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THE STRUCTURE OF (φ, ψ)-MODULE CONTRACTIBLE BANACH ALGEBRAS

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In the present article, we introduce and study the concepts of (φ, ψ)-module contractibility and (φ, ψ)-module biprojectivity for Banach algebras. Moreover, some known outcomes concerning the (module) contractibility of Banach algebras are generalized. As a main example, for a semigroup S with the set of idempotents E, we study these notions for $l^1(S)$ (as an $l^1(E)$-module) for arbitrary module actions, and extend the well-known results due to Selivanov and Helemskii.

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1. Introduction

The story of amenability and contractibility for Banach algebras was commenced by B. E. Johnson in [17], where he linked the amenability of Banach algebras and groups. One of the fundamental results was that the group algebra $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable locally compact group. Next, Selivanov showed that the Banach algebra $L^1(G)$ is contractible if and only if G is finite [25]. Recall from [17] that a Banach algebra $\mathcal{A}$ is called *contractible* (*amenable*) if $H^1(\mathcal{A}, X) = \{0\}$ (resp. $H^1(\mathcal{A}, X^*) = \{0\}$) for every Banach $\mathcal{A}$-bimodule X, where the left hand side in the equality is the *first cohomology group* of $\mathcal{A}$ with coefficient in X (resp. X^*). Johnson [18] showed that a Banach algebra $\mathcal{A}$ is contractible if and only if it has a diagonal, that is, there is an element $\mathbf{m} \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\omega(\mathbf{m})$ is an identity for $\mathcal{A}$ and $a \cdot \mathbf{m} = \mathbf{m} \cdot a$, for each $a \in \mathcal{A}$, where $\omega : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$; $a \otimes b \mapsto ab$ is the canonical morphism (for emphasis, $\omega_{\mathcal{A}}$). Obviously, ω is an $\mathcal{A}$-bimodule map (i.e. a bounded linear map which preserves the module operations) with respect to the canonical bimodule structure on the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$.

A Banach algebra $\mathcal{A}$ is called *biprojective* if ω has a bounded right inverse which is an $\mathcal{A}$-bimodule map. A Banach algebra $\mathcal{A}$ is said to be *biflat* if the adjoint $\omega^* : \mathcal{A}^* \longrightarrow (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ has a bounded left inverse which is an $\mathcal{A}$-bimodule map. It is obvious that every biprojective Banach algebra is biflat. Helemskii [14] studied the structure of Banach algebras through the concepts of biprojectivity and biflatness. In the main result, he showed that a Banach algebra $\mathcal{A}$ is contractible if and only if it is unital and biprojective (the basic properties of biprojectivity and biflatness can be found in [14]). In particular, $l^1(G)$ is biprojective if and only if G is finite [13, 14]. Biprojectivity is important notion in the category of commutative Banach algebras. For instance, each commutative Banach algebra has a discrete character space if it is biprojective, and the converse holds for all commutative C^*-algebras [24].

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M. Amini [1] and H. P. Aghababa [22] initiated the notion of module amenability and contractibility for a class of Banach algebras, respectively, which could be considered as some generalizations of the Johnson's results when the $\mathbb{C}$-module structure is replaced by a Banach algebra module structure. They showed that for an inverse semigroup S with the set of idempotents E, the semigroup algebra $l^1(S)$ is module amenable (resp. contractible), as a Banach module over $l^1(E)$, if and only if S (resp. $G_S \cong S/\approx$) is amenable (resp. finite), where G_S is an appropriate group homomorphic image of S. The structure of module contractible Banach algebras and some hereditary properties of module contractibility are investigated in [3] and [4]. In addition, Aghababa [22] studied the module contractible Banach algebras using module diagonals and showed that under some mild conditions, a Banach algebra is module contractible if and only if it has a module diagonal [22].

In [6], the third author and Amini introduced and studied a module biprojective Banach algebra which is a Banach module over another Banach algebra with compatible actions. For every inverse semigroup S with subsemigroup E of idempotents, they showed that $l^1(S)$ is module biprojective, as an $l^1(E)$-module, if and only if G_S is finite under the extra condition D_k of Duncan and Namioka [11] for some k on E. Recently, this condition is removed in [8] and the authors showed that $l^1(S)$ is module biprojective as an $l^1(E)$-module with trivial left action if and only if G_S if finite. Some module cohomological properties such as module biprojectivity and module biflaness of the projective tensor products are studied in [16].

In this paper, we study the notion of (φ, ψ)-module contractibility for a Banach algebra $\mathcal{A}$ which extends the concepts of module contractibility, where φ and ψ are two $\mathfrak{A}$-module homomorphisms on $\mathcal{A}$. We also find sufficient conditions for (φ, ψ)-module contractibility of $\mathcal{A}$ to be equivalent to (φ, ψ)-contractibility of $\mathcal{A}$. Moreover, we introduce the notions of (φ, ψ)-module biprojectivity and (φ, ψ)-module diagonal for Banach algebras and show that under which conditions these notions are equivalent. In particular, we present and prove some generalizations of the Johnson's and Helemskii's theorems. Furthermore, we improve the main result of [20] for arbitrary commutative compatible actions without the pseudo-unital condition. As a consequence of a semigroup S with the finite set of idempotents E, we generalize results due to Selivanov and Helemskii for a discrete group G.

2. (φ, ψ)-module contractibility

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha), \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a),$$

for all $a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X$ and similar for the right or two-sided actions. Then, we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, if $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}, x \in X$, then X is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. Note that $\mathcal{A}$ is not an $\mathcal{A}$-$\mathfrak{A}$-module in general because $\mathcal{A}$ does not satisfy the compatibility condition $a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. But when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathfrak{A}$-bimodules. Then a $\mathfrak{A}$-module homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a bounded map $T : \mathcal{A} \longrightarrow \mathcal{B}$ with $T(a \pm b) = T(a) \pm T(b)$, and is multiplicative, that is $T(ab) = T(a)T(b)$ for all $a, b \in \mathcal{A}$, and

$$T(\alpha \cdot a) = \alpha \cdot T(a), \ T(a \cdot \alpha) = T(a) \cdot \alpha, \quad (a, \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

We denote by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$, the space of all such homomorphisms and denote $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{A})$ by $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. Note that when $\mathfrak{A} = \mathbb{C}$, the set of complex numbers, then $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}) = \mathrm{Hom}(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Suppose that φ and ψ are in $\mathrm{Hom}_{\mathfrak{A}}(A)$. A bounded map $D : \mathcal{A} \longrightarrow X$ is called a *module (φ, ψ)-derivation* if

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha, (a \in \mathcal{A}, \quad \alpha \in \mathfrak{A})$$

and

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = \varphi(a) \cdot D(b) + D(a) \cdot \psi(b), \quad (a, b \in \mathcal{A}).$$

Recall that $D : \mathcal{A} \longrightarrow X$ is bounded if there exist $M > 0$ such that $||D(a)|| \leq M||a||$, $(a \in \mathcal{A})$. Although D and the elements of $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A}, \mathcal{B})$ are not necessarily linear, their boundedness still implies their norm continuity. When X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, then each $x \in X$ defines a module (φ, ψ)-derivation $ad_x^{(\varphi,\psi)} = \varphi(a) \cdot x - x \cdot \psi(a)$ on $\mathcal{A}$. These are called *(φ, ψ)-module inner derivations*. Derivations of these forms are studied in [2] and [5].

