NOISE-INDICATOR AUTOREGRESSIVE CONDITIONAL HETEROSKEDASTIC PROCESS WITH APPLICATION IN MODELING ACTUAL TIME SERIES

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This paper describes a modification of the ARCH-type models with threshold regime and two independent white noise innovations, obtained by using the so-called Noise Indicator. Basic stochastic properties of the modified, so-called NIN-ARCH model, has been researched. For estimation of parameters of the NIN-ARCH model, the conditional characteristic function (CCF) estimation method has been considered. Numerical simulation, along with the application of the model in the analysis dynamics of two actual time series is given as well.

Keywords: Noise–Indicator, ARCH processes, CCF method, parameters estimation, application

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1. Introduction. Definition of the model

The most conventional time series and models which have been used in finance assume a constant standard deviation in asset returns. With appearance of first Autoregressive Conditional Heteroskedastic (ARCH) model [1], the fitting models to describe conditional heteroskedasticity became widely discussed topic. To overcome drawbacks and weaknesses of the first ARCH model there was appeared a wide range of its extensions, such as generalization of ARCH (GARCH) [2], the exponential GARCH (EGARCH) [3], and the threshold GARCH (TGARCH) [4]. As soon as it became clear that volatility plays an important role in measures of risk of investments, these models have found successful application in asset pricing, portfolio optimizing and building the value-at-risk models to help provide a picture of risks with potential investments [5].

Further, in the volatility dynamics of some actual data series has been observed emphasized nonlinearity, manifested by sharp changes in a relatively short time intervals [6]. Since that kind of nonlinearity could not be explained by the standard (G)ARCH models, there was need for their further generalization as well as the creation of new related models that complement, in greater or lesser degree, their deficiencies. In cases where nonlinear behavior of volatilities is caused by emphatic fluctuations of current data series, a conditional heteroskedastic model called the Split-ARCH process can be used [7]. Similarly, the analog modification (and generalization) of the Stochastic Volatility Model (SV) was introduced in [8], as well as the integer-valued autoregressive time series in [9].

To describe the time dynamic and stochastic properties of the actual data series, a novel ARCH-type model, called *the Noise-Indicator AutoRegressive Conditional Heteroskedastic (NIN-ARCH) process*, is developed here. In that sense, we begin with the assumption that $(\Omega, \mathscr{F}, P)$ is the probability space, additionally expanded by filtration $F = (\mathscr{F}_t)$. Next, with (X_t) we denote the time series with known values at time $t \in \mathbf{Z}$ adapted to the filtration F. Given that we have a return series that is serially uncorrelated but admits higher order dependencies, such as volatility clustering, we

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define the NIN-ARCH process by following relations:

$$\begin{cases} X_t = \sigma_t \varepsilon_t, \\ \sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i X_{t-i}^2 + \sum_{j=1}^{q} b_j \eta_{t-j} \sigma_{t-j}^2, \end{cases} \tag{1}$$

where $t \in \mathbf{Z}$ and $a_i, b_j \geq 0$. Here, (σ_t) is the volatility, defined as series of $\mathscr{F}_{t-1}$ adaptive random variables (RVs), and (ε_t) is the white noise, i.e. the series of (0,1) independent identical distributed (i.i.d.) $\mathscr{F}_t$ adaptive RVs. Finally, $\eta_t(c) = I(\xi_t^2 \geq c)$ is *the Noise-Indicator*, and (ξ_t) is the i.i.d. $(0,1)$ noise series, mutually independent from the noise (ε_t).

