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TSENG-TYPE SUBGRADIENT METHODS FOR SOLVING NONMONOTONE VARIATIONAL INEQUALITIES

Xiangyao Li[1], Li-jun Zhu[2], Mihai Postolache[3]

In this article, we introduce two new approaches for solving variational inequalities without monotonicity. The first algorithm simplifies the projection region of each iteration in Ye and He [Comput. Optim. Appl., 60 (2015), 141-150], that is, it becomes the intersection of multiple half-spaces and no longer needs to be intersected with the feasible set. By a selection technique, the second algorithm replaces the projection on the common region of the feasible set and multiple half-spaces with a specific half-spaces in each iteration. The strong convergence of these two algorithms have been demonstrated under the assumption that the Minty variational inequality has a solution. Finally, some numerical examples are given to illustrate the advantages of the proposed algorithms.

Keywords: Variational inequalities, nonmonotone, projection method, Armijo line search.

MSC2020: 47H10, 58A05.

1. Introduction

Let H be a real Hilbert space and $C \subset H$ represent a nonempty convex closed set. Denote $\Gamma\colon H \longrightarrow H$ a continuous mapping, $\|\cdot\|$ and $\langle\cdot\rangle$ the norm and inner product in H, respectively. Variational inequalities (VI) problem is finding an element $u \in C$ such that

$$\langle \Gamma(u), x-u\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

We also know that the Minty variational inequality of (1) is formulated in the following form: find an element $u \in C$ such that

$$\langle \Gamma(x), x-u\rangle \geq 0, \ \forall x \in C.$$

Let S be the solution set of variational inequalities problem, and S_D be the solution set of the Minty variational inequality of (1). According to Karamardian [9] in 1976, we obtain that $S \subset S_D$ when Γ is pseudomonotone. Moreover, suppose that C is nonempty

[1] School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China, email li_xyao1@163.com

[2] Corresponding author. School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China & Health Big Data Research Institute of North Minzu University,Yinchuan 750021, China & The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China, email zljmath0@gmail.com

[3] Corresponding author. Health Big Data Research Institute of North Minzu University,Yinchuan 750021, China & Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest 050711, Romania and University "Politehnica" of Bucharest, Department of Mathematics and Informatics, Bucharest 060042, Romania, email mihai.postolache@upb.ro

convex closed and Γ represents a continuous mapping on C. Then the conclusion $S_D \subset S$ holds based on the Minty lemma. Hence, we have that $S = S_D$, when $C \subset H$ is nonempty convex closed and Γ represents a pseudomonotone and continuous mapping on C.

For the convergence analysis of many classical algorithms, the assumption that $S = S_D$ is very important. Such as the Goldstein-Levitin-Polyak projection algorithm [2], [3], proximal point algorithm [4], extragradient algorithm [5], subgradient extragradient algorithm [7], Tseng algorithm [8] and their variant algorithms (see [11], [12], [14], [17]). However, if the mapping Γ is quasimonotone or even nonmonotone, the condition $S = S_D$ no longer holds. In order to solve more general variational inequalities, we need to find new methods. In 2015, Ye and He [1] proposed a double projection approach for solving variational inequalities. This method guarantees convergence only under the assumption that S_D is nonempty and does not depend on the generalized monotonicity of Γ. However, for getting next iteration point x_{k+1} in [1], we need to project the current iteration point onto the common region of $k+1$ half-spaces and the feasible set. It is well known that the calculation of projection is complicated. Therefore, it will be very meaningful to cut down the computational cost of the approach in [1]. To this end, the first algorithm is introduced.

In 2021, Lei and He [19] proposed an improved extragradient algorithm for solving nonmonotone variational inequalities. Undoubtedly, this is a more efficient approach than [1]. Very recently, a modified Solodov-Svaiter method was introduced in [23]. In this method, the projection region in [1] is reduced to the common region of a certain half-space and the feasible set. However, both the methods in [19] and [23] need to project a vector onto the feasible set twice (or more complicated) in every iteration. Inspired by [7], [10], [11], [19] and [23], we proposed the second algorithm. The improved Tseng method and the subgradient method are combined, so as to achieve the purpose of convenient calculation. In fact, we can't prove theoretically which algorithm is more efficient, the first algorithm or the second algorithm. So, we have left both of them for interested readers to study. Throughout this article, we stipulate that the Minty variational inequality of problem (1) has a solution. The reader is referred to Ye [10] for more details on $S_D \neq \emptyset$.

The rest of this paper is structured as follows. In section 2, we recall some lemmas and properties for use in the following sections. In section 3, we propose two new iterative schemes to solve nonmonotone variational inequalities and perform convergence analysis on them. When the mapping imposed on variational inequalities is Lipschitz continuous, we can simplify the proposed two algorithms. In section 4, we demonstrate the efficiency of the introduced algorithms through several numerical examples.

2. **Preliminaries**

In this section, we present some properties and conclusions which will be useful for the following convergence analysis.

Let C represents a nonempty convex closed subset of H, and denote the distance from an element $x_1 \in H$ to C by $d(x_1, C)$; namely

$$d(x_1, C) := \inf\{\|x_1 - x_2\| : x_2 \in C\}.$$

Set $P_C(u_1)$ the projection of a vector u_1 onto C; namely

$$P_C(u_1) := \arg\min\{\|u_1 - u_2\| : u_2 \in C\}.$$

Because C represents a convex closed set, we get easily $d(u_1, C) = \|u_1 - P_C(u_1)\|$.

