

APPROXIMATING THE FINITE HILBERT TRANSFORM VIA SIMPSON TYPE INEQUALITIES AND APPLICATIONS

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In this paper, some approximations for the finite Hilbert transform of different classes of absolutely continuous functions via Simpson type inequalities are given. Some error bounds and numerical examples for the obtained approximations are also provided. These estimates are smaller than those obtained via trapezoid type inequalities.

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1. Introduction

The finite Hilbert transform, which is the Cauchy Principal Value (CPV) integral over a finite interval, plays a great role in many areas of physics and engineering. Let's start by looking at the CPV integral of a function $f(\tau)$ over an interval (a, b) defined by

$$(Tf)(a, b; t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\tau - t} d\tau, \quad (1)$$

where PV has the usual meaning of the Cauchy principle value. When the interval (a, b) coincides with the interval $(-\infty, \infty)$, this integral is called the Hilbert transform of $f(\tau)$, which appears frequently in tomographic image reconstruction [20], signal processing, disordered alloys, magnetic properties of disordered alloys, Kramers-Kronig equations of dielectric theory, and phase determination in bright-field electron microscopy [14].

There are some important approaches for evaluating finite Hilbert transforms, such as the Gaussian, Chebyshev, TANH, Iri-Moriguti-Takasawa, and double exponential quadrature methods. In [4], Dragomir et al. proved the following inequalities for the finite Hilbert transform of different classes of absolutely continuous functions via trapezoid type inequalities.

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Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$. Then the following bounds*

$$\begin{aligned} & \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{1}{2\pi} [f(b) - f(a) + f'(t)(b-a)] \right| \\ & \leq \begin{cases} \frac{\|f''\|_\infty}{4\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right], & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q\|f''\|_p}{2\pi(q+1)^{\frac{q+1}{q}}} \left[(t-a)^{\frac{q+1}{q}} + (b-t)^{\frac{q+1}{q}} \right], & \text{if } f'' \in L_p[a, b]; \\ \frac{\|f''\|_1(b-a)}{2\pi}, & \text{if } f'' \in L_1[a, b]; \end{cases} \end{aligned} \quad (2)$$

$$\begin{aligned} & \leq \begin{cases} \frac{\|f''\|_\infty(b-a)^2}{8\pi}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{q\|f''\|_p(b-a)^{\frac{q+1}{q}}}{2\pi(q+1)^{\frac{q+1}{q}}}, & \text{if } f'' \in L_p[a, b]; \\ \frac{\|f''\|_1(b-a)}{2\pi}, & \text{if } f'' \in L_1[a, b]; \end{cases} \end{aligned} \quad (3)$$

hold, for all $t \in [a, b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_p$ are the usual Lebesgue norms in $L_p[a, b]$ ($1 \leq p \leq \infty$).

Theorem 1.2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping whose second derivative is absolutely continuous on $[a, b]$. Then we have the bounds*

$$\begin{aligned} & \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{1}{2\pi} [f(b) - f(a) + f'(t)(b-a)] \right| \\ & \leq \begin{cases} \frac{\|f'''\|_\infty}{12\pi} (b-a) \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right], & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q\|f'''\|_p[B(q+1, q+1)]^{\frac{1}{q}}}{2\pi(2q+1)} \left[(t-a)^{\frac{2q+1}{q}} + (b-t)^{\frac{2q+1}{q}} \right], & \text{if } f''' \in L_p[a, b]; \\ \frac{\|f'''\|_1}{8\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right], & \text{if } f''' \in L_1[a, b]; \end{cases} \end{aligned} \quad (4)$$

$$\begin{aligned} & \leq \begin{cases} \frac{\|f'''\|_\infty(b-a)^3}{36\pi}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q\|f'''\|_p[B(q+1, q+1)]^{\frac{1}{q}}}{2\pi(2q+1)} (b-a)^{\frac{2q+1}{q}}, & \text{if } f''' \in L_p[a, b]; \\ \frac{\|f'''\|_1}{16\pi} (b-a)^2, & \text{if } f''' \in L_1[a, b]; \end{cases} \end{aligned} \quad (5)$$

for all $t \in [a, b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_p$ are the usual p -norms and $B(\cdot, \cdot)$ is Euler's beta mapping

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha, \beta > 0.$$

Theorem 1.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$, and

$$\Gamma = \sup_{t \in (a, b)} f''(t) < \infty; \quad \gamma = \inf_{t \in (a, b)} f''(t) > -\infty, \quad \Gamma > \gamma. \quad (6)$$

Then we have the following bound for all $t \in (a, b)$:

$$\begin{aligned} & \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{1}{2\pi} [f(b) - f(a) + f'(t)(b-a)] \right| \\ & \leq \frac{\Gamma - \gamma}{8\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right] \leq \frac{(\Gamma - \gamma)(b-a)^2}{16\pi} \end{aligned} \quad (7)$$

For classical results on the finite Hilbert transform, see [4, 5, 6, 15]. An extensive literature such as [1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19] deal with Simpson type inequalities.