The series (η_t), obtained by the additional noise (ξ_t), enables that the volatility terms (σ_{t-j}^2) in Eqs. (1) have the property of optionality. In more detail, if the noise (ξ_t) has 'small' preceding realized values (in some moment $t-j$), the NIN-ARCH model will follow standard ARCH regime. Conversely, in the case of 'greater' realizations of noise (ξ_t), the model introduces extended values of volatility (σ_{t-j}^2). In that way, series (X_t) follows the changes of volatility, but also shapely reacts on unexpected volatilitys changes. The level of significance in realizations of the series (ξ_t) determines the critical value of reaction $c > 0$, for which we denote:

$$m_c := E\left[I(\xi_t^2 \geq c)\right] = P\{\xi_t^2 \geq c\} = 1 - F(c),$$

where $F(\cdot)$ is the cumulative distribution function (CDF) of (ξ_t^2). Therefore, for the given value $c > 0$, the constant m_c can be determined, and vice versa. Moreover, in accordance with the real-based implementation of the NIN-ARCH process, we assume that RVs (ε_t) and (ξ_t) have absolutely continuous probability distributions.

2. Stochastic properties of the NIN-ARCH process

For purpose of researching the strong stationary conditions of the NIN-ARCH process, Eqs. (1) can be rewritten in form of following stochastic difference equation of order one:

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}, \quad t \in \mathbb{Z}. \tag{2}$$

Here, we denoted:

$$\mathbf{A}_t = \begin{pmatrix} \psi_{t-1} & \psi_{t-2} & \cdots & \psi_{t-r} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \mathbf{Y}_t = \begin{pmatrix} \sigma_t^2 \\ \sigma_{t-1}^2 \\ \vdots \\ \sigma_{t-r+1}^2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$r = \max\{p, q\}$, $\psi_{t-j} = a_j \varepsilon_{t-j}^2 + b_j \eta_{t-j}$, when $j = 1, \ldots r$, and $a_{p+i} = b_{q+j} = 0$, when $i = 1, \ldots, r-p$, $j = 1, \ldots, r-q$. The necessary and sufficient stationary conditions of the vector series $(\mathbf{Y}_t)$ can be specify in the following way:

Theorem 2.1. *Let vector series* $(\mathbf{Y}_t)$ *be defined by the recurrence relation* (2). *Then the following conditions are equivalent:*

(i) *The polynomial* $P(\lambda) = \lambda^r - \sum_{j=1}^{r} c_j \lambda^{r-j}$, *where*

$$c_j = \begin{cases} a_j + m_c b_j, & 1 \leq j \leq \min\{p, q\} \\ a_j, & q < j \leq p \\ b_j, & p < j \leq q \end{cases},$$

has the roots $\lambda_1, ..., \lambda_r$ *which satisfy the condition* $|\lambda_j| < 1$, *for any* $j = 1, \ldots, p$.

(ii) *Eq.* (2) *has the unique, strong stationary and ergodic solution*

$$\mathbf{Y}_t = \left(\mathbf{I} + \sum_{k=0}^{\infty} \mathbf{A}_t \cdots \mathbf{A}_{t-k}\right)\mathbf{B}, \tag{3}$$

where the sum above converges almost surely and in the mean square sense.

(iii) *The inequality* $\sum_{j=1}^{r} c_j = \sum_{i=1}^{p} a_i + m_c \sum_{j=1}^{q} b_j < 1$ *holds.*

Proof. $(i) \Rightarrow (ii)$: After some computations, it is easy to get equality:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^r P(\lambda),$$

where:

$$\mathbf{A} = E(\mathbf{A}_t) = \begin{pmatrix} c_1 & c_2 & \cdots & c_r \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Thus, it follows that the eigenvalues of matrix $\mathbf{A}$ are the roots of characteristics polynomial $P(\lambda)$. Then, assumption (i) implies the convergence $\mathbf{A}^k \longrightarrow \mathbf{O}_{r\times r}$, when $k \to \infty$. Following Francq et al. [10], the existence of almost sure unique, ergodic and stationary solution (3) of Eq. (2) is equivalent to the above convergence.