For a point $u \in H$ and a fixed number $\delta > 0$, we call $r(u, \delta)$ the residual function of the variational inequality (1); namely

$$r(u, \delta) = u - P_C(u - \delta\Gamma(u)). \tag{2}$$

Lemma 2.1 ([10])**.** *Let C represent a nonempty convex closed set. Then, the next propositions hold.*

(a) *for a vector $u_1 \in H$, we have*

$$v = P_C(u_1) \iff v \in C \text{ and } \langle u_1 - v, u_2 - v\rangle \le 0 \quad \text{for all } u_2 \in C.$$

(b) *the projection is nonexpansive, namely*

$$\|P_C(x_1) - P_C(x_2)\| \le \|x_1 - x_2\| \text{ for any } x_1, x_2 \in H.$$

(c) *set $v = P_C(u_1)$; then we have*

$$\|v - u_2\|^2 \le \|u_1 - u_2\|^2 - \|u_1 - v\|^2 \text{ for all } u_2 \in C.$$

Lemma 2.2 ([24, 25])**.** *Suppose that $r(u, \delta)$ is defined by* (2)*. Then the next two conclusions hold.*

(a) *a sufficient and necessary condition for $u \in S$ is that $\|r(u, \delta)\| = 0$ for any fixed $\delta > 0$.*

(b) *if we can find a constant $\delta > 0$ satisfying $\|r(u, \delta)\| = 0$, then u is an element in S.*

Lemma 2.3 ([23])**.** *Let $r(u, \delta)$ be defined by* (2) *and $u \in H$, then we have the next propositions.*

(a) *function $\delta \mapsto \|r(u, \delta)\|$ is nondecreasing whenever $\delta > 0$.*

(b) *function $\delta \mapsto \frac{\|r(u,\delta)\|}{\delta}$ is nonincreasing whenever $\delta > 0$.*

According to Lemma 2.3, we can get directly the following inequality and omit its proof. For any constant $\delta > 0$, we obtain

$$\min\{\delta, 1\}\|r(u, 1)\| \le \|r(u, \delta)\| \le \max\{\delta, 1\}\|r(u, 1)\|. \tag{3}$$

Lemma 2.4 ([10])**.** *Let $\{\alpha_k\}$ and $\{\beta_k\}$ represent two nonnegative real number sequences. If for any k, we have $\alpha_{k+1} \le \alpha_k + \beta_k$ and $\sum_{k=0}^{\infty} \beta_k < \infty$, then the sequence $\{\alpha_k\}$ is convergent.*

3. Main results

In this section, we propose two new approaches and their variant forms under Lipschitz continuity assumption for solving variational inequalities without monotonicity.

Algorithm 3.1

Step 0. Let Γ represent a continuous mapping on H; choose parameters $0 < \tilde{a} < \bar{a}$, $\theta, \sigma \in (0, 1)$, and choose $u_0 \in H$. Set $k = 0$.

Step 1. Compute $\vartheta_k = P_C(u_k - \alpha_k\Gamma(u_k))$ and $\alpha_k = \alpha_k^0\theta^{m_k}$, where $\alpha_k^0 \in [\tilde{a}, \bar{a}]$ and m_k represents the smallest nonnegative integer m satisfies

$$\alpha_k^0\theta^m\|\Gamma(u_k) - \Gamma(P_C(u_k - \alpha_k^0\theta^m\Gamma(u_k)))\| \leq \sigma\|u_k - P_C(u_k - \alpha_k^0\theta^m\Gamma(u_k))\|. \tag{4}$$

If $u_k = \vartheta_k$, then program stops. Else, it continues.

Step 2. Calculate

$$z_k = (1-\beta_k)u_k + \beta_k[\vartheta_k + \alpha_k(\Gamma(u_k) - \Gamma(\vartheta_k))],$$

where $\{\beta_k\} \subset [a, b] \subset (0, 2)$.

Step 3. Set $h_j = \{x \in H : \langle\Gamma(\vartheta_j), x - \vartheta_j\rangle \leq 0\}$, and let $H_k = \bigcap_{i=0}^k h_i$. Compute

$$u_{k+1} = P_{H_k}z_k.$$

Step 4. Let $k = k+1$ and return to Step 1.

Remark 3.1. If we remove the last part in Algorithm 3.1, namely projection on H_k, then the above algorithm collapses into the extrapolated Tseng algorithm in Bot [11].

Now, we show that the linesearch procedure is well-defined in Algorithm 3.1.

Lemma 3.1. *If Γ is continuous on H and $\alpha \in [\tilde{a}, \bar{a}]$ be a constant, then we can find a nonnegative integer m satisfying*

$$\alpha\theta^m\|\Gamma(u) - \Gamma(P_C(u - \alpha\theta^m\Gamma(u)))\| \leq \sigma\|u - P_C(u - \alpha\theta^m\Gamma(u))\|, \tag{5}$$

where $\theta, \sigma \in (0, 1)$.

Proof. If u is an element in the solution set of problem (1), then together with Lemma 2.2(a), we attain that $u - P_C(u - \alpha\Gamma(u)) = 0$. Take $m = 0$, and inequality (5) holds.

If $u \notin S$, based on Lemma 2.2(b), for every $\delta > 0$ we have $\|r(u, \delta)\| > 0$. Next, we prove that inequality (5) holds after a finite number of steps. Suppose to the contradict circumstance, for all m

$$\alpha\theta^m\|\Gamma(u) - \Gamma(P_C(u - \alpha\theta^m\Gamma(u)))\| > \sigma\|u - P_C(u - \alpha\theta^m\Gamma(u))\|. \tag{6}$$

Then, we discuss the problem in two possibilities.

