition 4.1, there exists a family $\{C_n : \ n \in \mathbb{N}\}$ of positive real numbers, such that for each finite set $F \subseteq \mathcal{A} \setminus I$, each $n \in \mathbb{N}$ and each $\varepsilon > 0$ there exists $a^{'} \in \mathcal{A}$ such that

$$p_n(a^{'}) \leq C_n \quad and \quad p_n(a - aa^{'}) < \frac{\varepsilon}{3k_m} \quad and \quad p_n(a - a^{'}a) < \frac{\varepsilon}{3k_m},$$

for each $a \in F$. Without loss of generality we may assume that k_m and $C_{n_0} > 1$ for each $m \in \mathbb{N}$. Now consider the family $\{k_m C_{n_0} : \ m \in \mathbb{N}\}$ of positive real numbers. Given a finite set $F^{'} \subseteq \mathcal{B} \setminus J$, $\varepsilon > 0$ and $m \in \mathbb{N}$, find a finite set $F \subseteq \mathcal{A} \setminus I$ such that, $q_m(\varphi(a) - b)) < \frac{\varepsilon}{3k_m C_{n_0}}$, for each $b \in F^{'}$ and $a \in F$. Now set $b^{'} = \varphi(a^{'})$. From our assumption it follows that

$$q_m(b^{'}) = q_m(\varphi(a^{'})) \leq k_m p_{n_0}(a^{'}) \leq k_m C_{n_0}$$

and

$$\begin{aligned} q_m(b - bb^{'}) &\leq q_m(\varphi(a - aa^{'})) + q_m(b - \varphi(a)) + q_m((\varphi(a) - b)b^{'}) \\ &\leq k_m p_{n_0}(a - aa^{'}) + \frac{\varepsilon}{3} + q_m(\varphi(a) - b) q_m(b^{'}) \\ &\leq k_m \frac{\varepsilon}{3k_m} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3k_m C_{n_0}} k_m C_{n_0} \\ &= \varepsilon, \end{aligned}$$

and similarly $q_m(b - b^{'}b) < \varepsilon$, for each $b \in F^{'}$. $\square$

By using Proposition 4.2 and [15, Remark 6.2] the following is immediate.

Corollary 4.1. *Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ has a locally bounded approximate identity modulo I, then $\frac{\mathcal{A}}{I}$ has a locally bounded approximate identity.*

An immediate cosequence of Proposition 4.2 and Remark 4.1 is the following result.

Corollary 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism with dense range. Suppose that I is a closed ideal of $\mathcal{A}$ and J is a closed ideal of $\mathcal{B}$ such that $\varphi(I) \subseteq J$. If $\mathcal{A}$ has a bounded approximate identity modulo I, then $\mathcal{B}$ has a bounded approximate identity modulo J.*

It is not hard to see the following lemma holds.

Lemma 4.1. *Let D be a dense subspace of normed algebra $\mathcal{A}$. If $\mathcal{A}$ has a bounded approximate identity $(e_\alpha)_\alpha$ modulo I, then $\mathcal{A}$ has a bounded approximate identity $(f_\mu)_\mu$ modulo I such that $f_\mu \in D$ for each μ.*

In the sequel we shall use the notation $I_n = \varphi_n(I)$, where $\varphi_n : \mathcal{A} \to \mathcal{A}_n$ is the canonical map.

Proposition 4.3. *Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. Then the following statements are equivalent;*

(i) *$\mathcal{A}$ has a locally bounded approximate identity modulo I.*
(ii) *For each Banach algebra $\mathcal{B}$ such that there exists a continuous homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ with dense range, $\mathcal{B}$ has a bounded approximate identity modulo J, where J is a closed ideal of $\mathcal{B}$ such that $\varphi(I) \subseteq J$.*

Proof. $(i) \Rightarrow (ii)$. This follows by using Proposition 4.2 and viewing Remark 4.1.
$(ii) \Rightarrow (i)$. Let $n \in \mathbb{N}$ be arbitrary and let $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$ and $I = \varprojlim \overline{I_n}$ be an Arens-Michael decomposition of I. Since $\varphi_n : \mathcal{A} \to \mathcal{A}_n$ is a continuous homomorphism with dense range, by assumption the Banach algebra $\mathcal{A}_n$ has a bounded approximate identity modulo $\overline{I_n}$. Hence by Lemma 4.1, the same is true for the dense subalgebra $\varphi_n(\mathcal{A}) \subseteq \mathcal{A}_n$. Therefore Proposition 4.1, yields the exitence of a bounded set $B \subseteq \mathcal{A}$ such that for each finite set $F' \subseteq \varphi_n(\mathcal{A}) \setminus \overline{I_n}$ and each $\varepsilon > 0$ there exists $b \in B$ such that