$(ii) \Rightarrow (iii)$: If suppose that the condition (ii) is true, according Eq. (3) follows:

$$E(\mathbf{Y}_t) = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = a_0 \cdot \left(1 - \sum_{i=1}^{p} a_i - m_c \sum_{j=1}^{q} b_j\right)^{-1} \cdot \mathbf{1}_{r\times 1}.$$

Therefore, the non-negative time series (X_t^2) and (σ_t^2) have the mean:

$$E\left(X_t^2\right) = E\left(\sigma_t^2\right) = a_0 \cdot \left(1 - \sum_{i=1}^{p} a_i - m_c \sum_{j=1}^{q} b_j\right)^{-1} > 0$$

and (iii) obviously holds.

$(iii) \Rightarrow (i)$: Let $\mathscr{S}_r(\mathbf{A}) = \max_j \{\lambda_j\}$ be the spectral radius of the matrix $\mathbf{A}$. Then, $\mathscr{S}_r(\mathbf{A}) \le \|\mathbf{A}\|_\infty$, where $\|\mathbf{A}\|_\infty = \max\left\{\sum_{j=1}^{r} c_j,\, 1\right\} = 1$. If assume that $\mathscr{S}_r(\mathbf{A}) = 1$, then for some $\varphi \in [0, 2\pi)$ there exists an eigenvalue $\lambda' = \mathrm{e}^{\mathrm{i}\varphi}$ which satisfies equality:

$$P(\lambda') = \mathrm{e}^{\mathrm{i}\varphi} - \sum_{j=1}^{r} c_j \mathrm{e}^{\mathrm{i}(r-j)\varphi} = 0.$$

After that, the inequality $\left|\mathrm{e}^{\mathrm{i}p\varphi}\right| \le \sum_{j=1}^{r} c_j \left|\mathrm{e}^{\mathrm{i}(r-j)\varphi}\right|$ implies $\sum_{j=1}^{r} c_j \ge 1$, which contradicts (iii). Hence, $\mathscr{S}_r(\mathbf{A}) < 1$ is valid, what is equivalent to (i). □

Further, we have considered in a more detail the simplest case of NIN-ARCH process, when $p = q = 1$. According to Eqs. (1), the volatility series is:

$$\sigma_t^2 = a_0 + a_1 X_{t-1}^2 + b_1 \eta_{t-1} \sigma_{t-1}^2 = \begin{cases} a_0 + a_1 X_{t-1}^2 + b_1 \sigma_{t-1}^2, & \xi_{t-1}^2 \ge c \\ a_0 + a_1 X_{t-1}^2, & \xi_{t-1}^2 < c, \end{cases} \quad t \in \mathbb{Z}, \tag{4}$$

and under condition $a_1 + m_c b_1 < 1$, the both series (X_t) and (σ_t^2) are strictly stationary. Also, Eq. (3) gives the following, stationary representation of the volatility:

$$\sigma_t^2 = a_0 \left(1 + \sum_{k=1}^{\infty} \psi_{t-1} \cdots \psi_{t-k}\right) = a_0 \left(1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \left(a_1 \varepsilon_{t-j}^2 + b_1 \eta_{t-j}\right)\right). \tag{5}$$

As the noise series (ε) and (η_t) are mutually independent, the series (ψ_t) is also the i.i.d. sequence of RVs. This fact is particularly important in researching the properties of the NIN-ARCH process. Namely, it is easy to show that (X_t) is series of uncorrelated RVs, with:

$$E(X_t) = E(\varepsilon_t) = 0, \qquad \mathrm{Var}(X_t) = E(X_t^2) = E(\sigma_t^2) = \frac{a_0}{1 - a_1 - b_1 m_c}.$$

Also, in the case of Gaussian noise series (ε_t), which will be further considered, according to Eqs. (4)-(5) and after some computation, the autocorrelation of series (X_t^2) is as follows:

$$\mathrm{Corr}(X_t^2, X_{t+k}^2) = \begin{cases} 1, & k = 0; \\ \dfrac{a_1(1 - a_1 b_1 m_c - b_1^2 m_c^2)}{1 - 2a_1 b_1 m_c - b_1^2 m_c^2} (a_1 + b_1 m_c)^{k-1}, & k \geq 1. \end{cases}$$

