POSITIVE SOLUTION FOR A FRACTIONAL SWITCHED SYSTEM INVOLVING RIEMANN-STIELTJES INTEGRAL

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We study positive solution for a fractional switched system involving Riemann-Stieltjes integral. Our results cover the fractional differential equation with switched nonlinearity, moreover, the Riemann-Stieltjes integral was involved in the boundary condition. We obtain positive solution for the above system according to the fixed point theorems for mixed monotone operators with perturbation. We obtain new iterative sequences to approach the positive solution. An example is given to illustrate the abstract results.

Keywords: Positive solution; fractional switched system; Riemann-Stieltjes integral; mixed monotone operator

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1. **Introduction**

In this article, we study fractional switched system involving Riemann-Stieltjes integral such as the following

$$\{\phi_p[D^{\alpha}_{0+}u(t)+\phi_q(\int_0^t g_{\sigma(s)}(s,I^{\alpha}_{0+}u(s),u(s))ds)]\}'+f_{\sigma(t)}(t,I^{\alpha}_{0+}u(t),u(t))=0,\quad t\in J=[0,1], \tag{1}$$

$$D^{\alpha}_{0+}u(0)=0,\quad u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\qquad u(1)=\int_0^1 u(s)dH(s), \tag{2}$$

which arises from some complex system of economic and engineering science, where $n-1<\alpha\le n,\ n\in N,\ n\ge 3,\ D^{\alpha}_{0+},\ I^{\alpha}_{0+}$ are the Riemann-Liouville fractional derivative and fractional integral of order α respectively, $\phi_p(s)=|s|^{p-2}s,\ p>1,\ (\phi_p)^{-1}=\phi_q,\ \frac{1}{p}+\frac{1}{q}=1$, the integrals from the boundary conditions are Riemann-Stieltjes integral with $H(t)$ is a bounded variation function. $\sigma: J\to M=\{1,2,\cdots,N\}$ is a finite switching signal which is a piecewise constant function depending on t. Corresponding to the switching signal $\sigma(t)$, we have the following switching sequence

$$\{(i_0,t_0),\cdots,(i_j,t_j),\cdots,(i_k,t_k)|i_j\in\{1,2,\cdots,N\},j=0,1,2,\cdots,k\},$$

which means that the i_j^{th} nonlinearity is activated when $t\in[t_j,t_{j+1})$ and the i_k^{th} nonlinearity is activated when $t\in[t_k,1]$. Here $x_0=0,\ t_0=0$.

The study of solutions for fractional problem associated to Riemann-Stieltjes integral has a long history. Its significance to mathematical physics is emphasized in applied fields

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such as population, dynamics, underground water flow, blood flow problems and chemical engineering, see references [1-6]. Some authors have studied fractional problems with the following boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \gamma u(\eta),$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \Sigma_{i=1}^{m-2} a_i u(\eta_i),$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)ds,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)ds + \gamma u(\eta).$$

We can see that the above boundary conditions are all special forms of (2), so it is very important to study the solutions of (1)-(2).

In [7], Hao et al. discussed positive solutions for the following n-th order boundary value problem with Riemann-Stieltjes integral

$$x^{(n)}(t) + a(t)f(t, x(t)) = 0, \quad 0 < t < 1,$$

$$x^{(k)}(0) = 0, \quad 0 \le k \le n-2, \; x(1) = \int_0^1 x(s)dA(s).$$

Zhang and Han in [8] studied positive solutions for the following fractional boundary value problem with Riemann-Stieltjes integral

$$D_{0+}^{\alpha} x(t) + f(t, x(t)) = 0, \quad n-1 < \alpha \le n, \quad 0 < t < 1,$$

$$x^{(k)}(0) = 0, \quad 0 \le k \le n-2, \; x(1) = \int_0^1 x(s)dA(s).$$

Haddouchi [9] investigated positive solutions for the following nonlocal boundary value problem with Riemann-Stieltjes integral

