GRAPHICAL METHOD FOR DETERMINING PARAMETERS OF DISCRETE PULSES GENERATED BY DELAY OPERATOR ON LIMITED TIME INTERVALS

Cristian TOMA[1]

This study presents basic properties of discrete pulses generated by the use of delay operator applied on certain dynamical system. It is shown that pulses with highly concentrated values of amplitude are generated in. when a delay operator is used as a binomial acting upon an initial unity state. This binomial model provides useful when pulses become shorter each time when they pass through a certain material medium. A graphical method for determining the parameters of this model based on tangent line in inflection point is also presented.

Keywords: binomial model, delay operator, time-limited pulses

1. Introduction

Many phenomena in physics, biology, or human behavior exhibit a pulse-like interaction. Hence, a certain characteristic time-function is (almost) restricted to a limited space-time interval. Different mathematical models were established for modelling these pulse-like phenomena. Starting from the wave equation, superpositions of waves with different frequencies (wave packets) were analyzed. Terms as phase velocity and group velocity were introduced to justify the envelope of pulses and propagation properties. However, in case of a suddenly emerging phenomenon on a limited space-time interval it is difficult to justify its propagation at long distances, due to distortion (the velocities corresponding to waves with different frequencies are not the same, the original shape of the pulse being altered). Attempts for coherent structural trapping are studied nowadays (see [1]). Waves considered from the very beginning as waveforms with effectively limited duration (wavelets) were also introduced to study propagating phenomena (see [2]). However, the shape of a Dirac delta distribution is difficult to be obtained through differential equations of evolution (see [3]) – as nonlinear differential equations are involved. Algebraic methods based on binomial model were involved mainly in probability [4], not in describing dynamics of phenomena. A systemic approach to delay functions and delay differential equations has been presented in [4].

[1] Assoc. Prof., Dept.of Physics, University POLITEHNICA of Bucharest, Romania, e-mail: cristian.toma@upb.ro

In a previous study [5], a certain delay operator has been denoted as τ and has been included in a binomial as $(1+\tau)$. Different parts of a dynamical system have been considered to start from an initial state S_i at an initial time moment. Describing the interaction generating pulses by an operator $(1+\tau)^n$ acting upon the initial state S_i of this system, it was shown that the corresponding time evolution of this system could be characterized by a state S determined through $S = (1+\tau)^n S_i$

Setting the initial state to a scalar value $S_i = 1$, expansion of $(1+\tau)^n$ yields

$$S = (1+\tau)^n S_i = \{C_n^0\tau^0 + C_n^1\tau^1 + \cdots + C_n^k\tau^k + \cdots + C_n^n\tau^n\}\,1 \qquad (1)$$

To use the operator τ as a delay operator, the initial time moment is set to zero. Therefore, the sum written above consists of terms $C_n^k\tau^k$ - this means as a sum of terms corresponding to the delay operator applied k - times. According to previously presented considerations regarding the use of the delay operator in a multiplicative manner, the monomial τ^k could be considered as a delay operator with a certain time interval τ applied k times - this means as a delay operator with the time difference $k\tau$.

Connecting the coefficient C_n^k (multiplying τ^k) to a certain amplitude, it follows that the term $C_n^k\tau^k$ can be associated to certain amplitude C_n^k which can be noticed at the time moment $k\tau$ (considered from an initial zero moment). Thus, a discrete-time function is generated which can be extended through interpolation in order to obtain a continuous function which can be easily represented graphically.

.

2. Square-root concentration of significant values for dynamical systems driven by binomial time-delay operator

Let us consider once again the combinatorial function $f(k) = C_n^k, k \in [0,n]$ which is a discrete function. However, for great values of n, its graph resembles a continuous function. Its minimum values corresponds to $k = 0, n$ and its maximum value corresponds approximately to $k_{max} = n/2$. In a similar manner as for continuous function, we shall study points where its first and second derivative (in fact its first and second order differences) are equal (or almost equal) to zero.