Finally, the fourth moment of the series (X_t) is

$$E(X_t^4) = 3E(\sigma_t^4) = \frac{3a_0^2(1 + a_1 + b_1 m_c)}{(1 - a_1 - b_1 m_c)(1 - 3a_1^2 - 2a_1 b_1 m_c - b_1^2 m_c^2)},$$

from here the stationarity value of kurtosis can be obtained as:

$$K := \frac{E(X_t^4)}{\left[E(X_t^2)\right]^2} = \frac{3\left(1 - (a_1 + b_1 m_c)^2\right)}{1 - 3a_1^2 - 2a_1 b_1 m_c - b_1^2 m_c^2} \geq 3.$$

Similar to (G)ARCH models, this value indicates the "peaked" distribution density of the series (X_t). Also, is valid $K = 3 \iff a_1 = 0$, when the NIN-ARCH process is reduced to a Gaussian white noise.

3. **Estimation of the model's parameters**

Due the threshold structure of the NIN-ARCH process, the estimation of its parameters $\theta = (a_0, a_1, b_1, m_c)'$ requires a more complex procedure. In order to get efficient estimators of the process parameters, here is proposed the novel estimation method, called *the conditional characteristic function (CCF) method*. In short, it is a hybrid method that combines two well-known estimation methods: the empirical characteristic function (ECF) method [11]-[14], and the conditional least squares (CLS) method [15]. The main aim of the CCF method is to minimize "distance" (in sense of a certain measure) between the CCFs $\varphi_t(r;\theta) := E\left(\exp(\mathrm{i}rX_t)|\mathscr{F}_{t-1}\right)$ and corresponding empirical characteristic functions (ECFs) $\widetilde{\varphi}_t(r) := \exp(\mathrm{i}rX_t)$. Notice that, when $\theta = \theta_0$ is a true value of the parameter, the CCFs and ECFs have the same means, which are equal the theoretical characteristic function $\varphi_X(r,\theta_0) := E[\widetilde{\varphi}_t(r)] = E[\varphi_t(r;\theta_0)]$. Therefore, here can be used a principle based on CLS method, where the distance between CCFs and ECFs is given by sum:

$$Q_T(r;\theta) = \frac{1}{T}\sum_{t=1}^{T}\left[\exp(\mathrm{i}rX_t) - E\left(\exp(\mathrm{i}rX_t)|\mathscr{F}_{t-1}\right)\right]^2 = \frac{1}{T}\sum_{t=1}^{T}\left[\widetilde{\varphi}_t(r) - \varphi_t(r;\theta)\right]^2,$$

and $\mathbf{X}_T := \{X_1, \ldots, X_T\}$ is the sample of length $T \in \mathbb{N}$ of the NIN-ARCH(1,1) process (X_t). Since the CCFs and ECFs depend on variable $r \in \mathbb{R}$, estimates based on the CCF method can be obtained by a minimization the following objective function:

$$S_T(\theta) := \int_{\mathbb{R}} Q_T(r;\theta) g(r)\,\mathrm{d}r,$$

where $g : \mathbb{R} \to \mathbb{R}^+$ is some weight function. Accordingly, CCF estimates are solutions of the minimization equation:

$$\hat{\theta}_T = \arg\min_{\theta \in \Theta} S_T(\theta), \tag{6}$$

where $\Theta \subseteq \mathbb{R}^4$ is the parameter space of the non-trivial, stationary NIN–ARCH$(1,1)$ process. The necessary conditions of strong consistency and asymptotic normality (AN) of the CCF estimates can be specified by the following statement.