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u''(0) = 0, \quad u(1) = \mu u(\eta) + \beta \int_0^1 u(s)dA(s).$$

In his monograph [10], Henderson considered the following system with the uncoupled integral boundary conditions

$$D_{0+}^{\alpha} u(t) + \lambda f(t, u(t), v(t)) = 0, \quad 0 < t < 1,$$

$$D_{0+}^{\alpha} v(t) + \mu f(t, u(t), v(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)dH(s),$$

$$v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0, \qquad v(1) = \int_0^1 v(s)dK(s).$$

On the other hand, switching systems have received attention from many scholars in recent years owing to their applications in the fields of electrical and chemical engineering, air traffic control, aircraft, automotive, etc. [11-13]. Switched systems are a particular kind of hybrid systems that consist of a set of subsystems and a switching signal selecting a subsystem to be active during an interval of time. Currently, research on switching systems mainly focuses on stability analysis [14], H-infinity control [15]. As is known to all, existence and uniqueness of the solutions is fundamental and crucial to a switched system. In [16], the authors considered the existence and uniqueness of solutions about switched Hamiltonian systems. In [17], Ahmad et al. considered the existence and uniqueness of solutions for coupled implicit $\psi-$ Hilfer fractional switched systems. Li et al. [18] derived the existence and uniqueness of its solutions under some time-varying switching law. We can see that the above literature discusses the existence of solutions for the switched system, but for practical problem, positive solutions are more meaningful. Until now, there are few literatures on the existence of positive solutions for the switched system [19-21].

Li et al. [19] studied the positive solutions for the following switched system

$$x''(t) + f_{\sigma(t)}(t, x(t)) = 0, \quad t \in J = [0, 1],$$

$$x(0) = 0, \quad x(1) = \int_0^1 a(s)x(s)ds.$$

Guo [20] concerned with the positive solutions for the following p-Laplacian switched system

$$D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}u(t)) = f_{\sigma(t)}(t, u(t), D_{0+}^{\gamma}u(t)), \quad t \in J = [0, 1],$$

$$u(0) = \mu\int_0^1 u(s)ds + \lambda u(\xi), \quad D_{0+}^{\alpha}u(0) = kD_{0+}^{\alpha}u(\eta), \quad \xi, \eta \in [0, 1].$$

However, positive solutions for the fractional switched system involving Riemann-Stieltjes integral have not been studied till now. In this article, we shall considered the existence of positive solutions for the system (1) (2) according to the fixed point theorems for mixed monotone operators.

2. **The preliminary lemmas**

Definition 2.1. *[23] The fractional integral of order $\alpha > 0$ of a function $g : (0, +\infty) \to R$ denoted by $I_{0+}^{\alpha}g$ is expressed as*

$$I_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \quad t > 0,$$

provided the right hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. *[23] For a function $g : (0, +\infty) \to R$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ denoted by $D_{0+}^{\alpha}g$ is expressed as*

$$D_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}ds, \quad t > 0,$$

here $n = [\alpha] + 1$, provided the right hand side is pointwise defined on $(0, +\infty)$.

Suppose that E is a real Banach space, P is a cone of E,θ represents the zero element in E. For all $x, y \in E$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, where N is a positive constant, we call P a normal cone. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ satisfing $\lambda x \le y \le \mu x$. Evidently, $\sim$ is an equivalence relation. Suppose $h > \theta$ ($i.e., h \ge \theta$ and $h \ne \theta$), the set P_h defined as $P_h = \{x \in E | x \sim h\}$. Clearly, $P_h \subset P$.

