U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 2, 2015 ISSN 1223-7027

SUPER d-ANTIMAGIC LABELING OF UNIFORM SUBDIVISION OF WHEEL [*]

Muhammad IMRAN[1], Muhammad Kamran SIDDIQUI[2], Muhammad NUMAN[3]

This paper deals with the problem of labeling the vertices, edges and faces of uniform subdivision of wheel. A weight of a face is the sum of the label of a face and the labels of the vertices and edges surrounding that face. In a super d-antimagic labeling the vertices receive the smallest labels and the weights of all s-sided faces constitute an arithmetic progression of difference d, for each s that appearing in the graph.

The paper examines the existence of super d -antimagic labelings of type $(1,1,1)$ *for uniform subdivision of wheel for certain differences d.*

Keywords : plane graph, *d*-antimagic labeling, uniform subdivision of wheel.
2010 Mathematics Subject Classification: Primary 05C78; Secondary 05C38

1. Introduction and Definitions

Let $G=(V,E,F)$ be a finite connected plane graph without loops and multiple edges, where $V(G)$, $E(G)$ and $F(G)$ are its vertex set, edge set and face set, respectively. General reference for graph-theoretic notions is [16].

A labeling of *type* $(1,1,1)$ assigns labels from the set $\{1,2,\ldots,|V(G)|+|E(G)|+|F(G)|\}$ to the vertices, edges and faces of plane graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label.

The *weight* of a face under a labeling of type $(1,1,1)$ is the sum of labels carried by that face and all the edges and vertices surrounding it.

A labeling of type $(1,1,1)$ of a plane graph G is called d*-antimagic* if for every positive integer s the set of weights of all s-sided faces is

[1]Assistant Professor, Department of Matheatics, School of Naatural Sciences (SNS), National University of Sciences and Technology (NUST), Sector H-12, Islamabad, Pakistan. Email: imrandhab@gmail.com
[2] PhD Scholar, Comsats Institute of Information Technology, Sahiwal, Pakistan .
Email: kamransiddiqui75@gmail.com
[3] PhD Scholar, Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan .
Email: numantng@gmail.com
[*] This research is partially supported by NUST Islamabad and Higher Education Commission of Pakistan.

$W_s = \{a_s, a_s + d, a_s + 2d, \dots, a_s + (f_s - 1)d\}$ for some integers a_s and $d \geq 0$, where f_s is the number of the s-sided faces. We allow different sets W_s for different s.
Somewhat related types of antimagic labelings were defined by Hartsfield and Ringel in [10] and by Bodendiek and Walther in [9].
The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit 1design, network design and communication design.
The concept of the d-antimagic labeling of plane graphs was defined in [7]. This labeling is a natural extension of the notion of *magic labeling* (i.e. 0-antimagic labeling) defined by Lih [12].
The d-antimagic labelings of type $(1,1,1)$ for the hexagonal planar maps, generalized Petersen graph $P(n,2)$ and grids can be found in [3], [5] and [6], respectively. Lin *et al.* [13] and Sugeng *et al.* [14] described d-antimagic labelings of type $(1,1,1)$ for prisms D_n.
A d-antimagic labeling is called *super* if the smallest possible labels appear on the vertices. The super d-antimagic labelings of type $(1,1,1)$ for antiprisms and for disjoint union of prisms are presented in [2] and [1]. The existence of super d-antimagic labeling of type $(1,1,1)$ for the plane graphs containing a special Hamilton path is examined in [4] and super d-antimagic labelings of type $(1,1,1)$ for disconnected plane graphs are investigated in [8].
In this paper we examine the existence of super d-antimagic labelings for uniform subdivision of wheel. The concept of uniform subdivision of wheel is given in [15].

2. Uniform subdivision of wheel

The wheel W_n, consist of a cycle C_n and a new vertex c called central vertex or the hub, adjacent with all vertices of C_n. Let $W_{n,m}$, $n \geq 3$, $m \geq 1$ be the graph obtained by inserting m new vertices on each edge of W_n or equivalently by subdividing each edge of W_n with m vertices. The graph $W_{n,m}$, $n \geq 3$, $m \geq 1$, has n vertices of degree 3, $2mn$ vertices of degree 2, one vertex of degree n, $2n(m+1)$ edges, n internal $3m+3$-sided faces and one external $mn+n$-side face. Let the vertex set, edge set and face set of $W_{n,m}$ be defined as follows (see Figure 1).

$$V(W_{n,m}) = \{c, x_i, y_j^i, z_j^i : 1 \leq i \leq n, 1 \leq j \leq m\},$$

$$E(W_{n,m}) = \{x_i y_1^i, cz_1^i, x_i z_m^i : 1 \le i \le n\} \cup \{y_j^i y_{j+1}^i, z_j^i z_{j+1}^i : 1 \le i \le n, 1 \le j \le m-1\}$$
$$\cup \{x_{i+1} y_m^i ; 1 \le i \le n-1\} \cup \{x_1 y_m^n\},$$
$$F(W_{n,m}) = \{f_1, f_2, \ldots, f_n, f_{ext}\}.$$

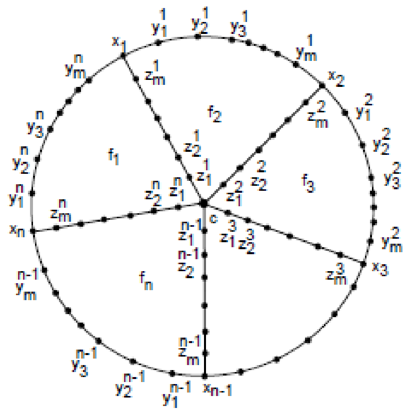


Fig 1. The graph of $W_{n,m}$

Note that under the study of d-antimagic labeling of $W_{n,m}$ we will consider a weight of the external $mn+n$-side face only if $m=1$ and $n=3$. In other cases the external $mn+n$-side face is unified and comparing its weight to weights of internal $3m+3$-sided faces is without sense. Let us denote