In this paper, motivated by [4], we shall point out some new inequalities for the finite Hilbert transform of different classes of absolutely continuous functions via Simpson type inequalities. Some error bounds and numerical examples for the obtained approximations will also be presented. These estimates are smaller than those obtained via trapezoid type inequalities.

2. Some error bounds

Theorem 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$. Then the following bounds

$$\begin{aligned} & \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right. \\ & \quad \left. - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right] \right| \\ & \leq \begin{cases} \frac{5\|f''\|_\infty}{36\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right], & \text{if } f'' \in L_\infty[a, b]; \\ \frac{2q\|f''\|_p \left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{q+1}{q}}} \left[(t-a)^{\frac{q+1}{q}} + (b-t)^{\frac{q+1}{q}} \right], & \text{if } f'' \in L_p[a, b]; \\ \frac{1}{3\pi} \|f''\|_1 (b-a), & \text{if } f'' \in L_1[a, b]; \end{cases} \end{aligned} \quad (8)$$

$$\leq \begin{cases} \frac{5\|f''\|_\infty(b-a)^2}{72\pi}, & \text{if } f'' \in L_\infty[a, b]; \\ \frac{2q\|f''\|_p \left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}}\right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{q+1}{q}}}(b-a)^{\frac{q+1}{q}}, & \text{if } f'' \in L_p[a, b]; \\ \frac{1}{3\pi}\|f''\|_1(b-a), & \text{if } f'' \in L_1[a, b] \end{cases} \quad (9)$$

hold, for all $t \in [a, b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_p$ ($1 \leq p \leq \infty$) are the usual Lebesgue norms in $L_p[a, b]$.

Proof. We start with the following elementary identity which can be proved using the integration by parts formula:

$$\begin{aligned} \int_\alpha^\beta g(u)du &= \frac{g(\alpha) + 4g\left(\frac{\alpha+\beta}{2}\right) + g(\beta)}{6}(\beta - \alpha) - \left[\int_\alpha^{\frac{\alpha+\beta}{2}} \left(u - \frac{5\alpha + \beta}{6}\right) g'(u)du \right. \\ &\quad \left. + \int_{\frac{\alpha+\beta}{2}}^\beta \left(u - \frac{\alpha + 5\beta}{6}\right) g'(u)du \right], \end{aligned} \quad (10)$$

provided that g is absolutely continuous on the interval $[\alpha, \beta]$, if $\alpha \leq \beta$ (or $[\beta, \alpha]$, if $\beta \leq \alpha$). As in [4], for the mapping $f : (a, b) \rightarrow \mathbb{R}$, and $f(t)=1$, $t \in (a, b)$, we can obtain the equalities

$$(Tf)(a, b; t) = \frac{1}{\pi} \ln \left(\frac{b-t}{t-a} \right), t \in (a, b),$$

$$\begin{aligned} (Tf)(a, b; t) &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau \\ &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{1}{\tau - t} d\tau, \end{aligned}$$

from which we get the equality

$$(Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau. \quad (11)$$

Now using (10), we obtain

$$\begin{aligned} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau &= PV \int_a^b \frac{\int_t^\tau f'(u)du}{\tau - t} d\tau \\ &= PV \int_a^b \frac{[f'(t) + 4f'\left(\frac{t+\tau}{2}\right) + f'(\tau)](\tau - t)}{6(\tau - t)} d\tau \\ &\quad - PV \int_a^b \frac{\int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6}\right) f''(u)du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6}\right) f''(u)du}{\tau - t} d\tau \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} PV \int_a^b \left[f'(t) + 4f' \left(\frac{t+\tau}{2} \right) + f'(\tau) \right] d\tau \\
&\quad - PV \int_a^b \frac{\int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right) f''(u) du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6} \right) f''(u) du}{\tau - t} d\tau \\
&= \frac{1}{6} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right] \\
&\quad - PV \int_a^b \frac{\int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right) f''(u) du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6} \right) f''(u) du}{\tau - t} d\tau,
\end{aligned}$$

and then, by (11), we can state the identity

$$\begin{aligned}
&(Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \\
&\quad - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right] \\
&= -\frac{1}{\pi} PV \int_a^b \frac{\int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right) f''(u) du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6} \right) f''(u) du}{\tau - t} d\tau. \quad (12)
\end{aligned}$$

Using the property of modulus, we have

$$\begin{aligned}
&\left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right. \\
&\quad \left. - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right] \right| \\
&\leq \frac{1}{\pi} PV \int_a^b \left| \frac{\int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right) f''(u) du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6} \right) f''(u) du}{\tau - t} \right| d\tau \\
&=: A(a, b; t). \quad (13)
\end{aligned}$$