For the first derivative it is obvious that this vanishes in the single point of maximum of this function. Yet the study of inflection points (where the second derivative vanishes) requires a more detailed analysis. For this purpose we shall use the recurrence formula

$$f(k) = \frac{n-k+1}{k}\, f(k-1) \qquad (2)$$

This implies

$$f(k-1) = \frac{k}{n-k+1}\, f(k),$$
$$f(k+1) = \frac{n-(k+1)+1}{k+1}\, f(k) = \frac{n-k}{k+1}\, f(k) \qquad (3)$$

and further

$$f(k+1) - f(k) = \left(\frac{n-k}{k+1} - 1\right) f(k) = \left(\frac{n-2k-1}{k+1}\right) f(k)$$
$$f(k) - f(k-1) = \left(1 - \frac{k}{n-k+1}\right) f(k) = \left(\frac{n-2k+1}{n-k+1}\right) f(k) \qquad (4)$$

The second order difference in point k results as

$$[f(k+1) - f(k)] - [f(k) - f(k-1)] =$$
$$\left\{\left(\frac{n-2k-1}{k+1}\right) - \left(\frac{n-2k+1}{n-k+1}\right)\right\} f(k) =$$
$$\left(\frac{n^2-4nk+4k^2-n-2}{(k+1)(n-k+1)}\right) f(k) \qquad (5)$$

Since $f(k)$ is a nonzero function (determined as a ratio of factorials), it results that the second order difference (corresponding to second order derivative for continuous functions) vanishes when the numerator of the ratio between round brackets above vanishes. This implies

$$n^2 - 4nk + 4k^2 - n - 2 = 4k^2 - 4nk + (n^2 - n - 2) = 0 \qquad (6)$$

(the numerator of the coefficient between round brackets is a product of nonzero numbers for k within the interval $(0, n)$, a distinct analysis being not necessary).

The equation above can be considered as a second order equation with k the unknown quantity (n supposed to be already known). Its solution corresponds to

$$k_{1,2} = \frac{4n \pm \sqrt{16n^2 - 4\cdot 4\,(n^2-n-2)}}{8} = \frac{4n \pm \sqrt{16n+32}}{8} = = \frac{n}{2} \pm \frac{\sqrt{n+2}}{2} = k_{max} \pm \frac{\sqrt{n+2}}{2} \qquad (7)$$

This implies that the Δk_{max} interval around k_{max} between these inflection points equals $\sqrt{n+2} \approx \sqrt{n}$ for great values of n. It can be easily shown that $\sqrt{n+2} - \sqrt{n}$ is trending to zero when n increases. By the other hand, the maximum of $f(k) = C_n^k$ for $k = n/2$ can be approximated (using $n! \sim \sqrt{2\pi n}\,(\frac{n}{e})^n$) by

$$Max_{f(k)} \approx \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} = \frac{\sqrt{2\pi n}\frac{n^n}{e^n}}{\left(\sqrt{2\pi\left(\frac{n}{2}\right)}\frac{\left(\frac{n}{2}\right)^{\frac{n}{2}}}{e^{\frac{n}{2}}}\right)^2} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{n}}\,2^n \qquad (8)$$

It can be easily notice that

$$\Delta k_{max} \cdot Max_{f(k)} \approx \sqrt{n}\left(\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{n}}\,2^n\right) = \sqrt{\frac{2}{\pi}}\,2^n \qquad (9)$$

(the area of a rectangle with height $Max_{f(k)}$ and width Δk_{max} defined around the middle of the interval $(0, n)$). In figure 1 is represented the graph with normalized values (C_n^k on Oy axis were divided by 2^n, k along Ox axis was divided by n) for $n = 50$. It can be noticed that significant values of C_n^k are concentrated within the normalized interval $\{50/2 \pm \sqrt{52}\}/50$, this means between the normalized values 0.428 and 0.572 along the Ox axis.