Definition 2.3. *[24] We call $T: P \times P \to P$ a mixed monotone operator if $T(x,y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i=1,2) \in P$, $u_1 \le u_2$, $v_1 \ge v_2$ imply $T(u_1, v_1) \le T(u_2, v_2)$. If x belongs to P and satisfies $T(x,x)=x$, we call x a fixed point of T.*

Lemma 2.1. *[24] Let P be a normal cone in E. Assume that $A, B: P \times P \to P$ are two mixed monotone operators and satisfy the following conditions:*
(i) for any $\lambda \in (0,1)$, there exists a number $\psi(\lambda) \in (\lambda, 1]$ such that

$$A(\lambda x, \lambda^{-1} y) \ge \psi(\lambda) A(x,y), \quad x, y \in P;$$

(ii) for any $\lambda \in (0,1)$, $x, y \in P$, $\quad B(\lambda x, \lambda^{-1} y) \ge \lambda B(x,y)$;
(iii) there exists $h \in P$ with $h > \theta$ such that $A(h,h) \in P_h$, $B(h,h) \in P_h$;
(iv) there exists a constant $\delta > 0$ such that for all $x, y \in P$, $A(x,y) \ge \delta B(x,y)$.
Then the operator equation $A(x,x) + B(x,x) = x$ has a unique solution $x^* \in P_h$, and for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}),$$

$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}), \quad n = 1, 2, \cdots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Lemma 2.2. *[8] Let $\Delta_1 = 1 - \int_0^1 s^{\alpha-1} dH(s) \neq 0$, for $h \in C(0,1) \cap L^1[0,1]$, the unique solution of the fractional switched system*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s) dH(s), \tag{4}$$

is given by

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{5}$$

here

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{\Delta_1} \int_0^1 g(\tau, s) dH(\tau), \quad (t,s) \in [0,1] \times [0,1], \tag{6}$$

and

$$g(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{7}$$

Lemma 2.3. *[8] The function $g(t,s)$ given by (7) satisfied:*
(i) $g(t,s) = g(1-s, 1-t)$, for $t, s \in [0,1]$;
(ii) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le \Gamma(\alpha) g(t,s) \le (\alpha-1) t^{\alpha-1}(1-t)$.

Lemma 2.4. *[8] Let $F = \max_{0 \le s \le 1} \int_0^1 g(\tau, s) dH(\tau)$, if $H: [0,1] \to R$ is a nondecreasing function and $\Delta_1 > 0$, the Green's function defined by (6) has the following properties:*
(i) $G(t,s) > 0$ for each $s, t \in (0,1)$;
(ii) $h(t) c \int_0^1 g(\tau, s) dH(\tau) \le G(t,s) \le h(t) d$ for $s, t \in [0,1]$;
where $c = \frac{1}{\Delta_1}$, $d = \frac{F}{\Delta_1} + \frac{1}{\Gamma(\alpha-1)}$, $h(t) = t^{\alpha-1}$.

Lemma 2.5. *The fractional switched system*

$$\{\phi_p[D_{0^+}^{\alpha}u(t)+\phi_q(\int_0^t g_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds)]\}'+f_{\sigma(t)}(t,I_{0^+}^{\alpha}u(t),u(t))=0,\quad t\in J=[0,1], \tag{8}$$

$$D_{0^+}^{\alpha}u(0)=0,\quad u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\qquad u(1)=\int_0^1 u(s)dH(s), \tag{9}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)[\phi_q(\int_0^s g_{\sigma(\tau)}(\tau,I_{0^+}^{\alpha}u(\tau),u(\tau))d\tau)+\phi_q(\int_0^s f_{\sigma(\tau)}(\tau,I_{0^+}^{\alpha}u(\tau),u(\tau))d\tau)]ds. \tag{10}$$

Proof. By integrating both sides of (8) from 0 to t, and considering condition (9), we have

$$\phi_p[D_{0^+}^{\alpha}u(t)+\phi_q(\int_0^t g_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds)]=-\int_0^t f_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds,$$

consequently,

$$D_{0^+}^{\alpha}u(t)+\phi_q(\int_0^t g_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds)=-\phi_q(\int_0^t f_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds),$$

so,

$$D_{0^+}^{\alpha}u(t)+\phi_q(\int_0^t g_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds)+\phi_q(\int_0^t f_{\sigma(s)}(s,I_{0^+}^{\alpha}u(s),u(s))ds)=0. \tag{11}$$