Now, obviously

$$\begin{aligned}
&\left| \int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right) f''(u) du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6} \right) f''(u) du \right| \\
&\leq \sup_{u \in [a, b]} |f''(u)| \left(\left| \int_t^{\frac{t+\tau}{2}} \left| u - \frac{5t+\tau}{6} \right| du \right| + \left| \int_{\frac{t+\tau}{2}}^\tau \left| u - \frac{t+5\tau}{6} \right| du \right| \right) \\
&= \|f''\|_\infty \left(\frac{5|t-\tau|^2}{72} + \frac{5|t-\tau|^2}{72} \right) = \|f''\|_\infty \frac{5|t-\tau|^2}{36},
\end{aligned}$$

holds, for all $t, \tau \in (a, b)$. Then we have the following inequality

$$\begin{aligned}
 A(a, b; t) &\leq \frac{1}{\pi} PV \int_a^b \|f''\|_{\infty} \frac{5|t - \tau|}{36} d\tau \\
 &= \frac{5\|f''\|_{\infty}}{36\pi} \lim_{\varepsilon \rightarrow 0^+} \left[\int_a^{t-\varepsilon} (t - \tau) d\tau + \int_{t+\varepsilon}^b (\tau - t) d\tau \right] \\
 &= \frac{5\|f''\|_{\infty}}{36\pi} \lim_{\varepsilon \rightarrow 0^+} \left[\frac{(t-a)^2}{2} + \frac{\varepsilon^2}{2} + \frac{(b-t)^2}{2} - \frac{\varepsilon^2}{2} \right] \\
 &= \frac{5\|f''\|_{\infty}}{36\pi} \frac{(t-a)^2 + (b-t)^2}{2} \\
 &= \frac{5\|f''\|_{\infty}}{36\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right],
 \end{aligned}$$

and the first bound in (8) is proved.

Using Hölder's integral inequality, we can state for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that

$$\begin{aligned}
 &\left| \int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right) f''(u) du + \int_{\frac{t+\tau}{2}}^{\tau} \left(u - \frac{t+5\tau}{6} \right) f''(u) du \right| \\
 &\leq \left| \int_t^{\frac{t+\tau}{2}} |f''(u)|^p du \right|^{\frac{1}{p}} \left| \int_t^{\frac{t+\tau}{2}} \left| u - \frac{5t+\tau}{6} \right|^q du \right|^{\frac{1}{q}} \\
 &\quad + \left| \int_{\frac{t+\tau}{2}}^{\tau} |f''(u)|^p du \right|^{\frac{1}{p}} \left| \int_{\frac{t+\tau}{2}}^{\tau} \left| u - \frac{t+5\tau}{6} \right|^q du \right|^{\frac{1}{q}} \\
 &\leq \left(\int_a^b |f''(u)|^p du \right)^{\frac{1}{p}} \left| \int_t^{\frac{5t+\tau}{6}} \left(\frac{5t+\tau}{6} - u \right)^q du + \int_{\frac{5t+\tau}{6}}^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6} \right)^q du \right|^{\frac{1}{q}} \\
 &\quad + \left(\int_a^b |f''(u)|^p du \right)^{\frac{1}{p}} \left| \int_{\frac{t+\tau}{2}}^{\frac{t+5\tau}{6}} \left(\frac{t+5\tau}{6} - u \right)^q du + \int_{\frac{t+5\tau}{6}}^{\tau} \left(u - \frac{t+5\tau}{6} \right)^q du \right|^{\frac{1}{q}} \\
 &= \|f''\|_p \frac{2 \left(\left| \frac{t-\tau}{6} \right|^{q+1} + \left| \frac{t-\tau}{3} \right|^{q+1} \right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} = 2\|f''\|_p |t - \tau|^{\frac{q+1}{q}} \frac{\left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}},
 \end{aligned}$$