Case 1: If $u \in C$, then we can get $P_C(u) = u$. By the fact $\theta \in (0, 1)$ and both $P_C(\cdot)$ and $\Gamma(\cdot)$ are continuous, we see

$$\|\Gamma(u) - \Gamma(P_C(u - \alpha\theta^m\Gamma(u)))\| \to 0 \text{ as } m \to \infty. \tag{7}$$

Moreover, together with $\theta \in (0, 1)$ and Lemma 2.3(b), for sufficient large m

$$\frac{\|u - P_C(u - \alpha\theta^m\Gamma(u))\|}{\alpha\theta^m} = \frac{\|r(u, \alpha\theta^m)\|}{\alpha\theta^m} \geq \frac{\|r(u, 1)\|}{1}.$$

Since $u \notin S$, we have $\|r(u, 1)\| > 0$. By $\sigma \in (0, 1)$ and (7), for sufficient large m

$$\sigma\frac{\|r(u, \alpha\theta^m)\|}{\alpha\theta^m} \geq \sigma\|r(u, 1)\| > \alpha\theta^m\|\Gamma(u) - \Gamma(P_C(u - \alpha\theta^m\Gamma(u)))\|.$$

It contradicts (6).

Case 2: If $u \notin C$, according to Lemma 2.2(b), we have $\sigma\|u - P_C(u)\| > 0$. Since $\Gamma(\cdot)$ and $P_C(\cdot)$ are continuous, we see that $\alpha\theta^m\|\Gamma(u) - \Gamma(P_C(u - \alpha\theta^m\Gamma(u)))\| \to 0$ and $\sigma\|u - P_C(u - \alpha\theta^m\Gamma(u))\| \to \sigma\|u - P_C(u)\| > 0$ as $m \to \infty$. This contradicts (6).

After all, the proof is completed. □

Set $\widetilde{H} = \bigcap_{i=0}^{\infty} h_i$. Because $S_D \neq \emptyset$, we can get clearly $\widetilde{H} \bigcap C \neq \emptyset$.

Proposition 3.1. *Let the sequence $\{u_k\}$ and $\{z_k\}$ be generated by Algorithm 3.1. Then for any $\omega^* \in \widetilde{H} \bigcap C$, we obtain that $\|u_{k+1} - \omega^*\| \leq \|z_k - \omega^*\| \leq \|u_k - \omega^*\|$ for any k.*

Proof. From the definition of z_k, we have

$$\begin{aligned}
\|z_k - \omega^*\|^2 &= \|(1-\beta_k)u_k + \beta_k\vartheta_k - \omega^*\|^2 + \alpha_k^2\beta_k^2\|\Gamma(u_k) - \Gamma(\vartheta_k)\|^2 \\
&+ 2\alpha_k\beta_k\langle(1-\beta_k)u_k + \beta_k\vartheta_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&= (1-\beta_k)\|u_k - \omega^*\|^2 + \beta_k\|\vartheta_k - \omega^*\|^2 - (1-\beta_k)\beta_k\|u_k - \vartheta_k\|^2 \\
&+ \alpha_k^2\beta_k^2\|\Gamma(u_k) - \Gamma(\vartheta_k)\|^2 + 2\alpha_k(1-\beta_k)\beta_k\langle u_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&+ 2\alpha_k\beta_k^2\langle\vartheta_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&= (1-\beta_k)\|u_k - \omega^*\|^2 + \beta_k\|\vartheta_k - u_k + u_k - \omega^*\|^2 \\
&- (1-\beta_k)\beta_k\|u_k - \vartheta_k\|^2 + \alpha_k^2\beta_k^2\|\Gamma(u_k) - \Gamma(\vartheta_k)\|^2 \\
&+ 2\alpha_k(1-\beta_k)\beta_k\langle u_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&+ 2\alpha_k\beta_k^2\langle\vartheta_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&= (1-\beta_k)\|u_k - \omega^*\|^2 + \beta_k(\|\vartheta_k - u_k\|^2 + \|u_k - \omega^*\|^2 + 2\langle\vartheta_k - u_k, u_k - \omega^*\rangle) \\
&- (1-\beta_k)\beta_k\|u_k - \vartheta_k\|^2 + \alpha_k^2\beta_k^2\|\Gamma(u_k) - \Gamma(\vartheta_k)\|^2 \\
&+ 2\alpha_k(1-\beta_k)\beta_k\langle u_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&+ 2\alpha_k\beta_k^2\langle\vartheta_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&= \|u_k - \omega^*\|^2 + \beta_k\|u_k - \vartheta_k\|^2 + 2\beta_k\langle\vartheta_k - u_k, u_k - \vartheta_k + \vartheta_k - \omega^*\rangle \\
&- (1-\beta_k)\beta_k\|u_k - \vartheta_k\|^2 + \alpha_k^2\beta_k^2\|\Gamma(u_k) - \Gamma(\vartheta_k)\|^2 \\
&+ 2\alpha_k(1-\beta_k)\beta_k\langle u_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&+ 2\alpha_k\beta_k^2\langle\vartheta_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&= \|u_k - \omega^*\|^2 - (2-\beta_k)\beta_k\|u_k - \vartheta_k\|^2 + 2\beta_k\langle\vartheta_k - u_k, \vartheta_k - \omega^*\rangle \\
&+ \alpha_k^2\beta_k^2\|\Gamma(u_k) - \Gamma(\vartheta_k)\|^2 + 2\alpha_k\beta_k^2\langle\vartheta_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle \\
&+ 2\alpha_k\beta_k(1-\beta_k)\langle u_k - \omega^*, \Gamma(u_k) - \Gamma(\vartheta_k)\rangle. \qquad (8)
\end{aligned}$$