$$\begin{aligned} wt_d(f_{i+1}) \;&= \varphi_d(c) + \sum_{j=1}^{m}[\varphi_d(z_j^i) + \varphi_d(z_j^{i+1}) + \varphi_d(y_j^i)] + \varphi_d(x_i) + \varphi_d(x_{i+1}) \\ &+ \varphi_d(cz_1^i) + \varphi_d(x_i z_m^i) + \varphi_d(x_i y_1^i) + \varphi_d(x_{i+1} y_m^i) + \varphi_d(x_{i+1} z_m^{i+1}) \\ &+ \varphi_d(f_{i+1}) + \quad \varphi_d(cz_1^{i+1}) + \sum_{j=1}^{m-1}[\varphi_d(z_j^i z_{j+1}^i) + \varphi_d(z_j^{i+1} z_{j+1}^{i+1}) + \varphi_d(y_j^i y_{j+1}^i)] \end{aligned}$$

as a weight of the internal $3m+3$-sided face f_{i+1}, $1 \le i \le n-1$, and

$$\begin{aligned} wt_d(f_1) \;&= \varphi_d(c) + \sum_{j=1}^{m}[\varphi_d(z_j^n) + \varphi_d(z_j^1) + \varphi_d(y_j^n)] + \varphi_d(x_n) + \varphi_d(x_1) \\ &+ \varphi_d(cz_1^n) + \varphi_d(x_n z_m^n) + \varphi_d(x_n y_1^n) + \varphi_d(x_1 y_m^n) + \varphi_d(x_1 z_m^1) \\ &+ \varphi_d(cz_1^1) + \varphi_d(f_1)) + \sum_{j=1}^{m-1}[\varphi_d(z_j^n z_{j+1}^n) + \varphi_d(z_j^1 z_{j+1}^1) + \varphi_d(y_j^n y_{j+1}^n)] \end{aligned}$$

as a weight of the internal $3m+3$-sided face f_1 and

$$wt_d(f_{ext}) = \sum_{i=1}^{n}[\varphi_d(x_i)+\varphi_d(x_i y_1^i)] + \sum_{j=1}^{m}\sum_{i=1}^{n}\varphi_d(y_j^i) + \varphi_d(x_1 y_m^n)$$
$$+\sum_{i=1}^{n-1}\varphi_d(x_{i+1}y_m^i) + \sum_{j=1}^{m-1}\sum_{i=1}^{n}\varphi_d(y_j^i y_{j+1}^i) + \varphi_d(f_{ext})$$

as a weight of the external $mn+n$-sided face.

The first theorem shows the existence of a super d-antimagic labeling of type $(1,1,1)$ for $W_{n,1}$.

Lemma. 1. 1 *For* $d \in \{0,1,2,3,4\}$, *the graph* $W_{3,1}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. Figure 2 shows the required result for $n=3$, and $d \in \{0,1,2,3,4\}$.

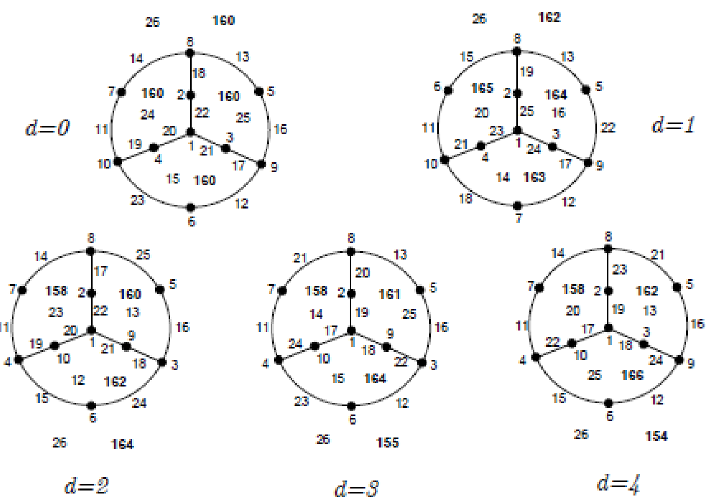


Fig 2. The graphs of $W_{3,1}$ with $d \in \{0,1,2,3,4\}$

Theorem. 1. 2 *For* $d \in \{0,1,2,3,4\}$ *and for every* $n \geq 4$, *the graph* $W_{n,1}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. First we define labelings φ_d of type $(1,1,1)$ which assign labels from the set $\{1,2,\dots,8n+2\}$ to the vertices, edges and faces of the graph $W_{n,1}$.

For $1 \leq i \leq n$,

$\varphi_d(c)=1$, $\varphi_d(x_i)=1+i$, $\varphi_d(y_1^i)=n+1+i$, $\varphi_d(z_1^i)=2n+1+i$,

$\varphi_d(x_i y_1^i)=4n+2-i$, $\varphi_d(x_1 y_1^n)=4n+2$, $\varphi_d(f_{ext})=8n+2$,

$\varphi_d(x_{i+1}y_1^i)=5n+2-i$, for $1 \leq i \leq n-1$,

$\varphi_0(cz_1^i)=\varphi_2(cz_1^i)=7n+2-i$, for $1 \leq i \leq n$,

$\varphi_0(x_i z_1^i) = \begin{cases} 6n+1-i, & if\ 1 \le i \le n-1 \\ 6n+1, & if\ i = n \end{cases}$; $\varphi_0(f_i) = \begin{cases} 7n+2+i, & if\ 1 \le i \le n-1 \\ 7n+2, & if\ i = n \end{cases}$

$\varphi_1(cz_1^i) = 8n+2-i$, for $1 \le i \le n$, $\varphi_1(x_i z_1^i) = \begin{cases} 7n-2i, & if\ 1 \le i \le n-1 \\ 7n, & if\ i = n \end{cases}$

$\varphi_1(f_i) = \begin{cases} 5n+3+2i, & if\ 1 \le i \le n-1 \\ 5n+3, & if\ i = n \end{cases}$; $\varphi_2(x_i z_1^i) = 5n+1+i$, for $1 \le i \le n$,