A Banach algebra $\mathcal{A}$ is called *(φ, ψ)-module contractible* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module (φ, ψ)-derivation $D : \mathcal{A} \longrightarrow X$ is (φ, ψ)-inner. One should remember that if φ and ψ are identity maps on $\mathcal{A}$, then every $(id_{\mathcal{A}}, id_{\mathcal{A}})$-module derivation is the same as module derivation, and therefore $(id_{\mathcal{A}}, id_{\mathcal{A}})$-module contractibility is the same as module contractibility. In addition, when $\mathfrak{A} := \mathbb{C}$, everything reduces to the classical case; see [7], [9], [12] and [21].

It is known that contractible Banach algebras and also every contractible commutative Banach $\mathfrak{A}$-modules are unital [3]. In the beginning of this section, we generalize this property as follows.

Proposition 2.1. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If $\mathcal{A}$ is (φ, ψ)-module contractible, then $\mathcal{A}$ has an identity for $\varphi(\mathcal{A}) \cap \psi(\mathcal{A})$. In particular $\mathcal{A}$ is unital, whenever both φ and ψ have dense ranges or are injective.*

Proof. Set $X := \mathcal{A}$. Then, X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, with the same actions of $\mathfrak{A}$ and the following actions of $\mathcal{A}$

$$a \cdot x = ax, \quad x \cdot a = 0, \quad (a \in \mathcal{A}, \ x \in X).$$

Define the mapping $D : \mathcal{A} \rightarrow X$ by $D(a) = \varphi(a)$ $(a \in \mathcal{A})$. Then, D is a (φ, ψ)-module derivation and there is $e_1 \in X$ such that $D = ad_{e_1}^{\varphi}$. Thus, e_1 is a right identity for $\varphi(\mathcal{A})$. Similarly, X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, with the same actions of $\mathfrak{A}$ and the following actions of $\mathcal{A}$

$$a \cdot x = 0, \quad x \cdot a = xa, \quad (a \in \mathcal{A}, \ x \in X).$$

Again, $D(a) := \psi(a)$ $(a \in \mathcal{A})$ is a (φ, ψ)-module derivation and there exists an $e_2 \in X$ such that $D = ad_{e_2}^{\psi}$ and therefore, e_2 is a left identity for $\psi(\mathcal{A})$. Consequently, $\mathcal{A}$ has an identity for $\varphi(\mathcal{A}) \cap \psi(\mathcal{A})$. □

Remark 2.1. Let $\mathcal{B}$ be a Banach algebra and $\sigma, \tau \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$. Consider $\mathfrak{A} := \mathbb{C}$, then $\mathcal{B}$ is automatically a commutative Banach $\mathbb{C}$-module and $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. Moreover, (σ, τ)-derivations and (σ, τ)-module derivations coincide, and so (σ, τ)-module contractibility is the same as (σ, τ)-contractibility for $\mathcal{B}$. Thus, if a Banach algebra $\mathcal{B}$ is (σ, τ)-contractible, then it has a right identity for $\sigma(\mathcal{B})$ and a left identity for $\tau(\mathcal{B})$ by Proposition 2.1.

Definition 2.1. *A Banach $\mathcal{A}$-bimodule X is called (φ, ψ)-unital if*

$$X = \{\varphi(a) \cdot x \cdot \psi(b) : \ a, b \in \mathcal{A}\}.$$

Lemma 2.1. *Let $\mathcal{A}$ has an identity for $\varphi(\mathcal{A}) \cup \psi(\mathcal{A})$. Suppose that either both φ and ψ are idempotents or have dense ranges. Then, $\mathcal{A}$ is (φ,ψ)-module contractible if and only if each (φ,ψ)-module derivation from $\mathcal{A}$ into any (φ,ψ)-unital commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module is (φ,ψ)-inner.*

Proof. Let $e \in \mathcal{A}$ be an identity for $\varphi(\mathcal{A})$ and $\psi(\mathcal{A})$. If both φ and ψ are idempotents, then $\varphi(e)$ and $\psi(e)$ are the identities for $\varphi(\mathcal{A})$ and $\psi(\mathcal{A})$, respectively. In the case that both φ and ψ have dense ranges, then $\varphi(e) = \psi(e) = e$. Assume that X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \longrightarrow X$ is a (φ,ψ)-module derivation. Consider

$$X_1 = \varphi(e) \cdot X \cdot \psi(e), \quad X_2 = (1 - \varphi(e)) \cdot X, \quad X_3 = X \cdot (1 - \psi(e)),$$

and $X_4 = (1 - \varphi(e)) \cdot X \cdot (1 - \psi(e))$, which are commutative Banach $\mathcal{A}$-$\mathfrak{A}$-modules by

$$\alpha \circ x_1 = x_1 \circ \alpha := \varphi(e) \cdot (\alpha.x) \cdot \psi(e), \ a \circ x_1 := \varphi(a) \cdot x_1, \ x_1 \circ a := x_1 \cdot \psi(a),$$

for each $x_1 = \varphi(e) \cdot x \cdot \psi(e) \in X_1$, $\alpha \in \mathfrak{A}$, $a \in \mathcal{A}$ and similarly for X_i $(2 \leq i \leq 4)$. Then $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$ and $D_i := p_i \circ D$ is a (φ,ψ)-module derivation, where $p_i : X \longrightarrow X_i$ is the canonical projection $(1 \leq i \leq 4)$. Since

$$D_i(\varphi(e)a) = D_i(a\psi(e)) = D_i(a), \quad (1 \leq i \leq 4, \ a \in \mathcal{A}),$$

we have $D_2 = ad^{(\varphi,\psi)}_{-D_2(\varphi(e))}$, $D_3 = ad^{(\varphi,\psi)}_{D_3(\psi(e))}$, and $D_4 = 0$. Thus, D is (φ,ψ)-inner if and only if D_1 is (φ,ψ)-inner. □

Lemma 2.2. *Let $\mathcal{A}$ has a left (or right) identity e and X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module such that $\varphi(e) \cdot x = x = x \cdot \psi(e)$, for all $x \in X$. If $D : \mathcal{A} \longrightarrow X$ is a (φ,ψ)-module derivation such that φ and ψ are $\mathbb{C}$-linear, then D is $\mathbb{C}$-linear.*