Considering the above equation (11), boundary condition (9) and lemma 2.2, we can conclude the proof. □

3. Main results

In this section, we consider (1)-(2) in the real Banach space $E=C[0,1]$, with the norm $\|u\|=\max\limits_{0\le t\le 1}|u(t)|$. Let $P=\{u\in E:\ u(t)\ge 0,\ t\in[0,1]\}$, then P is a normal cone of E. In the following, we denote $h(t)=t^{\alpha-1}$.

Define

$$F^{\sigma}(t,u,v)=\begin{cases} f_{i_0}(t,u,v), & t\in[0,t_1],\\ \vdots & \\ f_{i_j}(t,u,v), & t\in[t_j,t_{j+1}),\\ \vdots & \\ f_{i_k}(t,u,v), & t\in[t_k,1],\end{cases}\qquad G^{\sigma}(t,u,v)=\begin{cases} g_{i_0}(t,u,v), & t\in[0,t_1],\\ \vdots & \\ g_{i_j}(t,u,v), & t\in[t_j,t_{j+1}),\\ \vdots & \\ g_{i_k}(t,u,v), & t\in[t_k,1].\end{cases}$$

Remark 1 Define two operators $A,B:\ P\times P\to E$ by

$$A(u,v)(t)=\int_0^1 G(t,s)\phi_q(\int_0^s G^{\sigma}(\tau,I_{0^+}^{\alpha}u(\tau),v(\tau))d\tau)ds,$$

$$B(u,v)(t)=\int_0^1 G(t,s)\phi_q(\int_0^s F^{\sigma}(\tau,I_{0^+}^{\alpha}v(\tau),v(\tau))d\tau)ds.$$

Then Lemma 2.5 implies that a function $u\in E$ is a solution to fractional switched system (1)-(2) if and only if $u=A(u,u)+B(u,u)$.

Now we give some hypotheses of the switched system (1)-(2).

(H_1) $H:\ [0,1]\to R$ is a nondecreasing function, $\Delta_1=1-\int_0^1 s^{\alpha-1}dH(s)>0$;

(H_2) For any $i \in M$, $f_i, g_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous, and for all $t \in [0,1],\ f_i(t,0,1) \neq 0$;
(H_3) For any $i \in M$, $f_i(t,u,v), g_i(t,u,v)$ are increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$ and $v \in [0,+\infty)$, decreasing in $v \in [0,+\infty)$ for fixed $t \in [0,1]$ and $u \in [0,+\infty)$;
(H_4) For any $i \in M$, there exists $\psi(\lambda) \in (\lambda, 1)$, such that for all $t \in [0,1],\ u, v \in [0,+\infty)$,

$$g_i(t, \lambda u, \lambda^{-1} v) \geq (\psi(\lambda))^{p-1} g_i(t,u,v),$$

$$f_i(t, \lambda u, \lambda^{-1} v) \geq \lambda^{p-1} f_i(t,u,v), \text{ for all } \lambda \in (0,1),$$

(H_5) For any $i \in M$, there exists a constant $\delta > 0$, such that for all $t \in [0,1],\ u, v \in [0,+\infty)$,