Since $\vartheta_k = P_C(u_k - \alpha_k\Gamma(u_k))$ and $\omega^* \in C$, we have

$$\begin{aligned}
&\langle u_k - \alpha_k\Gamma(u_k) - \vartheta_k, \omega^* - \vartheta_k\rangle \leq 0 \\
&\Rightarrow \langle u_k - \vartheta_k, \omega^* - \vartheta_k\rangle \leq \alpha_k\langle\Gamma(u_k), \omega^* - \vartheta_k\rangle.
\end{aligned} \qquad (9)$$

Combining (8) and (9), we can get

$$\begin{aligned}
&\|u_{k+1}-\omega^*\|^2 = \|P_{H_k} z_k - P_{H_k}\omega^*\| \le \|z_k-\omega^*\|^2\\
&\le \|u_k-\omega^*\|^2 - (2-\beta_k)\beta_k\|u_k-\vartheta_k\|^2 + 2\alpha_k\beta_k\langle\Gamma(u_k),\omega^*-\vartheta_k\rangle\\
&+ \alpha_k^2\beta_k^2\|\Gamma(u_k)-\Gamma(\vartheta_k)\|^2 + 2\alpha_k\beta_k^2\langle\vartheta_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle\\
&+ 2\alpha_k\beta_k(1-\beta_k)\langle u_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle.\\
&\le \|u_k-\omega^*\|^2 - (2-\beta_k)\beta_k\|u_k-\vartheta_k\|^2 + 2\alpha_k\beta_k\langle\Gamma(u_k),\omega^*-\vartheta_k\rangle\\
&+ \sigma^2\beta_k^2\|u_k-\vartheta_k\|^2 + 2\alpha_k\beta_k^2\langle\vartheta_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle\\
&+ 2\alpha_k\beta_k(1-\beta_k)\langle u_k-\vartheta_k+\vartheta_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle.\\
&= \|u_k-\omega^*\|^2 - (2-\beta_k)\beta_k\|u_k-\vartheta_k\|^2 + 2\alpha_k\beta_k\langle\Gamma(u_k),\omega^*-\vartheta_k\rangle\\
&+ \sigma^2\beta_k^2\|u_k-\vartheta_k\|^2 + 2\alpha_k\beta_k^2\langle\vartheta_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle\\
&+ 2\alpha_k\beta_k\langle u_k-\vartheta_k,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle + 2\alpha_k\beta_k\langle\vartheta_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle\\
&- 2\alpha_k\beta_k^2\langle u_k-\vartheta_k,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle - 2\alpha_k\beta_k^2\langle\vartheta_k-\omega^*,\Gamma(u_k)-\Gamma(\vartheta_k)\rangle\\
&\le \|u_k-\omega^*\|^2 - (2-\beta_k)\beta_k\|u_k-\vartheta_k\|^2 - 2\alpha_k\beta_k\langle\Gamma(\vartheta_k),\vartheta_k-\omega^*\rangle\\
&+ \sigma^2\beta_k^2\|u_k-\vartheta_k\|^2 + 2\beta_k(1-\beta_k)\sigma\|u_k-\vartheta_k\|^2\\
&= \|u_k-\omega^*\|^2 - 2\alpha_k\beta_k\langle\Gamma(\vartheta_k),\vartheta_k-\omega^*\rangle\\
&+ [(\sigma-1)^2\beta_k^2 + 2(\sigma-1)\beta_k]\|u_k-\vartheta_k\|^2. \qquad (10)
\end{aligned}$$

By definition of $\widetilde{H}$, we can get

$$\langle\Gamma(\vartheta_k),\vartheta_k-\omega^*\rangle \ge 0. \qquad (11)$$

Set $f(\beta_k) = (\sigma-1)^2\beta_k^2 + 2(\sigma-1)\beta_k$. Then it is clear that $f(\beta_k) \le \max\{f(a), f(b)\} < 0$. Above all, we have for any k

$$\|u_{k+1}-\omega^*\| \le \|z_k-\omega^*\| \le \|u_k-\omega^*\|, \qquad (12)$$

and this concludes the proof. □

According to (12), we have that $\{u_k\}$ is a bound sequence. Due to the fact that Γ represents a continuous mapping, we see that $\{\Gamma(u_k)\}$ is bounded as well.

Moreover, based on (10) and (11) we also have for all k

$$\|u_{k+1}-\omega^*\|^2 \le \|u_0-\omega^*\|^2 + \sum_{i=0}^{k}[(\sigma-1)^2\beta_i^2 + 2(\sigma-1)\beta_i]\|u_i-\vartheta_i\|^2.$$

This means that

$$\lim_{k\to\infty}\|u_k-\vartheta_k\| = \lim_{k\to\infty}\|r(u_k,\alpha_k)\| = 0. \qquad (13)$$

Because $\{u_k\}$ is a bounded sequence, we can find a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\lim_{j\to\infty} u_{k_j} = \bar{u}$. Based on (13), we can get

$$\lim_{j\to\infty}\|u_{k_j}-\vartheta_{k_j}\| = 0. \qquad (14)$$

Since $\{\vartheta_k\} \subset C$ and C is closed, the equality (14) implies $\bar{u} \in C$.