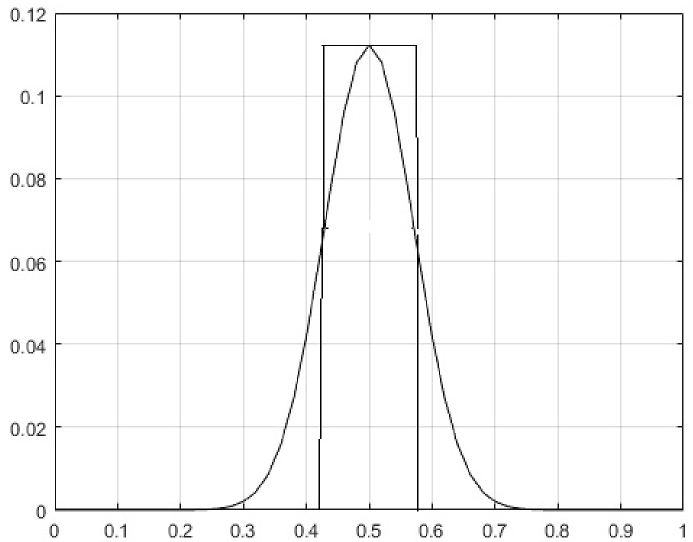

Fig. 1. Concentration of amplitude of discrete pulses versus time (normalized values) for binomial delay operator with n=50

Comparing this result with the well-known result from combinatorial analysis $\sum_{k=0}^{n} C_n^k = 2^n$, it results that the area of a rectangle graphically defined around k_{max}, more specific with height $Max_{f(k)}$ and width Δk_{max}, represents in fact the sum of all amplitudes of the pulse $f(k), k \in [0, n]$ multiplied by a factor $\sqrt{2/\pi}$. In fact, around 80% of the entire sum of $f(k)$ can be considered as consisting of the area of this graphically defined rectangle. (significant amplitudes are concentrated within this rectangle).

When n increases, the width of this rectangle becomes narrower as compared to the length of the working interval $[0, n]$, since the ratio $\sqrt{n}/n$ decreases.

3. Graphical aspects regarding the derivative of amplitude versus time function in inflection point

For determining the derivative of amplitude versus time function in first inflection point, we substitute $k = \frac{n}{2} - \frac{\sqrt{n+2}}{2}$, in formula (4). It results

Using formula

$$f(k) - f(k-1) = \left(1 - \frac{k}{n-k+1}\right) f(k) = \left(\frac{n-2k+1}{n-k+1}\right) f(k),\ k = \frac{n}{2} - \frac{\sqrt{n+2}}{2} \quad (10)$$

It results

$$f(k) - f(k-1) = \frac{2}{\sqrt{n+2}} f(k),, for\ k = \frac{n}{2} - \frac{\sqrt{n+2}}{2} \quad (11)$$

We shall note the value of amplitude versus time function in this inflection points with F, the distance between inflection poinsts with D and the derivative (the slope) of this function in the first inflection point with m Since the distance D between the two inflection points previously determined is $\sqrt{n+2}$ (according to formula (7)), it results that the tangent line at this graph in an inflection point "cuts' the vertical line on time axis drawn in the other inflection point at a height

$$H = F + mD = F + \frac{2}{\sqrt{n+2}} F \sqrt{n+2} = 3F \quad (12)$$

which is three times greater than the value of amplitude versus time function in an inflection point. This important geometrical feature can be used for validating the presence of a binomial model for a time-limited pulse.

The tangent line at this amplitude versus time function drawn in this inflection points can be extended also in opposite direction, towards smaller amplitudes. The value of this function in points where this extended line "cuts" Ox axis can be used for determining the value of parameter n for this binomial model. For this purpose, a supplementary function based on numerical computation is necessary.

4. Conclusions

This study has presented basic properties of highly concentrated and uniformly distributed discrete pulses obtained through the use of the delay operator. It has continued previous studies on interaction of laser pulses with specific materials [6], in connection with studies on shape optimization -see [7], [8].