$\varphi_2(f_i) = \begin{cases} 7n+2, & if\ i = 1 \\ 8n+3-i, & if\ 2 \le i \le n \end{cases}$; $\varphi_3(cz_1^i) = \varphi_4(cz_1^i) = 6n+2-i$,

$\varphi_3(x_i z_1^i) = 6n+2i$, for $1 \le i \le n$, $\varphi_3(f_i) = \begin{cases} 6n+3, & if\ i = 1 \\ 8n+5-2i, & if\ 2 \le i \le n \end{cases}$

$\varphi_4(x_i z_1^i) = \begin{cases} 7n+1+i, & if\ 1 \le i \le n-1 \\ 7n+1, & if\ i = n \end{cases}$; $\varphi_4(f_i) = \begin{cases} 6n+1+i, & if\ 1 \le i \le n-1 \\ 8n+1, & if\ i = n \end{cases}$

It is easy to verify that the weights of the internal $3m+3$-sided faces for $1 \le i \le n$, form an arithmetic sequence with the desired differences: $wt_0(f_i) = 47n+19$, $wt_1(f_i) = 49n+19-i$, $wt_2(f_i) = 46n+18+2i$, $wt_3(f_i) = 46n+17+3i$, $wt_4(f_i) = 46n+16+4i$.

This completes the proof. □

Theorem. 2. 3 *For every* $n \ge 3$*, the graph* $W_{n,1}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$ *for* $d \in \{5,6\}$.

Proof. First we define labelings φ_d, which assign labels from the set $\{1,2,\ldots,8n+2\}$ to the vertices, edges and faces of the graph $W_{n,1}$. For $1 \le i \le n$, $\varphi_d(c) = 1$, $\varphi_d(x_i) = 1+i$, $\varphi_d(cz_1^i) = 6n+2-i$, $\varphi_d(x_1 y_1^n) = 4n+2$, $\varphi_d(f_{ext}) = 8n+2$, $\varphi_d(x_{i+1} y_1^i) = 5n+2-i$, for $1 \le i \le n-1$,

$\varphi_d(y_1^i) = \begin{cases} 2n+2, & if\ i = 1 \\ n+1+i, & if\ 2 \le i \le n \end{cases}$ $\varphi_d(z_1^i) = \begin{cases} n+2, & if\ i = 1 \\ 2n+1+i, & if\ 2 \le i \le n \end{cases}$

$\varphi_d(x_i y_1^i) = \begin{cases} 3n+2+i, & if\ 1 \le i \le n-1 \\ 3n+2, & if\ i = n \end{cases}$; $\varphi_5(x_i z_1^i) = 6n+2i$, for $1 \le i \le n$,

$\varphi_5(f_i) = \begin{cases} 6n+3, & if\ i = 1 \\ 8n+5-2i, & if\ 2 \le i \le n \end{cases}$ $\varphi_6(x_i z_1^i) = \begin{cases} 7n+1+i, & if\ 1 \le i \le n-1 \\ 7n+1, & if\ i = n \end{cases}$

$\varphi_6(f_i) = \begin{cases} 6n+1+i, & if\ 1 \le i \le n-1 \\ 8n+1, & if\ i = n \end{cases}$

Case $d=5$.

In the labeling φ_5 the weights for the 6-sided faces constitute an arithmetic progression $45n+20, 45n+25,\dots,50n+15$. However if $n=3$, $wt_5(f_{ext})$ is not be a member of the arithmetic progression with difference $d=5$ of the weights of 6-sided faces. So if we swap the vertex value $\varphi_5(x_2)=3$ with the the vertex value $\varphi_5(z_1^2)=9$, and the edge value $\varphi_5(x_1y_1^n)=14$, with the face value $\varphi_5(f_1)=21$, then the weights of 6-sided faces form the arithmetic sequence with difference 5.

Case $d=6$.

In the labeling φ_6 the weights for the 6-sided faces constitute an arithmetic progression $45n+20, 45n+26,\dots,51n+14$. However if $n=3$, $wt_6(f_{ext})$ is not be a member of the arithmetic progression with difference $d=6$ of the weights of internal 6-sided faces. So if we swap the vertex value $\varphi_6(x_i)=i+1$ with $\varphi_6(z_1^i)=7+i$, for $2\le i\le 3$, then the weights of 6-sided faces are 149, 155, 161 and 167.

This completes the proof. □

Theorem. 3. 4 *For* $d\in\{7,8,9,10,11,12\}$ *and for every* $n\ge 3$*, the graph* $W_{n,1}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. First we define labelings φ_d of type $(1,1,1)$ which assign labels from the set $\{1,2,\dots,8n+2\}$ to the vertices, edges and faces of the graph $W_{n,1}$.

For $1\le i\le n$,

$\varphi_d(c)=1$, $\varphi_d(x_i)=1+i$, $\varphi_d(y_1^i)=n+1+i$, $\varphi_d(z_1^i)=2n+1+i$, $\varphi_d(x_1y_1^n)=4n+2$,

$$\varphi_d(x_iy_1^i)=\begin{cases}3n+2+i, & \text{if } 1\le i\le n-1\\ 3n+2, & \text{if } i=n\end{cases}$$

$$\varphi_7(x_iz_1^i)=\varphi_9(x_iz_1^i)=\varphi_{11}(x_iz_1^i)=6n+2i,\ \varphi_7(cz_1^i)=\varphi_8(cz_1^i)=6n+2-i,$$

$$\varphi_9(cz_1^i)=\varphi_{10}(cz_1^i)=\varphi_{11}(cz_1^i)=\varphi_{12}(cz_1^i)=5n+1+i,$$

For $1\le i\le n-1$,

$$\varphi_7(x_{i+1}y_1^i)=\varphi_8(x_{i+1}y_1^i)=\varphi_{11}(x_{i+1}y_1^i)=\varphi_{12}(x_{i+1}y_1^i)=4n+2+i,$$

$$\varphi_9(x_{i+1}y_1^i)=\varphi_{10}(x_{i+1}y_1^i)=5n+2-i,$$

$$\varphi_7(f_i) = \varphi_9(f_i) = \varphi_{11}(f_i) = \begin{cases} 6n+3, & if\ i=1 \\ 8n+5-2i, & if\ 2 \le i \le n \end{cases}$$

$$\varphi_8(f_i) = \varphi_{10}(f_i) = \varphi_{12}(f_i) = \begin{cases} 6n+1+i, & if\ 1 \le i \le n-1 \\ 8n+1, & if\ i=n \end{cases}$$

$$\varphi_8(x_i z_1^i) = \varphi_{10}(x_i z_1^i) = \varphi_{12}(x_i z_1^i) = \begin{cases} 7n+1+i, & if\ 1 \le i \le n-1 \\ 7n+1, & if\ i=n \end{cases}$$

Case $d = 7$.