Next, we show that $\bar{u}$ is an element in the solution set S.

Proposition 3.2. *Suppose that* $\{u_k\}$ *is defined by Algorithm* 3.1*, then any accumulation point of* $\{u_k\}$ *is a solution to problem* (1)*.*

Proof. We suppose, without loss of generality, that $\{u_{k_j}\}$ is a convergent subsequence of $\{u_k\}$ and $\lim_{k_j\to\infty} u_{k_j} = \bar{u}$. Denote $\bar{\alpha} = \inf_{k_j}\{\alpha_{k_j}\}$. We discuss the proposition in two cases.

Case 1: If $\bar{\alpha} > 0$, then for every k_j we have $\alpha_{k_j} \geq \bar{\alpha} > 0$. Based on (3), we have for every k_j

$$0 \leq \|r(u_{k_j}, 1)\| \leq \frac{\|r(u_{k_j}, \alpha_{k_j})\|}{\min\{\alpha_{k_j}, 1\}} \leq \frac{\|r(u_{k_j}, \alpha_{k_j})\|}{\min\{\bar{\alpha}, 1\}}.$$

Hence, together with (14), we have $\lim_{k_j\to\infty} \|r(u_{k_j}, 1)\| = 0 = \|r(\bar{u}, 1)\|$. This implies $\bar{u}$ a solution of problem (1).

Case 2: If $\bar{\alpha} = 0$, then we have $\lim_{k_j\to\infty} \alpha_{k_j} = 0$. By linesearch rule, for sufficiently large k_j

$$\alpha_{k_j}\theta^{-1}\|\Gamma(u_{k_j}) - \Gamma(P_C(u_{k_j} - \alpha_{k_j}\theta^{-1}\Gamma(u_{k_j})))\| > \sigma\|r(u_{k_j}, \alpha_{k_j}\theta^{-1})\|.$$

Hence, it follows that for sufficiently large k_j

$$\begin{aligned} &\|\Gamma(u_{k_j}) - \Gamma(P_C(u_{k_j} - \alpha_{k_j}\theta^{-1}\Gamma(u_{k_j})))\| > \frac{\sigma\|r(u_{k_j}, \alpha_{k_j}\theta^{-1})\|}{\alpha_{k_j}\theta^{-1}} \\ &\geq \frac{\sigma\|r(u_{k_j}, 1)\|}{1} = \sigma\|r(u_{k_j}, 1)\|. \end{aligned} \tag{15}$$

Since the projection is nonexpansive, we see that

$$\begin{aligned} &\|u_{k_j} - P_C(u_{k_j} - \alpha_{k_j}\theta^{-1}\Gamma(u_{k_j}))\| \\ &\leq \|r(u_{k_j}, \alpha_{k_j})\| + \|P_C(u_{k_j} - \alpha_{k_j}\Gamma(u_{k_j})) - P_C(u_{k_j} - \alpha_{k_j}\theta^{-1}\Gamma(u_{k_j}))\| \\ &\leq \|r(u_{k_j}, \alpha_{k_j})\| + \|u_{k_j} - \alpha_{k_j}\Gamma(u_{k_j}) - u_{k_j} - \alpha_{k_j}\theta^{-1}\Gamma(u_{k_j})\| \\ &= \|r(u_{k_j}, \alpha_{k_j})\| + \alpha_{k_j}(\theta^{-1} - 1)\|\Gamma(u_{k_j})\| \to 0 (k_j \to \infty), \end{aligned} \tag{16}$$

where the limit holds from the fact (14), the fact $\lim_{k_j\to\infty} \alpha_{k_j} = 0$ and $\{\Gamma(u_{k_j})\}$ is bounded.

Because Γ is continuous and together with (16), we can see that

$$\lim_{k_j\to\infty} \|\Gamma(u_{k_j}) - \Gamma(P_C(u_{k_j} - \alpha_{k_j}\theta^{-1}\Gamma(u_{k_j})))\| = 0.$$

Based on (15) and together with sandwich theorem, we can get $\|r(\bar{u}, 1)\| = 0$. This implies $\bar{u}$ is a point in S.

Above all, the proof is completed. □

In order to obtain the convergence of Algorithm 3.1, we prove that $\bar{u} \in \widetilde{H}$.

Proposition 3.3. *Suppose that* $\{u_k\}$ *represents the sequence defined by Algorithm* 3.1*. Then any cluster point of* $\{u_k\}$ *belongs to* $\widetilde{H}$*.*

Proof. By Lemma 2.1(c), for each $u^* \in H_k \bigcap C$ and any k

$$\|u_{k+1} - u^*\|^2 \leq \|z_k - u^*\|^2 - \|z_k - u_{k+1}\|^2.$$

Together with (12), we see that

$$\begin{aligned}\|z_k - u_{k+1}\|^2 &\le \|z_k - u^*\|^2 - \|u_{k+1} - u^*\|^2 \\ &\le \|u_k - u^*\|^2 - \|u_{k+1} - u^*\|^2.\end{aligned} \tag{17}$$

Notice that the right side of the latter inequality in (17) is summable. This implies

$$\lim_{k\to\infty} \|z_k - u_{k+1}\| = 0. \tag{18}$$

Since $\|z_k - u_{k+1}\| = d(z_k, H_k)$, we can get $z_k \in H_k$ as $k \to \infty$. Together with the structure of $\widetilde{H}$, we obtain $z_k \in \widetilde{H}$ as $k \to \infty$. Using (18) we can see $u_k \in \widetilde{H}$ as $k \to \infty$. This means that all cluster points of $\{u_k\}$ are in $\widetilde{H}$. □

Now, we set about proving that Algorithm 3.1 is global convergent.