Proof. It is clear that $D(e)$ is zero. For all $n \in \mathbb{N}$, additivity of D implies that

$$nD\left(\frac{1}{n}e\right) = D(e) = 0.$$

Thus, $D(re) = 0$ $(r \in \mathbb{Q})$. Hence by the continuity of D, $D(re) = 0$ $(r \in \mathbb{R})$. Besides, for the imaginary unit i, we have

$$0 = D(-e) = D\left(i^2 e\right) = \varphi(ie) \cdot D(ie) + D(ie) \cdot \psi(ie) = 2iD(ie).$$

Hence, $D(ie) = 0$, and so $D(\lambda e) = 0$ $(\lambda \in \mathbb{C})$. Consequently, for all $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$,

$$D(\lambda a) = \varphi(\lambda e) \cdot D(a) + D(\lambda e) \cdot \psi(a) = \lambda D(a).$$

This complete the proof. □

Lemma 2.3. *Let $\varphi, \psi \in Hom_{\mathfrak{A}}(\mathcal{A})$ be idempotents and $\mathcal{A}$ has an identity e for $\varphi(\mathcal{A})$ and $\psi(\mathcal{A})$. If $D : \mathcal{A} \longrightarrow X$ is a (φ,ψ)-module derivation and X is (φ,ψ)-unital, then*

$$D(ae) = D(a) = D(ea), \quad (a \in \mathcal{A}).$$

Proof. Since X is (φ,ψ)-unital, $\varphi(e) \cdot x = x = x \cdot \psi(e)$, for each $x \in X$. For all $a \in \mathcal{A}$,

$$\begin{aligned} D(\varphi(a)) &= \varphi^2(a) \cdot D(e) + D(\varphi(a)) \cdot \psi(e) \\ &= \varphi(a) \cdot D(e) + D(\varphi(a)), \end{aligned}$$

and

$$\begin{aligned} D(\psi(a)) &= \varphi(e) \cdot D(\psi(a)) + D(e) \cdot \psi^2(a) \\ &= D(\psi(a)) + D(e) \cdot \psi(a). \end{aligned}$$

Hence, $\varphi(a) \cdot D(e) = D(e) \cdot \psi(a) = 0$. Therefore,

$$D(ea) = \varphi(e) \cdot D(a) + D(e) \cdot \psi(a) = D(a),$$

for all $a \in \mathcal{A}$. Similarly, $D(a) = D(ae)$. □

The following corollary generalize [20, Theorem 3.3] and [3, Proposition 2.2]. This is done without extra assumption that $\mathcal{A}$ is essential or $\mathfrak{A}$ has an bounded approximate identity for $\mathcal{A}$.

Corollary 2.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\varphi, \psi \in \mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ such that one of the following is satisfied:*

(i) *φ and ψ are $\mathbb{C}$-linear with dense ranges,*
(ii) *$\varphi = \psi$ is idempotent.*

Then, (φ, ψ)-module contractibility of $\mathcal{A}$ follows from its (φ, ψ)-contractibility.

Proof. From Remark 2.1 $\mathcal{A}$ has an identity e for $\varphi(\mathcal{A}) \cap \psi(\mathcal{A})$. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \to X$ be a (φ, ψ)-module derivation. If (i) holds, then e is an identity for $\mathcal{A}$ and without loss of generality we may suppose that X is (φ, ψ)-unital, by Lemma 2.1. Hence, $\varphi(e) \cdot x = x = x \cdot \psi(e)$, for all $x \in X$. By Lemma 2.2, D is $\mathbb{C}$-linear and consequently (φ, ψ)-contractibility of $\mathcal{A}$ implies that D is (φ, ψ)-inner.
If (ii) is true, then $\mathcal{A}$ has an identity e for $\varphi(\mathcal{A})$ and we may suppose that X is (φ, φ)-unital. By Lemma 2.3, for each $a \in \mathcal{A}$, we have $D(e) \cdot \varphi(a) = 0$ and $D(ae) = D(a) = D(ea)$. Therefore, as in the proof of Lemma 2.2, we get $D(\lambda e) \cdot \varphi(a) = 0$, $(\lambda \in \mathbb{C}, a \in \mathcal{A})$. Thus,

$$D(\lambda a) = \varphi(\lambda e) \cdot D(a) + D(\lambda e) \cdot \varphi(a) = \lambda \varphi(e) \cdot D(a) = \lambda D(a), \quad (\lambda \in \mathbb{C}, a \in \mathcal{A}).$$

This means that, D is $\mathbb{C}$-linear. Consequently, (φ, φ)-contractibility of $\mathcal{A}$ implies that D is (φ, φ)-inner. □

Theorem 2.1. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module and (φ, ψ)-module contractible. Suppose that $\mathcal{A}$ is a $(\varphi|_K, \psi|_K)$-contractible for closed subalgebra K such that $\alpha \cdot e \in K$ $(\alpha \in \mathfrak{A})$, and one of the following is satisfied:*

(i) *φ and ψ are $\mathbb{C}$-linear with dense ranges and $e \in \mathcal{A}$ is an identity for $\mathcal{A}$,*
(ii) *φ and ψ are idempotents and $e \in \mathcal{A}$ is an identity for $\varphi(\mathcal{A}) \cup \psi(\mathcal{A})$.*

Then (φ, ψ)-contractibility of $\mathcal{A}$ follows from its (φ, ψ)-module contractibility.

Proof. Suppose that (i) holds and $D : \mathcal{A} \longrightarrow X$ is a (φ, ψ)-derivation for some Banach $\mathcal{A}$-bimodule X. Then, there is $x \in X$ such that

$$D(k) = \varphi(k) \cdot x - x \cdot \psi(k), \quad (k \in K),$$

that is $D = ad_x^{(\varphi,\psi)}$ on K. Let $\tilde{D} = D - ad_x^{(\varphi,\psi)}$ and Y be the $\mathcal{A}$-submodule of X generated by

$$\tilde{D}(\mathcal{A}) + \varphi(\mathcal{A}) \cdot \tilde{D}(\mathcal{A}) + \tilde{D}(\mathcal{A}) \cdot \psi(\mathcal{A}) + \varphi(\mathcal{A}) \cdot \tilde{D}(\mathcal{A}) \cdot \psi(\mathcal{A}).$$

Without loss of generality we may suppose that X is (φ, ψ)-unital. Then, $\tilde{D} : \mathcal{A} \longrightarrow Y \subseteq X$ is a (φ, ψ)-derivation such that the restriction of $\tilde{D}$ on K is zero. We now show that $\tilde{D}$ is (φ, ψ)-inner. To see this, we define compatible actions of $\mathfrak{A}$ on Y by

$$\alpha \circ y := \varphi(\alpha \cdot e) \cdot y, \ y \circ \alpha := y \cdot \psi(\alpha \cdot e), \quad (\alpha \in \mathfrak{A}, \ y \in Y).$$