For $n \ge 4$, if we swap the vertex value $\varphi_7(x_n) = n+1$ with the the vertex value $\varphi_7(y_1^n) = 2n$, and swapping the edge value $\varphi_7(x_1y_1^1) = 3n+3$, with the edge value $\varphi_7(x_1z_1^1) = 6n+2$, then the weights of 6-sided faces form the arithmetic sequence $44n+20, 44n+27, \ldots, 51n+13$. But for $n=3$, if modify the labels as follows, $\varphi_7(x_1) = 8$, $\varphi_7(z_1^1) = 2$, $\varphi_7(x_1z_1^1) = 25$, $\varphi_7(f_2) = 12$, $\varphi_7(x_1y_1^1) = 20$, $\varphi_7(x_2) = 3$, $\varphi_7(z_1^2) = 9$, $\varphi_7(y_1^2) = 10$, $\varphi_7(z_1^3) = 4$, $\varphi_7(x_3) = 6$, $\varphi_7(x_3y_1^3) = 21$, $\varphi_7(f_1) = 11$, then the weights of 6-sided faces form the arithmetic sequence with difference 7.

Case $d = 8$.

For $n \ge 4$, if we swap the edge value $\varphi_8(x_n z_1^n) = 7n+1$ with $\varphi_8(x_n y_1^{n-1}) = 5n+1$, then the weights of 6-sided faces are $wt_8(f_i) = 44n+12+8i$, for $1 \le i \le n$. But for $n=3$, $wt_8(f_{ext})$ is not be a member of the arithmetic progression with difference $d=8$ of the weights of internal 6-sided faces. So if we swap the label $\varphi_8(x_1) = 2$ with $\varphi_8(z_1^1) = 8$, $\varphi_8(x_1y_1^1) = 12$ with $\varphi_8(f_2) = 21$, $\varphi_8(x_2y_1^2) = 13$ with $\varphi_8(f_3) = 25$, $\varphi_8(x_3y_1^2) = 16$ with $\varphi_8(x_3z_1^3) = 22$, $\varphi_8(x_3y_1^3) = 11$ with $\varphi_8(f_1) = 20$, then the weights of 6-sided faces f_1, f_2, f_3 *and* f_{ext} are 152, 160, 168 and 176 respectively.

Case $d = 9$.

For $n \ge 4$, if we swap the edge value $\varphi_9(x_1y_1^1) = 3n+3$ with $\varphi_9(x_1z_1^1) = 6n+2$, and the edge value $\varphi_9(x_2y_1^1) = 5n+1$ with $\varphi_9(cz_1^1) = 5n+2$, then the weights of 6-sided faces are $wt_9(f_i) = 43n+11+9i$, for $1 \le i \le n$. But for $n=3$, $wt_9(f_{ext})$ is not be a member of the arithmetic progression with difference $d=9$ of the weights of internal 6-sided faces. So if we swap the label $\varphi_9(x_i) = 1+i$ with $\varphi_9(z_1^i) = 7+i$, for $1 \le i \le 3$, the label $\varphi_9(x_1z_1^1) = 20$ with $\varphi_9(x_1y_1^1) = 12$,

$\varphi_9(cz_1^1) = 17$ with $\varphi_9(x_2 y_1^1) = 16$, $\varphi_9(x_3 y_1^2) = 15$ with $\varphi_9(f_3) = 23$, $\varphi_9(x_1 y_1^3) = 14$ with $\varphi_9(f_1) = 21$, then the weights of 6-sided faces f_1, f_2, f_3 *and* f_{ext} are 149, 158, 167 and 176 respectively.

Case $d = 10$.

For $n \geq 4$, if we interchange the edge value $\varphi_{10}(x_n y_1^n) = 4n+1$ with $\varphi_{10}(x_n z_1^n) = 7n+1$, then the weights of 6-sided faces form an arithmetic sequence $43n+20, 43n+30, \ldots, 53n+10$. But for $n = 3$, if we swap the label $\varphi_{10}(x_i) = 1+i$ with $\varphi_{10}(z_1^i) = 7+i$, for $1 \leq i \leq 3$, the edge labels $\varphi_{10}(x_1 y_1^1) = 12$ with $\varphi_{10}(f_2) = 21$, $\varphi_{10}(x_3 z_1^3) = 22$ with $\varphi_{10}(x_2 y_1^2) = 13$, $\varphi_{10}(x_3 y_1^3) = 11$ with $\varphi_{10}(f_1) = 20$, then the weights of 6-sided faces form the arithmetic sequence with difference 10.

Case $d = 11$.

For $n \geq 4$, if we swap the edge value $\varphi_{11}(x_1 y_1^1) = 3n+3$ with the the vertex value $\varphi_{11}(x_1 z_1^1) = 6n+2$, and swapping the vertex value $\varphi_{11}(z_1^n) = 3n+1$, with $\varphi_{11}(y_1^{n-1}) = 2n$, then the weights of 6-sided faces form the arithmetic sequence $42n+20, 42n+31, \ldots, 53n+9$. But for $n = 3$, if modify the labels, $\varphi_{11}(x_1) = 8$, $\varphi_{11}(z_1^1) = 2$, $\varphi_{11}(x_1 y_1^1) = 20$, $\varphi_{11}(x_1 z_1^1) = 12$, $\varphi_{11}(f_2) = 15$, $\varphi_{11}(x_2 y_1^1) = 25$, $\varphi_{11}(z_1^3) = 6$, $\varphi_{11}(y_1^2) = 10$, $\varphi_{11}(f_3) = 23$, $\varphi_{11}(x_3 y_1^2) = 16$, then the weights of 6-sided faces f_1, f_2, f_3 *and* f_{ext} are 146, 157, 168 and 179 respectively.