Theorem 3.1. *Let θ_0 be the true value of the parameter $\theta \in \Theta$, and let $\hat{\theta}_T$, $T = 1,2,\ldots$ be solutions of the Eq.* (6)*. Additionally, we suppose that are satisfied the following regularity conditions:*

(i) *There exists the bounded set $\Theta' \subset \Theta$ so that $\theta_0, \hat{\theta}_T \in \Theta'$ for all $T \geq T_0 > 0$;*

(ii) $\dfrac{\partial^2 S_T(\theta_0)}{\partial\theta\,\partial\theta'}$ *is a regular matrix;*

(iii) $\dfrac{\partial\varphi_X(r;\theta_0)}{\partial\theta}\cdot\dfrac{\partial\varphi_X(r;\theta_0)}{\partial\theta'}$ *is a non-zero matrix, uniformly bounded by the strictly positive, g-integrable function $h : \mathbb{R} \to \mathbb{R}^+$.*

Then, $\hat{\theta}_T$ is strictly consistent and represents AN estimator for parameter θ.

Proof. Firstly, we prove the consistency of estimator $\hat{\theta}_T$. Notice that functions $S_T(\theta)$ and $Q_T(r;\theta)$ have the continuous partial derivatives up to the second order, for any component of the vector θ. Thus, according to the Taylor expansion of $\partial S_T(\theta)/\partial\theta$ at $\theta = \theta_0$, we have:

$$\frac{\partial S_T(\theta)}{\partial\theta} = \frac{\partial S_T(\theta_0)}{\partial\theta} + \frac{\partial^2 S_T(\theta_0)}{\partial\theta\partial\theta'}\cdot(\theta-\theta_0) + o(\theta-\theta_0),$$

and substituting θ with $\hat{\theta}_T$, under assumption (ii) and the fact that $\partial S_T(\hat{\theta}_T)/\partial\theta = 0$, we obtain:

$$\hat{\theta}_T - \theta_0 = -\left[\frac{\partial^2 S_T(\theta_0)}{\partial\theta\partial\theta'}\right]^{-1}\frac{\partial S_T(\theta_0)}{\partial\theta} + o(\hat{\theta}_T - \theta_0). \tag{7}$$

Using aforementioned properties of the function $S_T(\theta)$, it can be differentiated under the integral sign, e.g.,

$$\frac{\partial S_T(\theta)}{\partial\theta} = \int_{\mathbb{R}}\frac{\partial Q_T(r;\theta)}{\partial\theta}g(r)\,\mathrm{d}r = \frac{2}{T}\sum_{t=1}^{T}\int_{\mathbb{R}}\left[\varphi_t(r;\theta) - \widetilde{\varphi}_t(r)\right]\frac{\partial\varphi_t(r;\theta)}{\partial\theta}g(r)\,\mathrm{d}r, \tag{8}$$

$$\frac{\partial^2 S_T(\theta)}{\partial\theta\partial\theta'} = \frac{2}{T}\sum_{t=1}^{T}\int_{\mathbb{R}}\left[\frac{\partial\varphi_t(r;\theta)}{\partial\theta}\cdot\frac{\partial\varphi_t(r;\theta)}{\partial\theta'} + \left[\varphi_t(r;\theta)-\widetilde{\varphi}_t(r)\right]\frac{\partial^2\varphi_t(r;\theta)}{\partial\theta\,\partial\theta'}\right]g(r)\,\mathrm{d}r. \tag{9}$$

In that way, Eqs. (8)-(9) imply:

$$E\left[\frac{\partial S_T(\theta_0)}{\partial\theta}\right] = 0, \quad E\left[\frac{\partial^2 S_T(\theta_0)}{\partial\theta\,\partial\theta'}\right] = 2\mathbf{V}, \tag{10}$$

where $\mathbf{V} = \int_{\mathbb{R}}\left[\partial\varphi_X(r;\theta_0)/\partial\theta\right]\left[\partial\varphi_X(r;\theta_0)/\partial\theta'\right]g(r)\,\mathrm{d}r$, and under assumption (iii), the inequalities $0 < \|\mathbf{V}\| \leq \int_{\mathbb{R}} h(r)g(r)\,\mathrm{d}r < +\infty$ hold. Thus, Eqs. (10) and the strong law of large numbers give the almost sure convergence:

$$\left(\frac{\partial S_T(\theta_0)}{\partial\theta}, \frac{\partial^2 S_T(\theta_0)}{\partial\theta\,\partial\theta'}\right) \xrightarrow{\text{as}} (0, 2\mathbf{V}), \qquad T \to +\infty. \tag{11}$$