for all $t, \tau \in (a, b)$. Then

$$\begin{aligned}
 A(a, b; t) &\leq \frac{2}{\pi} PV \int_a^b \frac{\|f''\|_p |t - \tau|^{\frac{1}{q}} \left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} d\tau \\
 &= \frac{2}{\pi} \|f''\|_p \frac{\left(\frac{1}{6^{q+1}} + \frac{1}{3^{q+1}} \right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} PV \int_a^b |t - \tau|^{\frac{1}{q}} d\tau
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi} \|f''\|_p \frac{\left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \lim_{\varepsilon \rightarrow 0^+} \left[\int_a^{t-\varepsilon} (t-\tau)^{\frac{1}{q}} d\tau + \int_{t+\varepsilon}^b (\tau-t)^{\frac{1}{q}} d\tau \right] \\
&= \frac{2}{\pi} \|f''\|_p \frac{\left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \lim_{\varepsilon \rightarrow 0^+} \left[\frac{(t-a)^{\frac{q+1}{q}} - \varepsilon^{\frac{q+1}{q}}}{\frac{q+1}{q}} + \frac{(b-t)^{\frac{q+1}{q}} - \varepsilon^{\frac{q+1}{q}}}{\frac{q+1}{q}} \right] \\
&= \frac{2}{\pi} \|f''\|_p \frac{\left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \left[\frac{(t-a)^{\frac{q+1}{q}} + (b-t)^{\frac{q+1}{q}}}{\frac{q+1}{q}} \right] \\
&= \frac{2q \|f''\|_p \left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{\pi (q+1)^{\frac{q+1}{q}}} \left[(t-a)^{\frac{q+1}{q}} + (b-t)^{\frac{q+1}{q}} \right],
\end{aligned}$$

and the second bound in (8) is proved. Finally, we can observe that

$$\begin{aligned}
&\left| \int_t^{\frac{t+\tau}{2}} \left(u - \frac{5t+\tau}{6}\right) f''(u) du + \int_{\frac{t+\tau}{2}}^\tau \left(u - \frac{t+5\tau}{6}\right) f''(u) du \right| \\
&\leq \max \left\{ \sup_{u \in [t, \frac{t+\tau}{2}]} \left| u - \frac{5t+\tau}{6} \right|, \sup_{u \in [\frac{t+\tau}{2}, \tau]} \left| u - \frac{t+5\tau}{6} \right| \right\} \int_t^\tau |f''(u)| du \\
&\leq \frac{|t-\tau|}{3} \int_a^b |f''(u)| du \\
&= \frac{|t-\tau|}{3} \|f''\|_1.
\end{aligned}$$

for all $t \in (a, b)$ Consequently,

$$A(a, b; t) \leq \frac{1}{\pi} PV \int_a^b \frac{1}{3} \|f''\|_1 d\tau = \frac{1}{3\pi} \|f''\|_1 (b-a),$$

and the theorem is proved. $\square$

Remark 2.1. We note that inequality (8) we obtained here gives a smaller error bound than that of (2).

The best inequality we can obtain from (8) is the one for $t = \frac{a+b}{2}$, and then, we can get the following corollary.

Corollary 2.1. With the assumptions of Theorem 2.1, we have

$$\begin{aligned}
&\left| (Tf) \left(a, b; \frac{a+b}{2}\right) - \frac{1}{6\pi} \left[f(b) - f(a) + 8f\left(\frac{a+3b}{4}\right) \right. \right. \\
&\quad \left. \left. - 8f\left(\frac{3a+b}{4}\right) + f'\left(\frac{a+b}{2}\right) (b-a) \right] \right|
\end{aligned}$$

$$\leq \begin{cases} \frac{5\|f''\|_{\infty}(b-a)^2}{144\pi}, & \text{if } f'' \in L_{\infty}[a, b]; \\ \frac{2q\|f''\|_p \left(\frac{1}{6q+1} + \frac{1}{3q+1}\right)^{\frac{1}{q}}}{\pi(q+1)^{\frac{q+1}{q}} 2^{\frac{1}{q}}} (b-a)^{\frac{q+1}{q}}, & \text{if } f'' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (14)$$

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping whose second derivative is absolutely continuous on $[a, b]$. Then we have the bounds

$$\begin{aligned} & \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right. \\ & \quad \left. - \frac{1}{6\pi} \left[f(b) - f(a) + 8f\left(\frac{t+b}{2}\right) - 8f\left(\frac{t+a}{2}\right) + f'(t)(b-a) \right] \right| \\ & \leq \begin{cases} \frac{\|f'''\|_{\infty}(b-a)}{81\pi} \left[\left(t - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \right], & \text{if } f''' \in L_{\infty}[a, b]; \\ \frac{q\|f'''\|_p}{2\pi(2q+1)} \left[(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}} \right] \Lambda, & \text{if } f''' \in L_p[a, b]; \end{cases} \quad (15) \\ & \quad \frac{\|f'''\|_1}{24\pi} \left[\left(t - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{4} \right], & \text{if } f''' \in L_1[a, b], \end{cases} \end{aligned}$$

$$\leq \begin{cases} \frac{\|f'''\|_{\infty}(b-a)^3}{243\pi}, & \text{if } f''' \in L_{\infty}[a, b]; \\ \frac{q\|f'''\|_p}{2\pi(2q+1)} (b-a)^{2+\frac{1}{q}} \Lambda, & \text{if } f''' \in L_p[a, b]; \\ \frac{\|f'''\|_1}{48\pi} (b-a)^2, & \text{if } f''' \in L_1[a, b], \end{cases} \quad (16)$$

for all $t \in [a, b]$, where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_p$ ($1 \leq p \leq \infty$) are the usual p -norms and