Theorem 3.1. *Let Γ represent a continuous mapping on H and $S_D \neq \emptyset$. Assume that the sequence $\{u_k\}$ is defined by Algorithm* 3.1*, then $\{u_k\}$ converges to an element in S.*

Proof. Using (14) and Proposition 3.3, we can get $\bar{u} \in \widetilde{H} \bigcap C$. Together with Proposition 3.1, we get that

$$\|u_{k+1} - \bar{u}\| \le \|u_k - \bar{u}\| \text{ for all } k.$$

Based on Lemma 2.4, we can see that $\{\|u_k - \bar{u}\|\}$ is convergent. Moreover, $\bar{u}$ is also a accumulation point of $\{u_k\}$. Hence, we have

$$\lim_{k\to\infty} \|u_k - \bar{u}\| = 0.$$

From Proposition 3.2, the conclusion holds. □

In fact, if the mapping Γ in problem (1) is Lipschitz continuous with Lipschitz modulus L, then we can rewrite Algorithm 3.1 in a simpler form.

Algorithm 3.2

Step 0. Let Γ represents a L-Lipschitz continuous mapping. Choose a parameter $\lambda \in (0, \frac{1}{L})$, and $u_0 \in H$. Set $k = 0$.

Step 1. Compute

$$\vartheta_k = P_C(u_k - \lambda\Gamma(u_k)).$$

If $u_k = \vartheta_k$, the program stops. Else, continue.

Step 2. Calculate

$$z_k = (1 - \beta_k)u_k + \beta_k[\vartheta_k + \lambda(\Gamma(u_k) - \Gamma(\vartheta_k))],$$

where $\{\beta_k\} \subset [a, b] \subset (0, 2)$

Step 3. Set $h_j = \{x \in H : \langle\Gamma(\vartheta_j), x - \vartheta_j\rangle \le 0\}$, and let $H_k = \bigcap_{i=0}^{k} h_i$. Compute

$$u_{k+1} = P_{H_k} z_k.$$

Step 4. Let $k = k + 1$ and go to Step 1.

Let Γ represent a L-Lipschitz continuous mapping and set $\lambda \in (0, \frac{1}{L})$. Clearly, if we take $\alpha_k^0 = \lambda$ and $\sigma = \lambda L$ in Algorithm 3.1, then it degenerates into Algorithm 3.2. Together with Theorem 3.1, we can get the next corollary.

Corollary 3.1. *Let Γ represent a L-Lipschitz continuous mapping on H. Suppose $\{u_k\}$ is the sequence generated by Algorithm* 3.2. *Then $\{u_k\}$ converges to a point in S.*

Proof. Since Γ is L-Lipschitz continuous, we take $\alpha_k^0 = \lambda$ and $\sigma = \lambda L$ in Algorithm 3.1. In this case, the inequality (4) is satisfied with $m_k = 0$, and Algorithm 3.1 is equivalent to Algorithm 3.2. Together with Theorem 3.1, we have the conclusion is true. □

Next, we propose another algorithm and its variant forms under Lipschitz continuity assumption.

Algorithm 3.3

Step 0. Let Γ be a continuous mapping on H. Choose parameters $0 < \tilde{a} < \bar{a}$, $\theta, \sigma \in (0,1)$, and $u_0 \in H$. Set $k = 0$.

Step 1. Calculate $\vartheta_k = P_C(u_k - \alpha_k \Gamma(u_k))$ and $\alpha_k = \alpha_k^0 \theta^{m_k}$, where $\alpha_k^0 \in [\tilde{a}, \bar{a}]$ and m_k represents the smallest nonnegative integer m which satisfies

$$\alpha_k^0 \theta^m \|\Gamma(u_k) - \Gamma(P_C(u_k - \alpha_k^0 \theta^m \Gamma(u_k)))\| \leq \sigma \|u_k - P_C(u_k - \alpha_k^0 \theta^m \Gamma(u_k))\|.$$

If $u_k = \vartheta_k$, the program stops. Else, continue.

Step 2. Calculate

$$z_k = (1 - \beta_k) u_k + \beta_k [\vartheta_k + \alpha_k (\Gamma(u_k) - \Gamma(\vartheta_k))],$$

where $\{\beta_k\} \subset [a, b] \subset (0, 2)$.

Step 3. Set $h_j = \{x \in H : \langle \Gamma(\vartheta_j), x - \vartheta_j \rangle \leq 0\}$, and let

$$t_k \in \arg\max\{d(u_k, h_j) : 0 \leq j \leq k\} \text{ and } H_k = h_{t_k}.$$

Compute

$$u_{k+1} = P_{H_k} z_k.$$

Step 4. Let $k = k + 1$ and go to Step 1.

According to the structure of Algorithm 3.3, it is not difficult to see that Proposition 3.1 and Proposition 3.2 are also true for Algorithm 3.3. So, we will just give the following statements and leave out their proof.