Since $\tilde{D}|_K = 0$, we have

$$\begin{aligned} \tilde{D}(\alpha \cdot a) &= \tilde{D}((\alpha \cdot e)a), \\ &= \varphi(\alpha \cdot e) \cdot \tilde{D}(a) + \tilde{D}(\alpha \cdot e) \cdot \psi(a), \\ &= \varphi(\alpha \cdot e) \cdot \tilde{D}(a), \end{aligned}$$

for all $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$. Similarly, $\tilde{D}(a\cdot\alpha) = \tilde{D}(a)\cdot\psi(\alpha\cdot e)$. Thus, for $\alpha \in \mathfrak{A}$ and $a, b, c \in \mathcal{A}$, from $\mathfrak{A}$-commutativity of $\mathcal{A}$ it follows that

$$\begin{aligned}
\varphi(\alpha\cdot e)\cdot\big[\varphi(a)\cdot\tilde{D}(b)\cdot\psi(c)\big] &= \varphi(a)\cdot\big[\varphi(\alpha\cdot e)\cdot\tilde{D}(b)\cdot\psi(c)\big],\\
&= \varphi(a)\cdot\tilde{D}(\alpha\cdot b)\cdot\psi(c),\\
&= \varphi(a)\cdot\tilde{D}(b\cdot\alpha)\cdot\psi(c),\\
&= \varphi(a)\cdot\big[\tilde{D}(b)\cdot\psi(\alpha\cdot e)\big]\cdot\psi(c),\\
&= \big[\varphi(a)\cdot\tilde{D}(b)\cdot\psi(c)\big]\cdot\psi(\alpha.e).
\end{aligned}$$

Hence, $\varphi(\alpha\cdot e)\cdot y = y\cdot\psi(\alpha\cdot e)$ for all $\alpha \in \mathfrak{A}$ and $y \in Y$. Therefore, Y is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. On the other hand, $\tilde{D}(a\cdot\alpha) = \tilde{D}(\alpha\cdot a) = \alpha\circ\tilde{D}(a)$, for all $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$. Thus, $\tilde{D} : \mathcal{A} \longrightarrow Y$ is a (φ,ψ)-module derivation and there is $y \in Y$ such that

$$\tilde{D}(a) = \varphi(a)\cdot y - y\cdot\psi(a), \quad (a \in \mathcal{A}).$$

Consequently, $D = ad^{(\varphi,\psi)}_{x+y}$.
Assume that (ii) is valid and X is a (φ,ψ)-unital Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow X$ is a (φ,ψ)-derivation. We turn X into another Banach $\mathcal{A}$-bimodule via φ and ψ, i.e.,

$$a\circ x := \varphi(a)\cdot x, \;\; x\circ a := x\cdot\psi(a), \quad (a \in \mathcal{A}, x \in X). \tag{1}$$

It is clear that D is again a (φ,ψ)-derivation. By Lemma 2.3, we know that

$$D(ae) = D(a) = D(ea),$$

for all $a \in \mathcal{A}$. Now, as in the proof of (i), we obtain that D is (φ,ψ)-inner and therefore $\mathcal{A}$ is (φ,ψ)-contractible. □

In Colorally 2.1, we found sufficient conditions that (φ,ψ)-contractibility of $\mathcal{A}$ implies its (φ,ψ)-module contractibility. The next corollary may be considered as a converse of Colorally 2.1. This is done without extra assumption that the left action between $\mathfrak{A}$ and $\mathcal{A}$ is trivial.

Corollary 2.2. *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module where $\mathfrak{A}$ is contractible. Let $\varphi,\psi \in$Hom$_{\mathfrak{A}}(\mathcal{A})$ such that one of the following assertions hold.*

(i) *φ and ψ are $\mathbb{C}$-linear with dense ranges,*

(ii) *$\varphi = \psi$ is idempotent.*

Then, (φ,ψ)-contractibility of $\mathcal{A}$ follows from its (φ,ψ)-module contractibility.

Proof. Suppose that $e \in \mathcal{A}$ is an identity for $\varphi(\mathcal{A})\cap\psi(\mathcal{A})$, which exists by Proposition 2.1. Let K be the closed linear span of $\{\alpha\cdot\varphi(e) : \alpha \in \mathfrak{A}\}$. If $\varphi(e)^2 = \varphi(e)$, then K is a closed subalgebra of $\mathcal{A}$ under the following multiplication:

$$(\alpha\cdot\varphi(e))\cdot(\beta\cdot\varphi(e)) = (\alpha\beta)\cdot\varphi(e), \quad (\alpha,\beta \in \mathfrak{A}).$$

Consider $\theta : \mathfrak{A} \longrightarrow K$ be defined through $\theta(\alpha) = \alpha\cdot\varphi(e)$, for $\alpha \in \mathfrak{A}$. Then, θ is a continuous algebra homomorphism and $\theta(\mathfrak{A})$ is dense in K. Hence K is contractible (see Exercise 4.1.4 (i) of [24]). Now, if (i) holds, then $\varphi(e) = e = \psi(e)$ and by the definition of K, we have $\varphi|_K, \psi|_K \in$Hom$_{\mathfrak{A}}(K)$ and contractibility of K implies its $(\varphi|_K,\psi|_K)$-contractibility. Now, Theorem 2.1 (i) shows that $\mathcal{A}$ is (φ,ψ)-contractible.
If (ii) is true, then $\varphi(e)$ is a identity for $\varphi(\mathcal{A})$ and K is a closed subalgebra of $\mathcal{A}$ such that $\varphi|_K \in$Hom$_{\mathfrak{A}}(K)$. Therefore, K is $(\varphi|_K,\varphi|_K)$-contractible and satisfies the conditions of Theorem 2.1 (ii). □

Recall that there exists a commutative Banach $\mathfrak{A}$-module $\mathcal{A}$ such that it is $(id_{\mathcal{A}}, id_{\mathcal{A}})$-module contractible, but is not $(id_{\mathcal{A}}, id_{\mathcal{A}})$-contractible (see example in the end of [20]). This shows that the condition contractibility of $\mathfrak{A}$ in Corollary 2.2 is necessary.