Case $d = 12$.

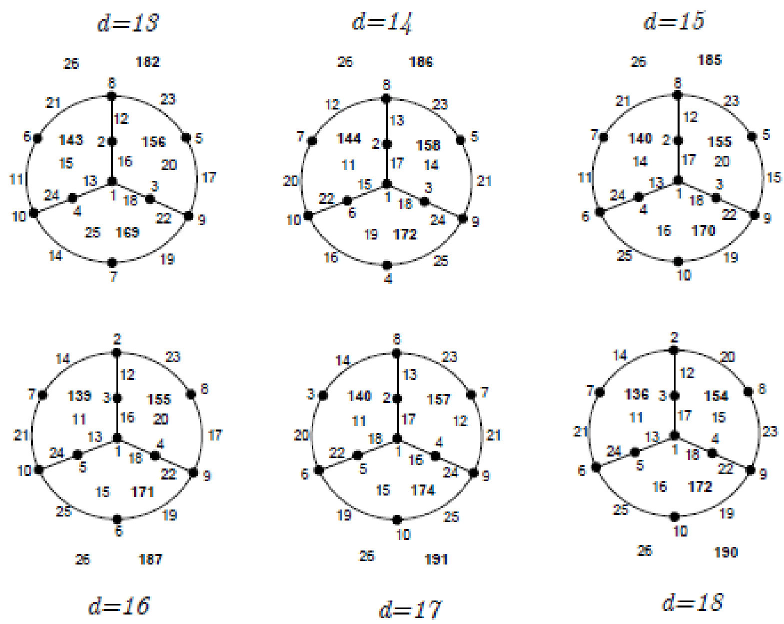
For $n \geq 4$, if we interchange the vertex value $\varphi_{12}(cz_1^n) = 6n+1$ with $\varphi_{12}(x_{n-1} y_1^{n-1}) = 4n+1$, $\varphi_{12}(x_n z_1^n) = 7n+1$ with $\varphi_{12}(x_n y_1^{n-1}) = 5n+1$, then the weights of 6-sided faces $wt_{12}(f_i) = 42n+8+12i$, for $1 \leq i \leq n$. But for $n = 3$, if we swap the label $\varphi_{12}(x_i) = 1+i$ with $\varphi_{12}(z_1^i) = 7+i$, for $1 \leq i \leq 3$, the edge labels $\varphi_{12}(x_1 y_1^1) = 12$ with $\varphi_{12}(f_2) = 21$, $\varphi_{12}(cz_1^3) = 19$ with $\varphi_{12}(x_2 y_1^2) = 13$, $\varphi_{12}(x_3 z_1^3) = 22$ with $\varphi_{12}(x_3 y_1^2) = 16$, $\varphi_{12}(x_3 y_1^3) = 11$ with $\varphi_{12}(f_1) = 20$, then the weights of 6-sided faces form the arithmetic sequence with difference 12.

So we have a super d-antimagic labeling of type $(1,1,1)$ for $W_{n,1}$ for every $n \geq 3$ and $d \in \{7,8,9,10,11,12\}$.

This conclude the proof. □

Lemma. 2. 5 *For* $d \in \{13,14,15,16,17,18\}$, *the graph* $W_{3,1}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. Figure 3 shows the required result for $n = 3$, and $d \in \{13,14,15,16,17,18\}$.



□

Fig 3. The graphs of $W_{3,1}$ with $d \in \{13,14,15,16,17,18\}$

Theorem. 4. 6 *For* $d \in \{13,14,15,16,17,18\}$ *and for every* $n \geq 4$, *the graph* $W_{n,1}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. First we define labelings φ_d of type $(1,1,1)$ which assign labels from the set $\{1,2,\ldots,8n+2\}$ to the vertices, edges and faces of the graph $W_{n,1}$.

For $1 \leq i \leq n$,

$$\varphi_d(c) = 1,\ \varphi_d(z_1^i) = 2n+1+i,\ \varphi_d(cz_1^i) = 5n+1+i,$$

$$\varphi_d(x_i y_1^i) = \begin{cases} 3n+2+i, & \text{if } 1 \leq i \leq n-1 \\ 3n+2, & \text{if } i = n \end{cases}$$

$$\varphi_{13}(x_i) = \varphi_{14}(x_i) = \varphi_{15}(x_i) = 1+i,\ \varphi_{13}(y_1^i) = \varphi_{14}(y_1^i) = \varphi_{15}(y_1^i) = n+1+i,$$

$$\varphi_{16}(x_i) = \varphi_{17}(x_i) = \varphi_{18}(x_i) = 2i,\ \varphi_{16}(y_1^i) = \varphi_{17}(y_1^i) = \varphi_{18}(y_1^i) = 1+2i,$$

$$\varphi_{13}(x_i z_1^i) = \varphi_{15}(x_i z_1^i) = \varphi_{16}(x_i z_1^i) = \varphi_{18}(x_i z_1^i) = 6n+2i,$$

$$\varphi_{13}(x_1 y_1^n) = \varphi_{15}(x_1 y_1^n) = \varphi_{16}(x_1 y_1^n) = \varphi_{18}(x_1 y_1^n) = 4n+2,$$

$\varphi_{13}(f_i)=\varphi_{16}(f_i)=6n+1+2i$, $\varphi_{14}(x_1y_1^n)=\varphi_{17}(x_1y_1^n)=3n+3$,