This convergence and Eq. (7) yield $\hat{\theta}_T - \theta_0 \xrightarrow{\text{a.s.}} 0$, when $T \to +\infty$, i.e. the estimator $\hat{\theta}_T$ is strictly consistent.

Now, we shall prove the AN of estimator $\hat{\theta}_T$. Notice that, using Eq. (7), we can write:

$$\sqrt{T}\left(\hat{\theta}_T - \theta_0\right) = \mathbf{N}_T^{-1}\cdot\mathbf{M}_T, \tag{12}$$

where:

$$\mathbf{M}_T = -\frac{\sqrt{T}}{2}\cdot\frac{\partial S_T(\theta_0)}{\partial\theta}, \quad \mathbf{N}_T = \frac{1}{2}\cdot\frac{\partial^2 S_T(\theta_0)}{\partial\theta\partial\theta'}.$$

According to Eq. (8), it is easily to prove that for any nonzero constant vector $\nu \in \mathbb{R}^4$, equality

$$E\left(\sqrt{T}\,\nu'\mathbf{M}_T|\mathscr{F}_{T-1}\right) = \sqrt{T-1}\,\nu'\mathbf{M}_{T-1}$$

holds. Hence, the series $T^{1/2}\nu'\mathbf{M}_T$ is a martingale, and by applying Billingsley's central limit theorem for martingales [16], we obtain:

$$\nu'\mathbf{M}_T \xrightarrow{d} \mathscr{N}(0, \nu' 4\mathbf{W}^2 \nu), \quad T \to +\infty,$$

where:

$$\mathbf{W}^2 = E\left(\frac{\partial S_T(\theta_0)}{\partial \theta} \cdot \frac{\partial S_T(\theta_0)}{\partial \theta'}\right).$$

Using the above convergence and Cramer-Wald's decomposition, we have:

$$\mathbf{M}_T \xrightarrow{d} \mathscr{N}(0, 4\mathbf{W}^2), \quad T \to +\infty.$$

Finally, this convergence and Eqs. (11)–(12) imply:

$$\sqrt{T}\left(\hat{\theta}_T - \theta_0\right) \xrightarrow{d} \mathscr{N}(0, \mathbf{V}^{-1}\mathbf{W}^2\mathbf{V}^{-1}), \quad T \to +\infty,$$

which ends the proof of Theorem. □

Remark 3.1. Let us assume, in following, Gaussian distribution $\mathscr{N}(0,1)$ of the noise innovations series (ε_t) and (ξ_t). In accordance to the fact that volatility (σ_t) is $\mathscr{F}_{t-1}$ adaptive time series, we can easy obtain the explicit expression of the CCFs:

$$\varphi_t(r;\theta) := E\left[\exp(\mathrm{i}rX_t)|\mathscr{F}_{t-1}\right] = \exp\left(\frac{-r^2\sigma_t^2}{2}\right).$$

Therefore, the CCFs represent the real-valued functions on $r \in \mathbb{R}$. Additionally, it means that appropriate ECF estimates are also real-valued functions Re $\widetilde{\varphi}_t(r) = \cos(rX_t)$.