3. Characterization of (φ,ψ)-module diagonal and (φ,ψ)-biprojectivity

Let $\mathcal{A}\hat{\otimes}\mathcal{A}$ be the projective tensor product of $\mathcal{A}$ by itself. Then $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module with the canonical actions. Consider the closed ideal $\mathcal{I}$ of $\mathcal{A}\hat{\otimes}\mathcal{A}$ generated by elements of the form $a\cdot\alpha\otimes b - a\otimes\alpha\cdot b$ ($\alpha\in\mathfrak{A}$, $a,b\in\mathcal{A}$). Let J be the closed ideal of $\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)b - a(\alpha\cdot b)$ ($\alpha\in\mathfrak{A}$, $a,b\in\mathcal{A}$). It is clear that J and $\mathcal{I}$ are both $\mathfrak{A}$-submodules and $\mathcal{A}$-submodules of $\mathcal{A}$ and $\mathcal{A}\hat{\otimes}\mathcal{A}$, respectively. Hence, the module projective tensor product $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}\cong(\mathcal{A}\hat{\otimes}\mathcal{A})/\mathcal{I}$ [23] and the quotient Banach algebra $\mathcal{A}/J$ are both Banach $\mathfrak{A}$-modules and Banach $\mathcal{A}$-modules. Now define $\tilde{\omega}\in\mathcal{L}(\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A},\mathcal{A}/J)$ by $\tilde{\omega}(a\otimes b+\mathcal{I}) = ab+J$, extended by linearity and continuity. Clearly, $\tilde{\omega}$ is an $\mathfrak{A}$-module morphism.

Suppose that $\varphi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$, $\alpha\in\mathfrak{A}$ and $a,b\in\mathcal{A}$. Then

$$\varphi((a\cdot\alpha)b - a(\alpha\cdot b)) = (\varphi(a)\cdot\alpha)\varphi(b) - \varphi(a)(\alpha\cdot\varphi(b))\in J,$$

and so $\varphi(J)\subseteq J$. Thus, if we use $\bar{a}$ to denote the coset of $a\in\mathcal{A}$ in $\mathcal{A}/J$, then we may define $\tilde{\varphi}:\mathcal{A}/J\longrightarrow\mathcal{A}/J$ by $\tilde{\varphi}(\bar{a}) = \overline{\varphi(a)}$.

The proof of the following lemma is the same as proof of Corollary 2.3 in [5], and so omitted.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-modules and $\varphi,\psi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$. If $\mathcal{A}$ is module contractible, then it is (φ,ψ)-module contractible.*

Lemma 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules, $\varphi,\psi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ and $\sigma,\tau\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{B})$. If $\theta\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$ has dense range such that $\theta\varphi=\sigma\theta$ and $\theta\psi=\tau\theta$, then (σ,τ)-module contractibility of $\mathcal{B}$ follows from (φ,ψ)-module contractibility of $\mathcal{A}$.*

Proof. Suppose that X is a commutative Banach $\mathcal{B}$-$\mathfrak{A}$-module and $D:\mathcal{B}\longrightarrow X$ is a (σ,τ)-module derivation. We turn X into a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule via θ. It is immediate that $D\circ\theta:\mathcal{A}\longrightarrow X$ is a (φ,ψ)-module derivation. Now, from (φ,ψ)-module contractibility of $\mathcal{A}$, there is $x\in X$ that $D\theta = ad_x^{(\varphi,\psi)}$. Thus, by density of range θ we get $D = ad_x^{(\sigma,\tau)}$. □

Lemma 3.3. *$\mathcal{A}$ is (φ,ψ)-module contractible if and only if $\mathcal{A}/J$ is $(\tilde{\varphi},\tilde{\psi})$-module contractible.*

Proof. Let $\mathcal{A}/J$ be $(\tilde{\varphi},\tilde{\psi})$-module contractible. Suppose that X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D:\mathcal{A}\longrightarrow X$ is a (φ,ψ)-module derivation. Clearly, $J\cdot X = X\cdot J = \{0\}$, and thus X is a commutative Banach $\mathcal{A}/J$-$\mathfrak{A}$-module by the same actions of $\mathfrak{A}$ and

$$\bar{a}\circ x = a\cdot x,\quad x\circ\bar{a} = x\cdot a,\quad a\in\mathcal{A}.$$

Since D vanishes on J, it induces a map $\tilde{D}$ from $\mathcal{A}/J$ to X which is $(\tilde{\varphi},\tilde{\psi})$-module derivation. Hence, there is $x\in X$ such that $\tilde{D} = ad_x^{(\tilde{\varphi},\tilde{\psi})}$. It is routine to check that $D = ad_x^{(\tilde{\varphi},\tilde{\psi})}$. The converse follows from Lemma 3.2 for the natural homomorphism $\pi:\mathcal{A}\longrightarrow\mathcal{A}/J$. □

Here, we give the concept of (φ,ψ)-module diagonal. It extends the notions of diagonal and module diagonal for Banach algebras.

Definition 3.1. *An element $\boldsymbol{m}\in\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is called a (φ,ψ)-module diagonal for $\mathcal{A}$ if*

$$\varphi(a)\cdot\boldsymbol{m} = \boldsymbol{m}\cdot\psi(a),\quad \tilde{\omega}(\boldsymbol{m}).\varphi(a) = \overline{\varphi(a)},\quad \psi(a)\cdot\tilde{\omega}(\boldsymbol{m}) = \overline{\psi(a)},\quad (a\in\mathcal{A}).$$

We note that $(id_{\mathcal{A}},id_{\mathcal{A}})$-module diagonal is exactly a module diagonal [3, 20]. Moreover, when $\mathfrak{A}:=\mathbb{C}$, everything reduces to the classical case [24].

Proposition 3.1. *Suppose that φ and ψ have dense renges such that $\tilde{\psi}^2 = \tilde{\psi}\tilde{\varphi}$. If $\mathcal{A}$ has a (φ,ψ)-module diagonal, then it is (φ,ψ)-module contractible.*

Proof. Assume that $D : \mathcal{A}/J \longrightarrow X$ is a $(\tilde{\varphi}, \tilde{\psi})$-module derivation, where X is a commutative Banach $\mathcal{A}/J$-$\mathfrak{A}$-module. Define

$$P : \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A} \longrightarrow X, \quad a \otimes b + \mathfrak{I} \mapsto D(\bar{a}) \cdot \tilde{\psi}(\bar{b}), \quad (a, b \in \mathcal{A}).$$

Let $\mathbf{m}$ be a (φ, ψ)-module diagonal for $\mathcal{A}$. Then, $\mathbf{m} = \sum_{n=1}^{\infty} a_n \otimes b_n + \mathfrak{I}$ where $(a_n)_n$ and $(b_n)_n$ are bounded sequences in $\mathcal{A}$. Since $\varphi(\mathcal{A})$ and $\psi(\mathcal{A})$ are dense in $\mathcal{A}$, $\tilde{\omega}(\mathbf{m}) = \sum_{n=1}^{\infty} \overline{a_n b_n}$ is an identity for $\mathcal{A}/J$ and $\sum_{n=1}^{\infty} \varphi(a)a_n \otimes b_n = \sum_{n=1}^{\infty} a_n \otimes b_n\psi(a)$ for all $a \in \mathcal{A}$. Therefore,

$$\sum_{n=1}^{\infty} D(\tilde{\varphi}(\bar{a})\overline{a_n}) \cdot \tilde{\psi}(\overline{b_n}) = \sum_{n=1}^{\infty} D(\overline{a_n}) \cdot \tilde{\psi}(\overline{b_n})\tilde{\psi}^2(\bar{a}), \quad (a \in \mathcal{A}).$$