For $1\le i\le n-1$,

$\varphi_{13}(x_{i+1}y_1^i)=\varphi_{16}(x_{i+1}y_1^i)=5n+2-i$,

$\varphi_{14}(x_{i+1}y_1^i)=\varphi_{17}(x_{i+1}y_1^i)=3n+3+2i$,

$\varphi_{15}(x_{i+1}y_1^i)=\varphi_{18}(x_{i+1}y_1^i)=4n+2+2i$,

$$\varphi_{15}(f_i)=\varphi_{18}(f_i)=\begin{cases}6n+3, & if\ i=1\\ 6n+1+2i, & if\ 2\le i\le n\end{cases}$$

$$\varphi_{14}(f_i)=\varphi_{17}(f_i)=\begin{cases}6n+1+i, & if\ 1\le i\le n-1\\ 8n+1, & if\ i=n\end{cases}$$

$$\varphi_{14}(x_iz_1^i)=\varphi_{17}(x_iz_1^i)=\begin{cases}7n+1+i, & if\ 1\le i\le n-1\\ 7n+1, & if\ i=n\end{cases}$$

Cases $d=13$ *and* $d=16$.

If we swap the edge value $\varphi_d(x_1y_1^1)=3n+3$ with $\varphi_d(x_1z_1^1)=6n+2$, $\varphi_d(x_2y_1^1)=5n+1$ with $\varphi_d(cz_1^1)=5n+2$, $\varphi_d(cz_1^n)=6n+1$ with $\varphi_d(x_{n-1}y_1^{n-1})=4n+1$, then the weights of 6-sided faces, $wt_{13}(f_i)=41n+7+13i$, for $1\le i\le n$. Moreover for $d=16$, we swap $\varphi_{16}(y_1^1)=3$ with $\varphi_{16}(z_1^1)=2n+2$ then the weights of 6-sided faces, $wt_{16}(f_i)=40n+3+16i$, for $1\le i\le n$.

Cases $d=14$ *and* $d=17$.

If we swap the edge value $\varphi_d(x_1y_1^1)=3n+4$ with $\varphi_d(x_1z_1^1)=7n+2$, $\varphi_{14}(cz_1^n)=6n+1$ with $\varphi_{14}(x_{n-1}y_1^{n-1})=5n$, $\varphi_{14}(x_n)=n$ with $\varphi_{14}(y_1^{n-1})=2n$, then the weights of 6-sided faces, $wt_{14}(f_i)=41n+7+14i$, for $1\le i\le n$. Moreover for $d=17$, we swap $\varphi_{17}(cz_1^n)=6n+1$ with $\varphi_{17}(x_ny_1^{n-1})=5n+1$ $\varphi_{17}(z_1^n)=3n+1$ with $\varphi_{17}(y_1^{n-1})=2n-1$ then the weights of 6-sided faces, $wt_{17}(f_i)=40n+3+17i$, for $1\le i\le n$.

Cases $d=15$ *and* $d=18$.

If we swap the edge value $\varphi_d(x_1y_1^1)=3n+3$ with $\varphi_d(x_1z_1^1)=6n+2$, $\varphi_d(cz_1^n)=6n+1$ with $\varphi_d(x_{n-1}y_1^{n-1})=4n+1$, $\varphi_d(z_1^n)=3n+1$ with $\varphi_d(y_1^{n-1})=2n$, then the weights of 6-sided faces, $wt_{15}(f_i)=40n+5+15i$, for $1\le i\le n$.

Moreover for $d=18$, we swap $\varphi_{18}(y_1^1)=3$ with $\varphi_{18}(z_1^1)=2n+2$ then the weights of 6-sided faces, $wt_{18}(f_i)=39n+1+18i$, for $1\le i\le n$.

So we have a super d-antimagic labeling of type $(1,1,1)$ for $W_{n,1}$ for every $n\ge 3$ and $d\in\{13,14,15,16,17,18\}$.

This conclude the proof. □

Theorem. 5. 7 *For* $d\in\{0,1,2,3,4\}$ *and for every* $n\ge 3$, $m\ge 2$, *the graph* $W_{n,m}$ *has a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. First we define labelings φ_d of type $(1,1,1)$ which assign labels from the set $\{1,2,\ldots,4mn+4n+2\}$ to the vertices, edges and faces of the graph $W_{n,m}$.

For $1\le i\le n$, $1\le j\le m$,

$\varphi_d(c)=1$, $\varphi_d(x_i)=1+i$, $\varphi_d(cz_1^i)=3n(m+1)+2-i$, $\varphi_d(x_iy_1^i)=2n(m+1)+2-i$,

$\varphi_d(y_j^i)=nj+1+i$, $\varphi_d(z_j^i)=mn+nj+1+i$, $\varphi_d(x_1y_m^n)=3mn+n+2$,

$\varphi_d(f_{ext})=4mn+4n+2$,

For $1\le i\le n$, $1\le j\le m-1$,

$$\varphi_d(y_j^iy_{j+1}^i)=2n(m+1)+nj+2-i,\ \varphi_d(z_j^iz_{j+1}^i)=3n(m+1)+nj+2-i,$$

$$\varphi_d(x_{i+1}y_m^i)=3mn+2n+2-i,\ \text{for } 1\le i\le n-1,$$

$$\varphi_0(x_iz_m^i)=\begin{cases}4mn+3n+1-i, & \text{if } 1\le i\le n-1\\ 4mn+3n+1, & \text{if } i=n\end{cases}$$

$$\varphi_0(f_i)=\begin{cases}4mn+3n+2+i, & \text{if } 1\le i\le n-1\\ 4mn+3n+2, & \text{if } i=n\end{cases}$$

$$\varphi_1(x_iz_m^i)=\begin{cases}4mn+4n-2i, & \text{if } 1\le i\le n-1\\ 4mn+4n, & \text{if } i=n\end{cases}$$

$$\varphi_1(f_i)=\begin{cases}4mn+2n+3+2i, & \text{if } 1\le i\le n-1\\ 4mn+2n+3, & \text{if } i=n\end{cases}$$