$$\begin{aligned} \Lambda = & \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Using the integration by parts formula, we can get the equality

$$\int_{\alpha}^{\beta} g(u) du = \frac{g(\alpha) + 4g\left(\frac{\alpha+\beta}{2}\right) + g(\beta)}{6} (\beta - \alpha)$$

$$\begin{aligned}
& -\frac{1}{2} \int_{\alpha}^{\frac{\alpha+\beta}{2}} (u-\alpha) \left(u - \frac{2\alpha+\beta}{3}\right) g''(u) du \\
& -\frac{1}{2} \int_{\frac{\alpha+\beta}{2}}^{\beta} (u-\beta) \left(u - \frac{\alpha+2\beta}{3}\right) g''(u) du, \tag{17}
\end{aligned}$$

where g is such that g' is absolutely continuous on $[\alpha, \beta]$ (if $\alpha < \beta$), or on $[\beta, \alpha]$ (if $\beta < \alpha$). By a similar procedure to that in Theorem 2.1, we have

$$\begin{aligned}
& (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \\
& - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right] \\
& = -\frac{1}{2\pi} PV \int_a^b \frac{\int_t^{\frac{t+\tau}{2}} (u-t) \left(u - \frac{2t+\tau}{3}\right) f'''(u) du + \int_{\frac{t+\tau}{2}}^{\tau} (u-\tau) \left(u - \frac{t+2\tau}{3}\right) f'''(u) du}{\tau-t} d\tau. \tag{18}
\end{aligned}$$

Using the property of modulus, we can obtain

$$\begin{aligned}
& \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right. \\
& \quad \left. - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right] \right| \\
& \leq \frac{1}{2\pi} PV \int_a^b \left| \frac{\int_t^{\frac{t+\tau}{2}} (u-t) \left(u - \frac{2t+\tau}{3}\right) f'''(u) du + \int_{\frac{t+\tau}{2}}^{\tau} (u-\tau) \left(u - \frac{t+2\tau}{3}\right) f'''(u) du}{\tau-t} \right| d\tau \\
& =: B(a, b; t).
\end{aligned}$$

First, let's observe that

$$\begin{aligned}
& \left| \int_t^{\frac{t+\tau}{2}} (u-t) \left(u - \frac{2t+\tau}{3}\right) f'''(u) du + \int_{\frac{t+\tau}{2}}^{\tau} (u-\tau) \left(u - \frac{t+2\tau}{3}\right) f'''(u) du \right| \\
& \leq \sup_{u \in [t, \tau]} |f'''(u)| \left(\left| \int_t^{\frac{t+\tau}{2}} |u-t| \left| u - \frac{2t+\tau}{3} \right| du \right| + \left| \int_{\frac{t+\tau}{2}}^{\tau} |u-\tau| \left| u - \frac{t+2\tau}{3} \right| du \right| \right) \\
& \leq \|f'''\|_{\infty} \frac{2|t-\tau|^3}{81},
\end{aligned}$$

and then

$$\begin{aligned}
B(a, b; t) & \leq \frac{2}{81} \frac{\|f'''\|_{\infty}}{2\pi} PV \int_a^b |t-\tau|^2 d\tau \\
& = \frac{\|f'''\|_{\infty}}{81\pi} \frac{(b-t)^3 + (t-a)^3}{3}
\end{aligned}$$

$$= \frac{\|f'''\|_\infty(b-a)}{81\pi} \left[\frac{(b-a)^2}{12} + \left(t - \frac{a+b}{2} \right)^2 \right],$$

which proves the first part of (15).

For the second part, we apply Hölder's integral inequality to obtain

$$\begin{aligned} & \left| \int_t^{\frac{t+\tau}{2}} (u-t) \left(u - \frac{2t+\tau}{3} \right) f'''(u) du + \int_{\frac{t+\tau}{2}}^\tau (u-\tau) \left(u - \frac{t+2\tau}{3} \right) f'''(u) du \right| \\ & \leq \left| \int_t^\tau |f'''(u)|^p du \right|^{\frac{1}{p}} \left| \int_t^{\frac{t+\tau}{2}} |u-t|^q \left| u - \frac{2t+\tau}{3} \right|^q du \right|^{\frac{1}{q}} \\ & \quad + \left| \int_t^\tau |f'''(u)|^p du \right|^{\frac{1}{p}} \left| \int_{\frac{t+\tau}{2}}^\tau |u-\tau|^q \left| u - \frac{t+2\tau}{3} \right|^q du \right|^{\frac{1}{q}} \\ & \leq \left(\int_a^b |f'''(u)|^p du \right)^{\frac{1}{p}} |t-\tau|^{2+\frac{1}{q}} \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right] \\ & = \|f'''\|_p |t-\tau|^{2+\frac{1}{q}} \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right], \end{aligned}$$