4. Numerical simulations & application of the model

The estimation procedure of the NIN-ARCH(1,1) parameters, based on the CCF method, has been describe more precisely in this section. Since the CCF estimates are obtaining by minimization of the objective function $S_T(\theta)$, the computation of integral given by Eq. (9) becomes the main problem. To overcome that problem, we have applied the numerical approximation of this integral by using Gauss-Hermitian N-point cubature formula:

$$I(f) := \int_{\mathbb{R}} f(r)g(r)\,\mathrm{d}r \approx C_N(f) := \sum_{j=1}^{N} \omega_j f(v_j).$$

Here, v_j are the quadrature nodes and ω_j are the corresponding weight coefficients, computed according to the exponential weight function $g(r) = \exp\left(-r^2\right)$. This function puts more weights around the origin according to the fact that CF in that point provides the most information about the probability distribution of some model. In our case, 30-points quadrature formula has been used, while minimization of the objective function $S_T(\theta)$ has been carried out by using Nelder-Mead optimization method, implemented in statistical programming language "R".

Two samples of different sizes: $T = 250$ (small sample) and $T = 2\,500$ (large sample)) have been considered. For both samples are generated 500 independent Monte Carlo simulations, i.e. the realizations $\{X_0, X_1, \dots, X_T\}$ of the NIN-ARCH(1,1) series (X_t) with Gaussian innovations, where $X_0 \stackrel{as}{=} 0$. In Table 1 are presented obtained numerical results, i.e. means (Mean), minima (Min.), maxima (Max.) and standard estimating errors (SEE). Table 1 also contains values of the objective function $S_T^{(2)}$ as the reference estimation errors. The convergence of the CCF estimates is evident, since the values of SEE and $S_T^{(2)}$ decrease with increasing the sample size. Finally, note that estimates of the critical value are obtained by solving the equation $P\{\varepsilon_t^2 \geq c\} = m_c$ with respect to c.

In following, the NIN-ARCH(1,1) process is applied in fitting of the probability distribution and the analysis dynamic of two real-based time series. First of considered series (Series A) includes the dynamics of the trading values of 15 the most liquid Serbian shares, integrated within the so-called BELEX15 financial index. This index was defined and methodologically processed by *the*

TABLE 1. Estimated parameters values of the NIN-ARCH(1,1) process. (True parameters are: $a_0 = 0.1$, $a_1 = b_1 = 0.5$, $c = 1$, $m_c \approx 0.317$.)

	Sample size: $T = 250$				$S_T^{(2)}$	Sample size: $T = 2500$				$S_T^{(2)}$
	a_0	a_1	b_1	m_c		a_0	a_1	b_1	m_c	
Min.	0.0539	0.3082	0.4272	0.2649	6.12E-4	0.0691	0.3924	0.4553	0.3111	1.47E-4
Mean	0.0928	0.4203	0.4752	0.3198	8.87E-3	0.0996	0.4996	0.4998	0.3169	8.06E-4
Max.	0.2369	0.5249	0.5481	0.3624	1.95E-2	0.2259	0.5208	0.5139	0.3337	1.86E-3
SEE	0.0312	0.0262	0.0215	0.0201	1.98E-2	0.0307	0.0252	0.0195	0.0196	1.49E-2

TABLE 2. Estimated parameters values and fitting errors statistics of the actual data.

Sample		Parameters estimates					Fitting errors		
		a_0	a_1	b_1	m_c	c	$S_T^{(2)}$	RMS	AIC
Series A	GARCH	2.56E-4	0.2463	0.7486	1.0000	0.0000	3.38E-9	0.0334	-18275.2
	NIN–ARCH	2.01E-4	0.2499	0.7501	0.4999	0.4550	3.41E-10	0.0275	-18275.3
Series B	GARCH	0.0713	0.2711	0.5438	1.0000	0.0000	6.84E-4	0.8553	-5315.88
	NIN–ARCH	0.0137	0.5904	0.7549	0.2475	1.3370	4.07E-11	0.7846	-5316.05