We also may assume that X is $(\tilde{\varphi}, \tilde{\psi})$-unital. Thus, $D(\tilde{\omega}(\mathbf{m})) = D(\sum_{n=1}^{\infty} \overline{a_n b_n}) = 0$. Put $x = \sum_{n=1}^{\infty} \tilde{\varphi}(\overline{a_n}).D(\overline{b_n})$. For each $a \in \mathcal{A}$, let $(a_i)_i$ be a sequence in $\mathcal{A}$ that $\varphi(a_i)$ converges to a. Then

$$\begin{aligned}
D(\bar{a}) &= \lim_i \sum_{n=1}^{\infty} D(\tilde{\varphi}(\overline{a_i})\overline{a_n b_n}),\\
&= \lim_i \sum_{n=1}^{\infty} \left[\tilde{\varphi}(\overline{\varphi(a_i)a_n}).D(\overline{b_n}) + D(\overline{\varphi(a_i)a_n}).\tilde{\psi}(\overline{b_n})\right],\\
&= \lim_i \tilde{\varphi}^2(\bar{a}_i) \cdot x + \lim_i \sum_{n=1}^{\infty} D(\overline{a_n}) \cdot \tilde{\psi}(\overline{b_n})\tilde{\psi}^2(\bar{a}_i),\\
&= \lim_i \tilde{\varphi}^2(\bar{a}_i) \cdot x + \lim_i D(\sum_{n=1}^{\infty} \overline{a_n b_n}) \cdot \tilde{\psi}^2(\bar{a}_i) - \lim_i x \cdot \tilde{\psi}^2(\bar{a}_i),\\
&= \tilde{\varphi}(\lim_i \tilde{\varphi}(\bar{a}_i)) \cdot x - x \cdot \tilde{\psi}(\lim_i \tilde{\varphi}(\bar{a}_i))\\
&= \tilde{\varphi}(\bar{a}) \cdot x - x \cdot \tilde{\psi}(\bar{a}).
\end{aligned}$$

Hence, $D = ad_x^{(\tilde{\varphi},\tilde{\psi})}$. Therefore, $\mathcal{A}/J$ is $(\tilde{\varphi}, \tilde{\psi})$-module contractible and (φ, ψ)-module contractibility of $\mathcal{A}$ follows from Lemma 3.3. □

The upcoming outcome may be considered as a converse version of Proposition 3.1.

Proposition 3.2. *Suppose that $\mathcal{A}/J$ and $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ are commutative Banach $\mathfrak{A}$-modules, and $\varphi, \psi \in$Hom$_{\mathfrak{A}}(\mathcal{A})$ such that one of the following is satisfied:*

(i) *φ and ψ have dense ranges, such that $\tilde{\varphi}^2 = \tilde{\psi}\tilde{\varphi}$ or $\tilde{\psi}^2 = \tilde{\varphi}\tilde{\psi}$.*
(ii) *$\varphi = \psi$.*

Then, $\mathcal{A}$ has a (φ, ψ)-module diagonal, whenever it is (φ, ψ)-module contractible.

Proof. Assume that (i)is valid. It follows from Proposition 2.1 and Lemma 3.3 that there exists $e \in \mathcal{A}$ such that $\bar{e}$ is an identity for $\mathcal{A}/J$. Consider $D(a) := ad_{e\otimes e+\mathfrak{I}}^{(\varphi,\psi)} : \mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$. For all $a \in \mathcal{A}$, we have

$$\tilde{\omega}\big(D(\varphi(a))\big) = \tilde{\omega}\big(\varphi(\varphi(a)) \cdot e \otimes e + \mathfrak{I} - e \otimes e \cdot \psi(\varphi(a)) + \mathfrak{I}\big) = \tilde{\varphi}^2(\bar{a}) - \tilde{\psi}(\tilde{\varphi}(\bar{a})),$$

and

$$\tilde{\omega}\big(D(\psi(a))\big) = \tilde{\omega}\big(\varphi(\psi(a)) \cdot e \otimes e + \mathfrak{I} - e \otimes e \cdot \psi(\psi(a)) + \mathfrak{I}\big) = \tilde{\varphi}(\tilde{\psi}(\bar{a})) - \tilde{\psi}^2(\bar{a}).$$

By assumptions, $\tilde{\omega}\big(D(\varphi(a))\big) = 0$ or $\tilde{\omega}\big(D(\psi(a))\big) = 0$. Hence, $\tilde{\omega}\big(D(a)\big) = 0$ for all $a \in \mathcal{A}$. Thus, $ad_{e\otimes e+\mathfrak{I}}^{(\varphi,\psi)} : \mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is a (φ, ψ)-module derivation into ker$\tilde{\omega}$. Since $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, so is ker$\tilde{\omega}$. Therefore, there is $n + \mathfrak{I} \in$ ker$\tilde{\omega}$ such that

$ad^{(\varphi,\psi)}_{e\otimes e+\mathfrak{I}} = ad^{(\varphi,\psi)}_{n+\mathfrak{I}}$. It is easy to verify that $(e\otimes e - n)+\mathfrak{I}$ is a (φ,ψ)-module diagonal for $\mathcal{A}$. Let (ii) holds and $\bar{e}$ be an identity for $\tilde{\varphi}(\mathcal{A}/J)$. Then

$$\tilde{\omega}\big(\varphi(a)\cdot e\otimes e+\mathfrak{I} - e\otimes e\cdot\varphi(a)+\mathfrak{I}\big) = 0_{\mathcal{A}/J} \quad (a\in\mathcal{A}).$$

Similarly, $(e\otimes e - n)+\mathfrak{I}$ is a (φ,φ)-module diagonal for $\mathcal{A}$. □

The next example shows that the concept of (φ,ψ)-module contractibility for a Banach algebra $\mathcal{A}$ is not equivalent to the existence of a (φ,ψ)-module diagonal for it, whenever φ and ψ are arbitrary elements of $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.