$$\varphi_2(x_iz_m^i)=3mn+4n+1+i,\ \text{for } 1\le i\le n,\quad \varphi_2(f_i)=\begin{cases}3mn+5n+2, & \text{if } i=1\\ 4mn+4n+3-i, & \text{if } 2\le i\le n\end{cases}$$

$$\varphi_3(x_iz_m^i)=4mn+2n+2i,\ \text{for } 1\le i\le n,\quad \varphi_3(f_i)=\begin{cases}4mn+2n+3, & \text{if } i=1\\ 4mn+4n+5-2i, & \text{if } 2\le i\le n\end{cases}$$

$$\varphi_4(x_iz_m^i)=\begin{cases}4mn+3n+1+i, & \text{if } 1\le i\le n-1\\ 4mn+3n+1, & \text{if } i=n\end{cases}$$

$$\varphi_4(f_i)=\begin{cases}4mn+2n+1+i, & \text{if } 1\le i\le n-1\\ 4mn+4n+1, & \text{if } i=n\end{cases}$$

It is easy to verify that the weights of the internal $3m+3$-sided faces for $1 \le i \le n$, form an arithmetic sequence with the desired differences:

$$wt_0(f_i) = 13m^2n + 23mn + 9m + 11n + 10,$$
$$wt_1(f_i) = 13m^2n + 23mn + 9m + 12n + 10 - i,$$
$$wt_2(f_i) = 13m^2n + 23mn + 9m + 14n + 9 + 2i,$$
$$wt_3(f_i) = 13m^2n + 23mn + 9m + 10n + 8 + 3i,$$
$$wt_4(f_i) = 13m^2n + 23mn + 9m + 10n + 7 + 4i.$$

This completes the proof. □

Theorem. 5. ***8*** *For* $m \ge 2$, $n \ge 3$ *and* $d \in \{5,6\}$, *the graph* $W_{n,m}$ *admits a super* d*-antimagic labeling of type* $(1,1,1)$.

Proof. Define the labeling

$\varphi_d : V(W_{n,m}) \cup E(W_{n,m}) \cup F(W_{n,m}) \to \{1,2,3,\ldots,4mn+4n+2\}$ as follows:

For $1 \le i \le n$, $1 \le j \le m$

4mm $\varphi_d(c) = 1$, $\varphi_d(z_j^i) = mn + 1 + nj + i$, $\varphi_d(cz_1^i) = 3mn + 2 + 3n - i$,

4mm $\varphi_d(x_1 y_m^n) = 2mn + 2 + n$, $\varphi_d(x_{i+1} y_m^i) = 2mn + 2 + 2n - i$,

For $1 \le i \le n-1$

4mm $\varphi_d(z_j^i z_{j+1}^i) = 3mn + 2 + 3n + nj - i$, for $1 \le i \le n$, $1 \le j \le m-1$

4mm $\varphi_5(x_i) = 2i$, $\varphi_5(x_i y_1^i) = 3mn + 2n + 3 - 2i$, for $1 \le i \le n$

4mm $\varphi_6(x_i) = 3i - 1$, $\varphi_6(x_i y_1^i) = 3mn + 2n + 4 - 3i$, for $1 \le i \le n$

4mm $\varphi_d(f_{ext}) = 4mn + 4n + 2$.

$$\varphi_d(x_i z_m^i) = \begin{cases} 4mn + 3n + 1 + i, & \text{if } 1 \le i \le n-1 \\ 4mn + 3n + 1, & \text{if } i = n \end{cases};$$

$$\varphi_d(f_i) = \begin{cases} 4mn + 2n + 1 + i, & \text{if } 1 \le i \le n-1 \\ 4mn + 4n + 1, & \text{if } i = n \end{cases}$$

$$\varphi_5(y_j^i) = \begin{cases} 2i + 1, & \text{if } j = 1, 1 \le i \le n \\ nj + 1 + i, & \text{if } 2 \le j \le m, 1 \le i \le n \end{cases}$$

$$\varphi_6(y_j^i) = \begin{cases} 3i + j - 1, & \text{if } j = 1,2, 1 \le i \le n \\ nj + 1 + i, & \text{if } 3 \le j \le m, 1 \le i \le n \end{cases}$$

$$\varphi_5(y_j^i y_{j+1}^i) = \begin{cases} 3mn + 2n + 2 - 2i, & \text{if } j = 1, 1 \le i \le n \\ 3mn + 2n + 2 - nj - i, & \text{if } 2 \le j \le m-1, 1 \le i \le n \end{cases}$$

$$\varphi_6(y_j^i y_{j+1}^i) = \begin{cases} 3mn + 2n + 4 - j - 3i, & \text{if } j = 1,2, 1 \le i \le n \\ 3mn + 2n + 2 - nj - i, & \text{if } 3 \le j \le m-1, 1 \le i \le n \end{cases}$$

We can observe that the labeling φ_d, $d \in \{5,6\}$, uses each integer from the given set exactly once and also it is easy to verify that the weights of the internal $3m+3$-sided faces form an arithmetic sequence with the desired differences:
$wt_5(f_i) = 13m^2n + 23mn + 9m + 10n + 6 + 5i$, for $1 \le i \le n$
$wt_6(f_i) = 13m^2n + 23mn + 9m + 10n + 5 + 6i$, for $1 \le i \le n$. □

In the next theorem we show the existence of d-antimagic labelings of the graph $W_{n,m}$ for all $d \ge 7$ but with restriction for m.

Theorem. 6. *9 If $m+2 \ge d \ge 7$ and $n \ge 3$ then the graph $W_{n,m}$ has a super d-antimagic labeling of type* $(1,1,1)$.