It was studied the concentration of significant values of discrete pulses generated by the use of the delay operator as a binomial acting upon an initial unitary state. It has been shown that the interval along time axis connected to significant amplitudes corresponds to the time interval between inflection points on the graph amplitude versus time (continuous graph obtained by extending the discrete distribution). The width of this interval being proportional to square root of n (the power of binomial operator), the width of this interval decreases as related to the length of the working interval when power n increases.

It was also shown that the tangent line drawn at the amplitude versus time function can be used: (i) for checking the validity of this model for already registered time-limited pulses and (ii) for determining the parameters of this model.

This model provides a useful tool for analyzing time-limited pulses, especially in the case when pulses are shrinking after each path through an active medium (as in High Energy Photonics). It will be aslo check whether this model can be extended for risk treatment in systems with planned evolution, as in [10] using advanced mathematical tools as matrix factorization for multidimensional domains, as in [9].

Acknowledgements

This research was funded by the project: Advanced Infrastructure for Nuclear Photonics research experiments at ELI-NP / ELI-INFRA (Bucharest-Magurele).

R E F E R E N C E S

[1]. *Henrik T. Lemke, Kasper S. Kjær, Robert Hartsock, Tim B. van Driel, Matthieu Chollet, James M. Glownia, Sanghoon Song, Diling Zhu, Elisabetta Pace, Samir F. Matar, Martin M. Nielsen,Maurizio Benfatto, Kelly J. Gaffney, Eric Collet, Marco Cammarata,*,“ Coherent structural trapping through wave packet dispersion during photoinduced spin state switching”, in Nature Communications., **vol. 8**, 2017, Article number: 15342, https://doi.org/10.1038/ncomms15342

[2]. *FMM. Ghaini, C. Cattani, MH. Heydari, MR Hoshmandasl,* " Wavelets method for the time fractional diffusion-wave equation”, in Physics Letters A 379., **vol. 3**, 2015, pp. 71-76

[3]. *San-Li Chung, Pai-Ta Shih,* ,“ Generalized Cox-Ross-Rubinstein Binomial Models”, in Management Science 53., **vol. 3**, 2007, pp. 508-520

[4]. *C.N.. Angstmann, S.J.M. Burney, B.I. Henry, B.A. Jacobs, Z. Xu,* “A Systematic Approach to Delay Functions”, in Mathematics,, **vol. 11**, no. 3, 2023, pp. 4526, https://doi.org/10.3390/math11214526

[5]. *Bogdan Dumitru, Cristian Toma, Mihaela Ghelmez,* “Cooperative Combinatorial Model Based on Discrete Pulses generated within Limited Time Intervals”, in Scientific Bulletin Politehnica University, Bucharest, Series A-Applied Mathematics and Physics., **vol. 85**, no. 3, 2023, pp. 189-200

[6]. *Mihaela Ghelmez, Bogdan Dumitru, Cristian Toma,* “Simple Systems of Fatty Acids under Laser Action”, in Scientific Bulletin Politehnica University, Bucharest, Series A-Applied Mathematics and Physics., **vol. 79**, no. 2, 2017, pp. 243-254

[7]. *Y.S.Grasimov,* “Some shape optimization problems for eigenvalues”, in . Journal of Physics A: Mathematical and Theoretical,., **vol. 41**, no. 5, 2008, 055202, DOI:10.1088/1751-8113/41/5/055202

[8]. *Y.S.Grasimov,* “On a shape design problem for one spectral functional.”, in . Journal of Inverse and Ill-Posed Problems, **vol. 21**, no. 5, 2013, pp.629-637

[9]. *D.A.Juraev, Y..S.Grasimov,* “On the regularization Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain”, in Azerbaijan Journal of Mathematics, **vol. 12**, no. 1, 2022, pp.142-161

[10]. *Bogdan Dumitru, Mihaela Ghelmez, Cristian Toma,* “Physical Methods for Analysis and Treatment of Risks in a System with Planned Evolution”, in Scientific Bulletin Politehnica University, Bucharest, Series A-Applied Mathematics and Physics., **vol. 83**, no. 4, 2021, pp. 231-242