Proposition 3.4. *Let the sequence $\{u_k\}$ and $\{z_k\}$ be generated by Algorithm* 3.3. *Then for any $\omega^* \in \widetilde{H} \bigcap C$ and all k, we have that the statement $\|u_{k+1} - \omega^*\| \leq \|z_k - \omega^*\| \leq \|u_k - \omega^*\|$ holds.*

Proposition 3.5. *Assume that $\{u_k\}$ is the sequence defined by Algorithm* 3.3. *Then all cluster points of $\{u_k\}$ are the solution of problem* (1).

Now, we only need to show that any accumulation point of $\{u_k\}$ belongs to $\widetilde{H}$. Then we obtain that Algorithm 3.3 is convergent.

Proposition 3.6. *Let $\{u_k\}$ represent the sequence defined by Algorithm* 3.3. *Then any accumulation point of $\{u_k\}$ is in $\widetilde{H}$.*

Proof. From the structure of $\widetilde{H}$, we attain $\widetilde{H} \subset h_i$ for each $i \geq 0$. According to Lemma 2.1(c) and Proposition 3.4, for all $\omega^* \in \widetilde{H} \bigcap C$, we have

$$\|z_k - u_{k+1}\|^2 \leq \|u_k - \omega^*\|^2 - \|u_{k+1} - \omega^*\|^2 \text{ for any } k. \tag{19}$$

Notice that the left hand of the inequality (19) is summable. This implies

$$\lim_{k\to\infty} \|z_k - u_{k+1}\| = 0. \tag{20}$$

Since $\|z_k - u_{k+1}\| = d(z_k, H_k)$, we can get $\lim_{k\to\infty} d(z_k, H_k) = 0$. From the definition of H_k, we have $0 \leq d(z_k, h_i) \leq d(z_k, H_k)$ for all $i \leq k$. This means that for any fixed i, $\lim_{k\to\infty} d(z_k, h_i) = 0$. Together with (20), we obtain that $d(\bar{u}, h_i) = 0$ for any fixed i, and the proposition is proved. □

According to Proposition 3.4-3.6, we can give the following theorem and omit its proof.

Theorem 3.2. *Let Γ be continuous on H. The iterative procedure of Algorithm* 3.3 *either terminates in finite number of steps or converges to an element in S.*

Similarly, if Γ is L-Lipschitz continuous in (1), then we can remove the line search procedure in Algorithm 3.3 and rewrite it in a simpler form.

Algorithm 3.4

Step 0. Let Γ represent a L-Lipschitz continuous mapping, choose a parameter $\lambda \in (0, \frac{1}{L})$, and choose $u_0 \in H$. Set $k = 0$.

Step 1. Calculate

$$\vartheta_k = P_C(u_k - \lambda\Gamma(u_k)).$$

If $u_k = \vartheta_k$, the program stop; Else, continue.

Step 2. Calculate

$$z_k = (1 - \beta_k)u_k + \beta_k[\vartheta_k + \lambda(\Gamma(u_k) - \Gamma(\vartheta_k))],$$

where $\{\beta_k\} \subset [a, b] \subset (0, 2)$.

Step 3. Set $h_j = \{x \in H :< \Gamma(\vartheta_j), x - \vartheta_j >\leq 0\}$, and let $t_k \in \arg\max\{d(u_k, h_j) : 0 \leq j \leq k\}$ and $H_k = h_{t_k}$.

Compute $u_{k+1} = P_{H_k} z_k$.

Step 4. Let $k = k + 1$ and return to Step 1.

Since the convergence analysis between Algorithm 3.4 and Algorithm 3.2 have no difference, we give directly the following conclusion and omit its proof.

Theorem 3.3. *Assume that Γ is L-Lipschitz continuous on H, and $\{u_k\}$ represent the sequence defined by Algorithm 3.4, then $\{u_k\}$ converges to a point in S.*

4. **Numerical Examples**

We illustrate the convergence of our proposed iterative schemes through several examples. Denote the Algorithm 3 in [23] by Alg. B, and compare it with Algorithm 3.2 (for short Alg. 2) and Algorithm 3.4 (for short Alg. 4). Assume that $C = \{u \in \mathbb{R}^n : Au \leq b\}$, where $A \in \mathbb{R}^{m\times n}$ is a matrix and $b \in \mathbb{R}^m$ is a vector. According to [23], projecting a vector x onto the polyhedron C is equivalent to solving the following quadratic programming problem:

$$\arg\min\left\{\frac{1}{2}u^T u - x^T u : u \in C\right\}. \tag{21}$$

Since a half-space is also a polyhedron, the intersection of half-spaces and C is still a polyhedron, we can use the matlab optimization toolbox to calculate the projection.

Example 4.1 ([23])**.** Let be given $C = [-1,1]^4$ and $\Gamma(u) = (u_1^2, u_2^2, u_3^2, u_4^2)^T$. If we set $u = (-1,-1,-1,-1)^T$, then for all $x \in C$, we see that $\langle\Gamma(x), x-u\rangle \geq 0$. Moreover, it is easily verified that Γ is Lipschitz continuous with Lipschitz modulus 4. Take the stopping criteria $error = \|r(u_k,\lambda)\|^2 \leq 10^{-7}$, we use Alg. 2, Alg. 4 with parameter $\lambda = 0.2$ and Alg. B with parameter $\lambda = \frac{0.9999}{4}$. The initial point is $u_0 = (-0.5,-0.5,-0.5,-0.5)^T$, and we illustrate the result in Table 1 and Fig. 1.