Proof. Let $n \ge 3$ and $m+2 \ge d \ge 7$. For vertices, edges and faces of $W_{n,m}$ we define the labeling ψ_d such that

For $1 \le i \le n$,

$$\psi_d(c) = 1, \psi_d(x_i) = 2 + (d-3)(i-1),\ \psi_d(x_i y_1^i) = 3mn + 2n - 2 + d - (d-3)i,$$

$$\psi_d(cz_1^i) = 3mn + 2 + 3n - i,$$

$$\psi_d(z_j^i) = mn + 1 + nj + i, \text{ for } 1 \le i \le n,\ 1 \le j \le m,$$

$$\psi_d(z_j^i z_{j+1}^i) = 3mn + 2 + 3n + nj - i, \text{ for } 1 \le i \le n,\ 1 \le j \le m-1$$

$$\psi_d(x_1 y_m^n) = 2mn + 2 + n,\ \psi_d(x_{i+1} y_m^i) = 2mn + 2 + 2n - i, \text{ for } 1 \le i \le n-1$$

$$\psi_d(y_j^i) = \begin{cases} 2 + (d-3)(i-1) + j, & \text{if } 1 \le j \le d-4, 1 \le i \le n \\ 1 + i + nj, & \text{if } d-3 \le j \le m, 1 \le i \le n \end{cases}$$

$$\psi_d(y_j^i y_{j+1}^i) = \begin{cases} 3mn + 2n + 1 - j - (d-3)(i-1), & \text{if } 1 \le j \le d-4, 1 \le i \le n \\ 3mn + 2n + 2 - i - jn, & \text{if } d-3 \le j \le m-1, 1 \le i \le n \end{cases}$$

$$\psi_d(x_i z_m^i) = \begin{cases} 4mn + 3n + 1 + i, & \text{if } 1 \le i \le n-1 \\ 4mn + 3n + 1, & \text{if } i = n \end{cases}$$

$$\psi_d(f_i) = \begin{cases} 4mn + 2n + 1 + i, & \text{if } 1 \le i \le n-1 \\ 4mn + 4n + 1, & \text{if } i = n \end{cases}$$

We can see that the labeling ψ_d is a bijective function. The weights of the internal $3m+3$-sided faces constitute the set $\{wt_d(f_i) = 13m^2n + 23mn + 9m + 10n + 4 + di : 1 \le i \le n\}$ which implies that the labeling ψ_d is a super d-antimagic labeling of type $(1,1,1)$. □

3. Concluding remarks

In the foregoing section, we studied super d-antimagic labelings of type $(1,1,1)$ for $W_{n,m}$ and we showed the existence of such labelings for different values of

$d \geq 0$, but with restrictions. We are not able to prove or disprove the existence of such labelings for $d \geq 19$, when $m = 1$ and for $m+1 \leq d$ if $d \geq 7$. So, we conclude the paper with the following open problems.

Open Problem. 1. 10 *For the graph* $W_{n,1}$, $n \geq 3$, *determine if there is a super* d *-antimagic labeling of type* $(1,1,1)$ *with* $d \geq 19$.

Open Problem. 2. 11 *For the graph* $W_{n,m}$, $3 \leq m+1 \leq d$, $n \geq 3$, *determine if there is a super* d *-antimagic labeling of type* $(1,1,1)$ *with* $d \geq 7$.

R E F E R E N C E S

[1]. *G. Ali, M. Bača, F. Bashir, and A. Semaničová-Feňovčíková*, On face antimagic labelings of disjoint union of prisms. Utilitas Math**. vol. 85**, 2011, pp. 97-112.

[2]. *M. Bača, F. Bashir and A. Semaničová-Feňovčíková,* Face antimagic labelings of antiprisms. Utilitas Math . **vol. 84**, 2011, pp. 209-224.

[3]. *M. Bača, E.T. Baskoro, S. Jendrol̆ and M. Miller*, Antimagic labelings of hexagonal planar maps. Utilitas Math . **vol. 66**, 2004, pp. 231-238.

[4]. *M. Bača, L. Brankovic and A. Semaničová-Feňovčíková*, Labelings of plane graphs containing Hamilton path. Acta Math. Sin. (Engl. Ser.,) **vol. 27**, 2011, pp. 701-714.

[5]. *M. Bača, S. Jendrol̆, M. Miller and J. Ryan*, Antimagic labelings of generalized Petersen graphs that are plane. Ars Combin*.,* **vol. 73**, 2004, pp. 115-128.

[6]. *M. Bača, Y. Lin and M. Miller,* Antimagic labelings of grids. Utilitas Math*.,* **vol. 72**, 2007, pp. 65-75.

[7]. *M. Bača and M. Miller,* On d -antimagic labelings of type $(1,1,1)$ for prisms. J. Combin. Math. Combin. Comput.*,* **vol. 44**, 2003, pp. 199-207.

[8]. *M. Bača, M. Miller, O. Phanalasy and A. Semaničová-Feňovčíková,* Super d-antimagic labelings of disconnected plane graphs. Acta Math. Sin. (Engl. Ser.,) **vol. 26(12)**, 2010, pp. 2283-2294.

[9]. *R. Bodendiek and G. Walther*, On number theoretical methods in graph labelings. Res. Exp. Math., **vol. 21**, 1995, pp. 3-25.

[10]. *N. Hartsfield and G. Ringel,* Pearls in Graph Theory. Academic Press, Boston - San Diego - New York – London 1990.

[11]. *A. Krishnaa and M. S. Dulawat*, Algorithm for inner magic and inner antimagic labelings of some planar graphs, Informatica.*,* **vol. 17**, 2006, pp. 393-406.

[12]. *K.W. Lih,* On magic and consecutive labelings of plane graphs. Utilitas Math., **vol. 24**, 1983, pp. 165-197.

[13]. *Y. Lin, Slamin, M. Bač̆a and M. Miller*, On d-antimagic labelings of prisms. Ars Combin.*,* **vol. 72**, 2004, pp. 65-76.

[14]. *K.A. Sugeng, M. Miller, Y. Lin and M. Bača,* Face antimagic labelings of prisms. Utilitas Math., **vol. 71**, 2006, pp. 269-286.

[15]. *I. Tomescu, A. Riasat*, On metric dimension of uniform subdivisions of the wheel, Utilitas Math., (in press).

[16]. *D.B. West* , An Introduction to Graph Theory. Prentice - Hall, 1996.