for all $t, \tau \in (a, b)$. Indeed, if we assume that $\tau > t$, then by $u = (1-s)t + s\tau$, $s \in [0, 1]$, we can obtain the following equalities

$$\begin{aligned} & \int_t^{\frac{t+\tau}{2}} |u-t|^q \left| u - \frac{2t+\tau}{3} \right|^q du \\ & = \int_t^{\frac{2t+\tau}{3}} (u-t)^q \left(\frac{2t+\tau}{3} - u \right)^q du + \int_{\frac{2t+\tau}{3}}^{\frac{t+\tau}{2}} (u-t)^q \left(u - \frac{2t+\tau}{3} \right)^q du \\ & = |t-\tau|^{2q+1} \left[\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right], \end{aligned}$$

$$\int_{\frac{t+\tau}{2}}^\tau |u-\tau|^q \left| u - \frac{t+2\tau}{3} \right|^q du$$

$$\begin{aligned}
&= \int_{\frac{t+\tau}{2}}^{\frac{t+2\tau}{3}} (\tau - u)^q \left(\frac{t+2\tau}{3} - u \right)^q du + \int_{\frac{t+2\tau}{3}}^{\tau} (\tau - u)^q \left(u - \frac{t+2\tau}{3} \right)^q du \\
&= |t - \tau|^{2q+1} \left[\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right].
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&B(a, b; t) \\
&\leq \frac{\|f'''\|_p}{2\pi} PV \int_a^b |t - \tau|^{1+\frac{1}{q}} d\tau \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right] \\
&= \frac{\|f'''\|_p}{2\pi} \frac{(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}}}{2+\frac{1}{q}} \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right] \\
&= \frac{q\|f'''\|_p}{2\pi(2q+1)} \left[(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}} \right] \\
&\quad \times \left[\left(\int_0^{\frac{1}{3}} s^q \left(\frac{1}{3} - s \right)^q ds + \int_{\frac{1}{3}}^{\frac{1}{2}} s^q \left(s - \frac{1}{3} \right)^q ds \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^{\frac{2}{3}} (1-s)^q \left(\frac{2}{3} - s \right)^q ds + \int_{\frac{2}{3}}^1 (1-s)^q \left(s - \frac{2}{3} \right)^q ds \right)^{\frac{1}{q}} \right],
\end{aligned}$$

and the second part of (15) holds. For the last part, we observe that

$$\begin{aligned}
&\left| \int_t^{\frac{t+\tau}{2}} (u-t) \left(u - \frac{2t+\tau}{3} \right) f'''(u) du + \int_{\frac{t+\tau}{2}}^{\tau} (u-\tau) \left(u - \frac{t+2\tau}{3} \right) f'''(u) du \right| \\
&\leq \max \left\{ \sup_{u \in [t, \frac{t+\tau}{2}]} \left| (u-t) \left(u - \frac{2t+\tau}{3} \right) \right|, \sup_{u \in [\frac{t+\tau}{2}, \tau]} \left| (u-\tau) \left(u - \frac{t+2\tau}{3} \right) \right| \right\} \\
&\quad \times \int_t^{\tau} |f'''(u)| du \leq \frac{|t-\tau|^2}{12} \int_a^b |f'''(u)| du = \|f'''\|_1 \frac{|t-\tau|^2}{12},
\end{aligned}$$

since two simple calculations show that

$$\sup_{u \in [t, \frac{t+\tau}{2}]} \left| (u-t) \left(u - \frac{2t+\tau}{3} \right) \right| = \frac{|t-\tau|^2}{12},$$

$$\sup_{u \in [\frac{t+\tau}{2}, \tau]} \left| (u - \tau) \left(u - \frac{t + 2\tau}{3} \right) \right| = \frac{|t - \tau|^2}{12}.$$

Thus, we can get the following inequality:

$$\begin{aligned} B(a, b; t) &\leq \frac{\|f'''\|_1}{24\pi} PV \int_a^b |t - \tau| d\tau \\ &= \frac{\|f'''\|_1}{24\pi} \frac{(b - t)^2 + (t - a)^2}{2} \\ &= \frac{\|f'''\|_1}{24\pi} \left[\left(t - \frac{a + b}{2} \right)^2 + \frac{(b - a)^2}{4} \right], \end{aligned}$$

and the theorem is proved. $\square$

Remark 2.2. We note that inequality (15) we obtained here gives a smaller error bound than that of (4).

Taking into account the fact that all the mappings depending on t from the right-hand side of (15) are convex on the interval (a, b) , it is obvious that the best estimator from (15) is the following one for which $t = \frac{a+b}{2}$.