Example 3.1. Let $\mathfrak{A}$ and $\mathcal{A}$ be the Banach algebras of complex 2×2 matrices of the form

$$\mathfrak{A} = \left\{\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix} : a\in\mathbb{C}\right\},\ \mathcal{A} = \left\{\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} : a,b\in\mathbb{C}\right\}.$$

Then, $J=0$, $\mathcal{A}$ and $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ are commutative Banach $\mathfrak{A}$-modules. $\mathcal{A}$ is unital and finite-dimensional. Therefore, it is contractible and so is module contractible by Corollary 2.1. Define $\varphi,\psi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ through

$$\varphi\left(\begin{bmatrix} a & 0\\ 0 & d\end{bmatrix}\right) = \begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix},\quad \psi\left(\begin{bmatrix} a & 0\\ 0 & d\end{bmatrix}\right) = \begin{bmatrix} 0 & 0\\ 0 & d\end{bmatrix}.$$

It is clear that φ and ψ are idempotents, but neither φ nor ψ have dense ranges. From Lemma 3.1, $\mathcal{A}$ is (φ,ψ)-module contractible.
Now, if $\mathbf{m} = \sum_{i=1}^n \begin{bmatrix} x_i & 0\\ 0 & z_i\end{bmatrix}\otimes\begin{bmatrix} u_i & 0\\ 0 & w_i\end{bmatrix} + \mathfrak{I}$ is a (φ,ψ)-module diagonal for $\mathcal{A}$, then

$$e_{11} = \varphi(e_{11}) = \tilde{\omega}(\boldsymbol{m})\cdot\varphi(e_{11}). = \sum_{i=1}^n \begin{bmatrix} x_iu_i & 0\\ 0 & z_iw_i\end{bmatrix}\cdot\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix} = \sum_{i=1}^n\begin{bmatrix} x_iu_i & 0\\ 0 & 0\end{bmatrix}.$$

and

$$e_{22} = \psi(e_{22}) = \psi(e_{22})\cdot\tilde{\omega}(\boldsymbol{m}) = \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}\cdot\sum_{i=1}^n\begin{bmatrix} x_iu_i & 0\\ 0 & z_iw_i\end{bmatrix} = \sum_{i=1}^n\begin{bmatrix} 0 & 0\\ 0 & z_iw_i\end{bmatrix}.$$

Thus, $\sum_{i=1}^n x_iu_i = \sum_{i=1}^n z_iw_i = 1$ and hence $\tilde{\omega}(\mathbf{m})$ is the identity matrix. This implies that

$$e_{11} = \varphi(e_{11})\cdot\tilde{\omega}(\boldsymbol{m}) = \tilde{\omega}(\varphi(e_{11})\cdot\boldsymbol{m}) = \tilde{\omega}(\boldsymbol{m}\cdot\psi(e_{11})) = 0,$$

which is a contradiction. Therefore, there is no any (φ,ψ)-module diagonal for $\mathcal{A}$.

Here, we give the notion of (φ,ψ)-module biprojectivity as a generalization of the earlier notions of biprojectivity and module biprojectivity [6] for Banach algebras.

Definition 3.2. *A Banach $\mathfrak{A}$-module $\mathcal{A}$ is called (φ,ψ)-module biprojective if there is an $\mathfrak{A}$-module morphism $\rho:\mathcal{A}/J\longrightarrow\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ such that $\tilde{\omega}\rho$ is the identity map on $\tilde{\psi}(\mathcal{A}/J)$ and for all $a,b\in\mathcal{A}$,*

$$\varphi(a)\cdot\rho(\tilde{\varphi}(\bar{b})) = \rho(\tilde{\varphi}(\overline{ab})) = \rho(\tilde{\varphi}(\bar{a}))\cdot\psi(b).$$

We note that if φ and ψ are the identity maps, then $(id_{\mathcal{A}}, id_{\mathcal{A}})$-module biprojectivity overlaps on module biprojectivity [6, 20]. Moreover, in the case where $\mathfrak{A}:=\mathbb{C}$, then $id_{\mathcal{A}}$-module biprojectivity and biprojectivity coincide [10, 24].

Theorem 3.1. *Let $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. If there is $e\in\mathcal{A}$ such that $\bar{e}$ is an identity for $\tilde{\varphi}(\mathcal{A}/J)\cup\tilde{\psi}(\mathcal{A}/J)$ and $\tilde{\varphi}(\bar{e}) = \bar{e} = \tilde{\psi}(\bar{e})$, then $\mathcal{A}$ is (φ,ψ)-module biprojective, if and only if $\mathcal{A}$ has a (φ,ψ)-module diagonal.*

Proof. Since $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module, as in the proof of Lemma 3.3, $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is a commutative Banach $\mathcal{A}/J$- $\mathfrak{A}$-module. Let $\mathbf{m} \in \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be a (φ,ψ)-module diagonal for $\mathcal{A}$. For all $a,b,c \in \mathcal{A}$, we find

$$\tilde{\omega}\big(\bar{a} \circ (b \otimes c + \mathfrak{I})\big) = \tilde{\omega}(ab \otimes c + \mathfrak{I}) = \overline{abc} = \bar{a} \cdot \tilde{\omega}(b \otimes c + \mathfrak{I}).$$

Therefore, for all $a \in \psi(\mathcal{A})$, we have $\tilde{\omega}\big(\bar{a} \circ \mathbf{m}\big) = \bar{a} \cdot \tilde{\omega}(\mathbf{m})$. Define $\rho : \mathcal{A}/J \longrightarrow \mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ via $\rho(\bar{a}) = \bar{a} \circ \mathbf{m}$ $(a \in \mathcal{A})$. Then, ρ satisfies the conditions of Definition 3.2. Consequently, $\mathcal{A}$ is (φ,ψ)-module biprojective.
Conversely, let $\mathcal{A}$ be (φ,ψ)-module biprojective and ρ be as in Definition 3.2. Then

$$\rho(\tilde{\varphi}(\overline{a})\tilde{\varphi}(\overline{e})) = \rho(\tilde{\varphi}(\overline{e})\tilde{\varphi}(\overline{a})).$$

This implies that

$$\varphi(a) \cdot \rho(\tilde{\varphi}(\overline{e})) = \rho(\tilde{\varphi}(\overline{e})) \cdot \psi(a), \quad (a \in \mathcal{A}).$$

Now, it is routinely checked that $\mathbf{m} := \rho(\bar{e})$ is a (φ,ψ)-module diagonal for $\mathcal{A}$. □

Corollary 3.1. *Let $\mathcal{A}\hat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module and $\varphi,\psi \in$*Hom$_{\mathfrak{A}}(\mathcal{A})$. *Suppose that one of the following assumptions holds.*

(i) *φ and ψ have dense ranges.*
(ii) *$\varphi = \psi$ is idempotent.*

Then, $\mathcal{A}$ has a (φ,ψ)-module diagonal, if and only if $\mathcal{A}$ is (φ,ψ)-module biprojective and $\mathcal{A}/J$ has an identity for $\tilde{\varphi}(\mathcal{A}/J) \cup \tilde{\psi}(\mathcal{A}/J)$.