Example 4.2 ([23])**.** Suppose $C = [-1,1]\times[-2,2]\times[-3,3]$, and for any $u = (u_1,u_2,u_3)^T$, set $\Gamma(u) = (u_1 - u_2, \frac{u_3}{1+\|u\|^2}, \frac{u_3}{1+\|u\|^2})^T$. We take $u = (0,-\frac{1}{2},\frac{1}{4})^T$ and $x = (\frac{1}{2},1,\frac{1}{4})^T$, then it is not difficult to verify that Γ is not quasimonotone. Moreover, we can get clearly $\|\Gamma(u)-\Gamma(x)\| < \sqrt{6}\|u-x\|$. Take the stopping criteria $error = \|r(u_k,\lambda)\|^2 \leq 10^{-7}$, we use Alg. 2, Alg. 4, Alg. B with parameter $\lambda = \frac{1}{\sqrt{8}}$. The initial point is $u_0 = (0.5,1.5,2)^T$, and we illustrate the result in Table 2 and Fig. 2.

Table 1. Result of Example 1

u_0	Alg. 2		Alg. 4		Alg. B	
$(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})^T$	CPU(s)	iter.	CPU(s)	iter.	CPU(s)	iter.
	0.1875	21	0.20312	23	2.4531	311

Table 2. Result of Example 2

u_0	Alg. 2		Alg. 4		Alg. B	
$(0.5,1.5,2)^T$	CPU(s)	iter.	CPU(s)	iter.	CPU(s)	iter.
	0.60938	35	0.35938	46	0.59375	55

5. **Conclusions**

In this article, we introduce two new approaches to solve variational inequalities without monotonicity. The strong convergence of the two algorithms is proved. In comparison, the convergence range of algorithms proposed in this paper is larger than [23]. Finalliy, we demonstrate the advantages of our proposed algorithms through some numerical examples.

FIGURE 1. $u_0 = (-0.5, -0.5, -0.5, -0.5)$

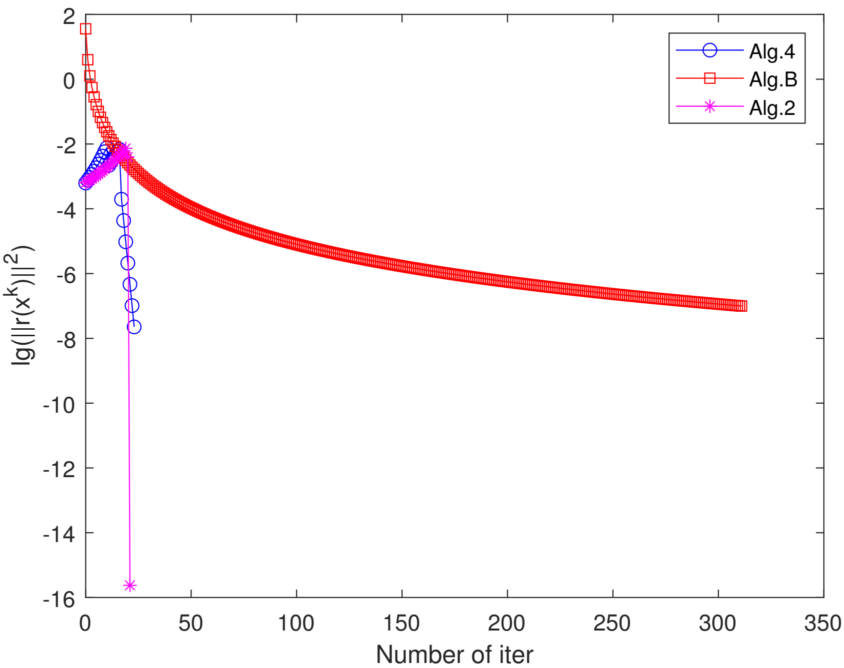
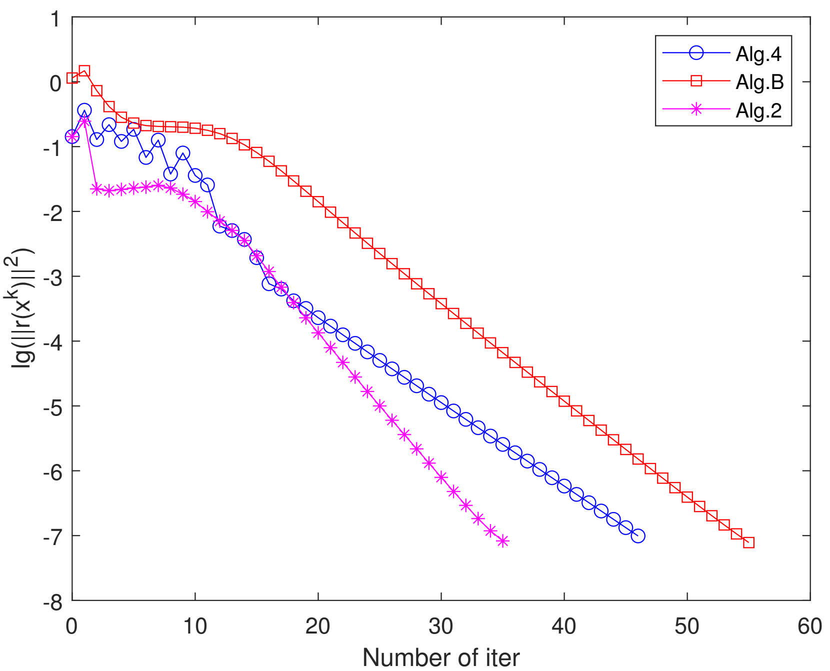


FIGURE 2. $u_0 = (0.5, 1.5, 2)$



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