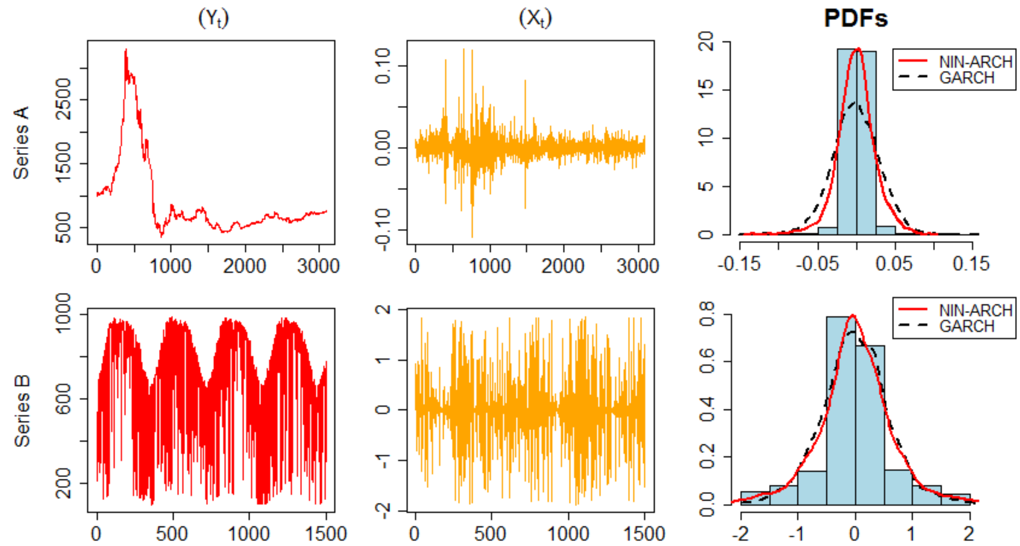

FIGURE 1. Dynamics of the actual time series, along with their empirical fitted PDFs: Series A (graphs above) and Series B (graphs below).

Belgrade Stock Exchange [17] at the beginning of October 2005. By collecting all of its changes until the end of 2017., here is obtained the time series with total of $T = 3087$ data. In contrast, the Series B contains data of the quantities of solar radiation between January 1998. and August 2016., supplied by *the National Centers for Environmental Information (NCEI)* [18]. The sample size of this series is $T = 2922$ data. We point out that values of both of the datasets have been measured at discrete, daily time intervals. Therefore, they represent univariate time series, with continuously distributed variables, denoted as (Y_t).

The series of the so-called log-returns $X_t = \ln(Y_t/Y_{t-1})$ has been introduced for both samples. In that way, the condition $E[X_t] = 0$ is fulfilled, so they can be modeled by the NIN–ARCH process. For comparison, the same estimation procedure was applied on the standard GARCH(1,1) process. The values of the estimated parameters of both processes are shown in Table 2. In addition, the probability density functions (PDFs) of the empirical and simulated data are compared, and the efficiency of both models is checked. For this purpose, two typical statistics of goodness are computed: the Root Mean Squares (RMS) of differences between observed and predicted values, as well as the Akaike Information Criterion (AIC). All calculated values indicate that the ECF estimates of NIN-ARCH process have lesser fitting errors, i.e. that the proposed model has higher efficiency. Some of these facts are illustrated in Fig. 1, which shows the dynamics of both actual time series. In this figure also are shown the empirical PDFs (histograms) of series A and B, along with PDFs obtained by fitting with CCF estimates. As it can be easily seen, in both cases, the NIN-ARCH process provides better matches with the appropriate empirical PDFs than the corresponding GARCH one.

5. **Conclusion**

In this paper, the novel nonlinear model of ARCH-type, named NIN-ARCH process, is developed. The model has been checked in fitting of an econometric and one physically-based actual data series. The estimation of the model's parameters has been performed using the CCF method in both cases. Thus obtained results show that NIN-ARCH process can be used for estimation and fitting the various kinds of non-linear time series. In the other words, the developed process can be an efficient tool for exploring the dynamics of real-based time series and their empirical analysis.

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