$$\|b'\varphi_n(b) - b'\|_n < \varepsilon \quad \textit{and} \quad \|\varphi_n(b)b' - b'\|_n < \varepsilon,$$

for each $b' \in F'$. Note that if $F = \{a_1, a_2, ..., a_m\} \subseteq \mathcal{A} \setminus I$ is a finite set, then there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $a_n^i \notin \overline{I_n}$, where $a_i = (a_n^i)$ and $1 \leq i \leq m$. In fact if $a_i \notin I$, then there exists $n_i \in \mathbb{N}$, such that $a_{n_i}^i \notin \overline{I_{n_i}}$. So for each $n \geq n_i$, $a_n^i \notin \overline{I_n}$. Now put $n_0 = \max\{n_i : \ 1 \leq i \leq m\}$. Therefore for each $n \geq n_0$, $a_n^i \notin \overline{I_n}$. In the sequel take a finite set $F \subseteq \mathcal{A} \setminus I$, $k \in \mathbb{N}$ and $\varepsilon > 0$. Then for each $n \geq \max\{k, n_0\}$, we can find a finite set $F' = \varphi_n(F) \subseteq \varphi_n(\mathcal{A}) \setminus \overline{I_n}$, such that

$$\begin{aligned} p_k(ab - a) &\leq p_n(ab - a) \\ &= \|\varphi_n(ab - a)\|_n \\ &= \|\varphi_n(a)\varphi_n(b) - \varphi_n(a)\|_n \\ &< \varepsilon \end{aligned}$$

and

$$\begin{aligned} p_k(ba-a) &\leq p_n(ba-a) \\ &= \|\varphi_n(ba-a)\|_n \\ &= \|\varphi_n(b)\varphi_n(a)-\varphi_n(a)\|_n \\ &< \varepsilon, \end{aligned}$$

for each $a \in F$. Since $b \in B$ and B is a bounded set, for each $k \in \mathbb{N}$ there exists $C_k > 0$ such that $p_k(b) \leq C_k$. By Definition 4.1, this completes the proof. □

Corollary 4.3. *Let $(\mathcal{A}, p_n)$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. Suppose that $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$ and $I = \varprojlim \overline{I_n}$ is an Arens-Michael decomposition of I. Then $\mathcal{A}$ has a locally bounded approximate identity modulo I if and only if each $\mathcal{A}_n$ has a bounded approximate identity modulo $\overline{I_n}$.*

Proof. Since $\varphi_n : \mathcal{A} \to \mathcal{A}_n$ is a continuous homomorphism with dense range, Proposition 4.3 implies that if $\mathcal{A}$ has a locally bounded approximate identity modulo I, then $\mathcal{A}_n$ has a bounded approximate identity modulo $\overline{I_n}$.
Conversely, suppose that each $\mathcal{A}_n$ has a bounded approximate identity modulo $\overline{I_n}$ and $\mathcal{B}$ be a Banach algebra such that there exists a continuous homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ with dense range. Arguing as in the proof of Theorem 3.1, we deduce that there exists a continuous homomorphism with dense range $\Phi_n : \mathcal{A}_n \to \mathcal{B}$, for some $n \in \mathbb{N}$. On the other hand if J is a closed ideal of $\mathcal{B}$ such that $\Phi(I) \subseteq J$, we have $\Phi_n(I_n) = \Phi(I) \subseteq J$. Therefore by Corollary 4.2, $\mathcal{B}$ has a bounded approximate identity modulo J such that $\Phi(I) \subseteq J$. Thus by using Proposition 4.3, $\mathcal{A}$ has a locally bounded approximate identity modulo I. □

Now we are in position to prove [16, Theorem 4] for Fréchet algebras.

Corollary 4.4. *Let $\mathcal{A}$ be a Fréchet algebra and I be a closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is amenable modulo I, then $\mathcal{A}$ has a locally bounded approximate identity modulo I.*

Proof. Suppose that $\mathcal{A} = \varprojlim \mathcal{A}_n$ be an Arens-Michael decomposition of $\mathcal{A}$ and $I = \varprojlim \overline{I_n}$ be an Arens-Michael decomposition of I. Since $\mathcal{A}$ is amenable modulo I, each $\mathcal{A}_n$ is amenamble modulo $\overline{I_n}$, by Theorem 3.1 . So $\mathcal{A}_n$ has a bounded approximate identity modulo $\overline{I_n}$ for each $n \in \mathbb{N}$, by [16, Theorem 4]. In view of Corollary 4.3, this completes the proof. □

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