4. **Applications to Semigroup algebras**

Let S be an arbitrary semigroup and E be the set idempotents of S which it is a commutative subsemigroup of S (see [15]).Then, $l^1(E)$ could be regarded as a subalgebra of $l^1(S)$ and thereby $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module with proper compatible actions. It is possible to consider arbitrary actions of $l^1(E)$ on $l^1(S)$ and prove certain module amenability results. In the results of this section we do not restrict ourselves to any particular action.
This following example shows that the class of (φ,ψ)-module contractible Banach algebras is large than the category of module contractible Banach algebras.

Example 4.1. Let $S = (\mathbb{N}, \wedge)$ be the inverse semigroup of positive integers with the minimum operation, $\mathcal{A} = l^1(S)$ and $\mathfrak{A} = l^1(E)$. Then, $\mathcal{A}$ is a commutative Banach algebra that is not unital, and hence $\mathcal{A}$ not contractible. Assume that the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is a bounded approximate identity for $\mathcal{A}$. Consider $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-module under the actions defined by the algebra multiplication. Let $\varphi(f) = \delta_1 * f$, for all $f \in \mathcal{A}$. Clearly, $\varphi \in$ Hom$_{\mathfrak{A}}(\mathcal{A})$ is $\mathbb{C}$-linear and idempotent, but has not dense range. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and D be a (φ,φ)-module derivation from $\mathcal{A}$ into X. Then

$$\begin{aligned} D(\delta_n) &= D(\delta_n * \delta_n), \\ &= \varphi(\delta_n) \cdot D(\delta_n) + D(\delta_n) \cdot \varphi(\delta_n), \\ &= \delta_1 \cdot D(\delta_n) + D(\delta_n) \cdot \delta_1, \\ &= D(\delta_1 * \delta_n) + D(\delta_n * \delta_1), \\ &= 2D(\delta_1) \quad (n \in \mathbb{N}). \end{aligned}$$

In particular, $D(\delta_1) = 2D(\delta_1)$ and hence $D(\delta_1) = 0$. Thus, $D(\delta_n) = 0$ for all $n \in \mathbb{N}$. Since $\mathcal{A} = \mathfrak{A}$,

$$D(f) = \lim_n D(\delta_n * f) = \lim_n D(\delta_n) \cdot f = 0,$$

for all $f \in \mathcal{A}$. It follows that D is zero. Consequently, $\mathcal{A}$ is (φ,φ)-module contractible. On the other hand, $\mathcal{A}$ is not unital and so not module contractible [22].

If $l^1(S)$ is a commutative Banach $l^1(E)$-module, then so is $l^1(S)\hat{\otimes}_{l^1(E)}l^1(S)$ and moreover $J=0$. Therefore, by Corollary 3.1, Proposition 3.2, Proposition 3.1, Corollary 2.1 and Corollary 2.2, we get the following result.

Theorem 4.1. *Let $l^1(S)$ be a commutative Banach $l^1(E)$-module. If $\varphi,\psi\in\mathrm{Hom}_{l^1(E)}(l^1(S))$ have dense ranges, then the following statements are true.*

(i) *If $l^1(E)$ is contractible, then $l^1(S)$ is (φ,ψ)-contractible, if and only if it is (φ,ψ)-module contractible.*

(ii) *If $\varphi=\psi$, then $l^1(S)$ is (φ,ψ)-module contractible, if and only if it has a (φ,ψ)-module diagonal.*

(iii) *$l^1(S)$ has a (φ,ψ)-module diagonal, if and only if it is unital and (φ,ψ)-module biprojective.*

Note that $L^1(G)$ is semisimple, when G is a locally compact group (Corollary 2.7.9 of [19]). Thus, the following corollary extends the previously known results duo to Selivanov [25] and Helemskii [14] (see also Theorem 3.3.32 of [10]).

Corollary 4.1. *Let S be a semigroup with finitely many idempotents. If $\varphi\in\mathrm{Hom}_{l^1(E)}(l^1(S))$ has dense range and $l^1(S)$ is a commutative Banach $l^1(E)$-module, then the following statements are equivalent:*

(i) *$l^1(S)$ is semisimple and S is finite.*

(ii) *$l^1(S)$ is (φ,φ)-module contractible.*

(iii) *$l^1(S)$ has a (φ,φ)-module diagonal.*

(iv) *$l^1(S)$ is unital and (φ,φ)-module biprojective.*

Proof. It is obvious that E is an inverse semigroup and finite semilattice. Thus, $l^1(E)$ is finite-dimensional amenable Banach algebra by Theorem 8 of [11], and so it is contractible. Hence (ii), (iii) and (iv) are equivalent by Theorem 4.1. Furthermore, $l^1(S)$ is (φ,φ)-module contractible if and only if it is (φ,φ)-contractible. Since φ has dense range, $l^1(S)$ is (φ,φ)-contractible if and only if it is contractible, and so (i) and (ii) are equivalent by Theorem 1.9.21 in [10]. □

This next example shows that condition of density of $\varphi(\mathcal{A})$ is necessary in Corollary 4.1 and this corollary is not valid for arbitrary φ in $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$.

Example 4.2. Let $S=(\mathbb{Z},.)$ be the semigroup of integers with the common multiplication. Then, S is a commutative, infinite semigroup, with idempotents $E=\{0,1\}$. Since E is a commutative subsemigroup of S, $\mathcal{A}:=l^1(S)$ is a commutative Banach $\mathfrak{A}:=l^1(E)$-module under the actions defined by the algebra multiplication. Let $\varphi(f)=f*\delta_0$, for all $f\in\mathcal{A}$. Clearly, $\varphi\in\mathrm{Hom}_{\mathfrak{A}}(\mathcal{A})$ is $\mathbb{C}$-linear that has not dense range. If X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and D is a (φ,φ)-module derivation from $\mathcal{A}$ into X, then

$$\begin{aligned} D(\delta_n) &= D(\delta_1*\delta_n),\\ &= \varphi(\delta_1)\cdot D(\delta_n)+D(\delta_1)\cdot\varphi(\delta_n),\\ &= \delta_0\cdot D(\delta_n)+D(\delta_1)\cdot\delta_0,\\ &= D(\delta_0*\delta_n)+D(\delta_1*\delta_0),\\ &= 2D(\delta_0), \end{aligned}$$

for all $n\in\mathbb{Z}$. In particular, $D(\delta_0)=2D(\delta_0)$, and thus $D(\delta_0)=0$. Hence, for all $f=\sum_n\alpha_n\delta_n$ in $\mathcal{A}$ we have

$$D(f)=D\left(\sum_n\alpha_n\delta_n\right)=D\left(\sum_n\alpha_n\delta_1*\delta_n\right)=\sum_n\alpha_n\delta_1\cdot D(\delta_n)=0.$$

This means that D is zero. Therefore, $\mathcal{A}$ is (φ,φ)-module contractible while S is not finite.

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