Corollary 2.2. Let f be as in Theorem 2.2. Then we can obtain the inequality

$$\begin{aligned} &\left| (Tf) \left(a, b; \frac{a+b}{2} \right) - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{a+3b}{4} \right) \right. \right. \\ &\quad \left. \left. - 8f \left(\frac{3a+b}{4} \right) + f' \left(\frac{a+b}{2} \right) (b-a) \right] \right| \\ &\leq \begin{cases} \frac{\|f'''\|_\infty (b-a)^3}{972\pi}, & \text{if } f''' \in L_\infty[a, b]; \\ \frac{q\|f'''\|_p}{2^{2+\frac{1}{q}}\pi(2q+1)} (b-a)^{2+\frac{1}{q}} \Lambda, & \text{if } f''' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{\|f'''\|_1 (b-a)^2}{96\pi}, & \text{if } f''' \in L_1[a, b]. \end{cases} \quad (19) \end{aligned}$$

3. Further error bounds

In [2], Bencze and Zhao proved the following Simpson type inequality:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b g(x) dx - \frac{g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)}{6} \right| \\ &\leq \frac{M-m}{2(b-a)} \left(\int_a^{\frac{a+b}{2}} \left| x - \frac{5a+b}{6} \right| dx + \int_{\frac{a+b}{2}}^b \left| x - \frac{a+5b}{6} \right| dx \right) \\ &= \frac{5}{72} (M-m)(b-a), \end{aligned} \quad (20)$$

provided that $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and $M = \sup_{x \in (a, b)} g'(x) < \infty$, $m = \inf_{x \in (a, b)} g'(x) > -\infty$, and $M > m$.

Using the above inequality, we can get the following theorem.

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[a, b]$, and*

$$\Gamma = \sup_{t \in (a, b)} f''(t) < \infty; \quad \gamma = \inf_{t \in (a, b)} f''(t) > -\infty, \quad \Gamma > \gamma. \quad (21)$$

Then we have the bound

$$\begin{aligned} & \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right. \\ & \quad \left. - \frac{1}{6\pi} \left[f(b) - f(a) + 8f\left(\frac{t+b}{2}\right) - 8f\left(\frac{t+a}{2}\right) + f'(t)(b-a) \right] \right| \\ & \leq \frac{5(\Gamma - \gamma)}{72\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right] \\ & \leq \frac{5(\Gamma - \gamma)(b-a)^2}{144\pi}, \end{aligned} \quad (22)$$

for all $t \in (a, b)$.

Proof. Applying the inequality (20) written for $f'(\cdot)$ in the following version:

$$\left| \frac{1}{\tau - t} \int_t^\tau f'(u) du - \frac{f'(t) + 4f'\left(\frac{t+\tau}{2}\right) + f'(\tau)}{6} \right| \leq \frac{5}{72}(\Gamma - \gamma)|t - \tau|,$$

we can state the inequality

$$\left| \frac{f(\tau) - f(t)}{\tau - t} - \frac{f'(t) + 4f'\left(\frac{t+\tau}{2}\right) + f'(\tau)}{6} \right| \leq \frac{5}{72}(\Gamma - \gamma)|t - \tau|, \quad (23)$$

for all $t, \tau \in [a, b]$, $t \neq \tau$. The following property of the Cauchy-principle value follows by the properties of integral, modulus, and limit:

$$\left| PV \int_a^b A(t, s) ds \right| \leq PV \int_a^b |A(t, s)| ds \quad (24)$$

holds, assuming that the PV involved exist for all $t \in (a, b)$. Using (23) and (24), we can obtain

$$\begin{aligned} & \left| PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - PV \int_a^b \frac{f'(t) + 4f'\left(\frac{t+\tau}{2}\right) + f'(\tau)}{6} d\tau \right| \\ & \leq PV \int_a^b \left| \frac{f(\tau) - f(t)}{\tau - t} - \frac{f'(t) + 4f'\left(\frac{t+\tau}{2}\right) + f'(\tau)}{6} \right| d\tau \end{aligned}$$

$$\leq PV \int_a^b \frac{5}{72} (\Gamma - \gamma) |t - \tau| d\tau, \quad (25)$$

and since that

$$\begin{aligned} \frac{1}{\pi} \int_a^b \frac{f(\tau) - f(t)}{\tau - t} &= (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right), \\ \frac{1}{\pi} \int_a^b \frac{f'(t) + 4f' \left(\frac{t+\tau}{2} \right) + f'(\tau)}{6} d\tau \\ &= \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right], \end{aligned}$$

and as

$$\frac{1}{\pi} \int_a^b |t - \tau| d\tau = \frac{(t-a)^2 + (b-t)^2}{2\pi} = \frac{1}{\pi} \left[\left(t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right],$$

then by (25) we deduce the desired inequality (22). $\square$

Remark 3.1. We note that inequality (22) we obtained here gives a smaller error bound than that of (7).

The midpoint $t = \frac{a+b}{2}$ gives the best estimator, getting the result below:

Corollary 3.1. With the above assumptions, we have the following inequality:

$$\begin{aligned} \left| (Tf) \left(a, b; \frac{a+b}{2} \right) - \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{a+3b}{4} \right) \right. \right. \\ \left. \left. - 8f \left(\frac{3a+b}{4} \right) + f' \left(\frac{a+b}{2} \right) (b-a) \right] \right| \leq \frac{5(\Gamma - \gamma)(b-a)^2}{288\pi}. \end{aligned}$$

4. Some numerical examples

For a differentiable function $f : (a, b) \rightarrow \mathbb{R}$, consider the expression

$$\begin{aligned} E(f; a, b, t) &= \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + \frac{1}{6\pi} \left[f(b) - f(a) + 8f \left(\frac{t+b}{2} \right) \right. \\ &\quad \left. - 8f \left(\frac{t+a}{2} \right) + f'(t)(b-a) \right]. \end{aligned} \quad (26)$$

As shown in Theorem 2.1-3.1, $E(f; a, b, t)$ provides an approximation for the finite Hilbert transform $T(f)(a, b, t)$. We remark that for small intervals, i.e., $(b-a) \rightarrow 0$, the approximation is accurate (with the order two or three, etc.).

If we consider the function $f : [1, 2] \rightarrow \mathbb{R}$, $f(x) = \sqrt{x^2 + 1}$ then the plots of $E(f; a, b, t)$ and $T(f)(a, b, t)$ are very close to each other (see Figure 1). The second plot in Figure 2 will show the magnitude of the closeness.

If we consider the function $f : [2.5, 3] \rightarrow \mathbb{R}$, $f(x) = \sin x$, then Figure 3 contains the plots of both $E(f; a, b, t)$ and $T(f)(a, b, t)$, while Figure 4 shows the magnitude of the closeness.

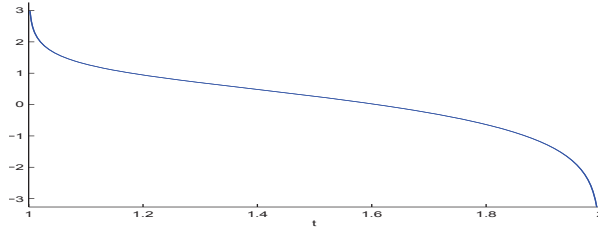


FIGURE 1. the plots of $E(f; a, b, t)$ and $(Tf)(a, b, t)$ on $[1, 2]$, when $f(x) = \sqrt{x^2 + 1}$.

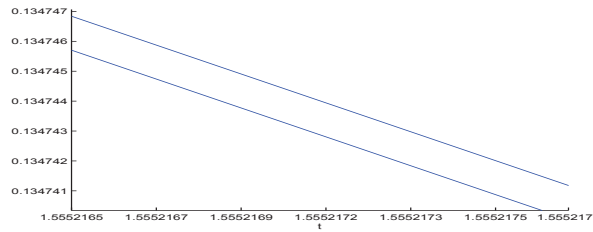


FIGURE 2. the plots of $E(f; a, b, t)$ and $(Tf)(a, b, t)$ on $[1.5552165, 1.5552177]$, when $f(x) = \sqrt{x^2 + 1}$.

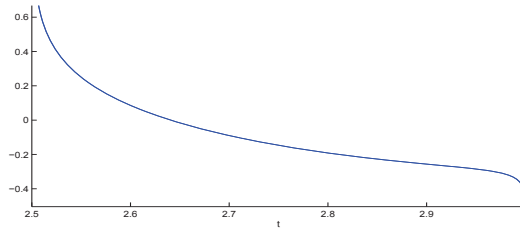


FIGURE 3. the plots of $E(f; a, b, t)$ and $(Tf)(a, b, t)$ on $[1, 2]$, when $f(x) = \sin x$.

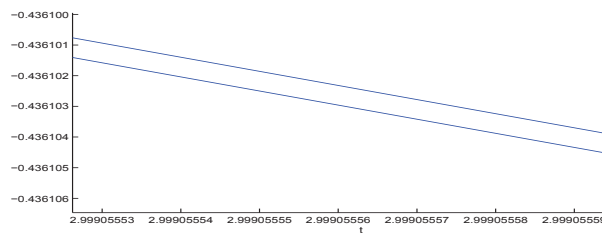


FIGURE 4. the plots of $E(f; a, b, t)$ and $(Tf)(a, b, t)$ on $[2.99905553, 2.99905559]$, when $f(x) = \sin x$.

With the above two numerical examples, we can observe that the obtained inequalities give more accurate estimates than those have been proved in previous literature [4], in which the inequalities were obtained via trapezoid type inequalities. For example, Figure 2 shows that the magnitude of the closeness is smaller than that of [4].

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