$$g_i(t,u,v) \geq \delta f_i(t,u,v).$$

Remark 2 Conditions (H_1)-(H_5) imply the following conditions of $F^\sigma(t,u,v)$ and $G^\sigma(t,u,v)$:
(H_1') $H:\ [0,1] \to R$ is a nondecreasing function, $\Delta_1 = 1 - \int_0^1 s^{\alpha-1} dH(s) > 0$;
(H_2') $F^\sigma, G^\sigma : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous, and for all $t \in [0,1],\ F^\sigma(t,0,1) \neq 0$;
(H_3') $F^\sigma(t,u,v), G^\sigma(t,u,v)$ are increasing in $u \in [0,+\infty)$ for fixed $t \in [0,1]$ and $v \in [0,+\infty)$, decreasing in $v \in [0,+\infty)$ for fixed $t \in [0,1]$ and $u \in [0,+\infty)$;
(H_4') there exists $\psi(\lambda) \in (\lambda, 1)$, such that for all $t \in [0,1],\ u, v \in [0,+\infty)$,

$$G^\sigma(t, \lambda u, \lambda^{-1} v) \geq (\psi(\lambda))^{p-1} G^\sigma(t,u,v),$$

$$F^\sigma(t, \lambda u, \lambda^{-1} v) \geq \lambda^{p-1} F^\sigma(t,u,v), \text{ for all } \lambda \in (0,1),$$

(H_5') For any $i \in M$, there exists a constant $\delta > 0$, such that for all $t \in [0,1],\ u, v \in [0,+\infty)$,

$$G^\sigma(t,u,v) \geq \delta F^\sigma(t,u,v).$$

Theorem 3.1. *Suppose that hypotheses (H_1)-(H_5) hold. Then for any finite switching signal $J \to M$, the problem (1)-(2) has a unique positive solution $u^* \in P_h$, here $h(t) = t^{\alpha-1},\ t \in [0,1]$, and for any $u_0, v_0 \in P_h$, constructing successively the sequence as follows*

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 G(t,s)[\phi_q(\int_0^s G^\sigma(\tau, I_{0+}^\alpha u_n(\tau), v_n(\tau))d\tau) \\ &\quad + \phi_q(\int_0^s F^\sigma(\tau, I_{0+}^\alpha u_n(\tau), v_n(\tau))d\tau)]ds,\ n = 0,1,2,\cdots, \end{aligned}$$

$$\begin{aligned} v_{n+1}(t) &= \int_0^1 G(t,s)[\phi_q(\int_0^s G^\sigma(\tau, I_{0+}^\alpha v_n(\tau), u_n(\tau))d\tau) \\ &\quad + \phi_q(\int_0^s F^\sigma(\tau, I_{0+}^\alpha v_n(\tau), u_n(\tau))d\tau)]ds,\ n = 0,1,2,\cdots, \end{aligned}$$

thus we have $\|u_n - u^*\| \to 0$ and $\|v_n - u^*\| \to 0$ as $n \to \infty$, that is, $\{u_n(t)\}$ and $\{v_n(t)\}$ both converges to $u^*(t)$ uniformly for all $t \in [0,1]$.

Proof. From hypothesis (H_2) and the properties of the function $G(t,s)$, it can be concluded that $A:\ P \times P \to P$ and $B:\ P \times P \to P$. Thus we set out to prove that A, B satisfy all the assumptions of Lemma 2.1.

Firstly, we prove that A and B are two mixed monotone operators. Let $u_i, v_i \in P, i = 1,2$ with $u_1 \geq u_2,\ v_1 \leq v_2$, this is equal to $u_1(t) \geq u_2(t),\ v_1(t) \leq v_2(t),\ t \in [0,1]$, thus we have

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u_1(s) ds \geq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u_1(s) ds,$$

which imply

$$I_{0+}^{\alpha}u_1(t) \geq I_{0+}^{\alpha}u_2(t).$$

Considering hypothesis (H_3), we have

$$\begin{array}{l}\int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, I_{0+}^{\alpha}u_1(\tau), v_1(\tau))d\tau)ds \\ \geq \int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, I_{0+}^{\alpha}u_2(\tau), v_2(\tau))d\tau)ds,\ i = 1, 2, \cdots, N,\end{array}$$

$$\begin{array}{l}\int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, I_{0+}^{\alpha}u_1(\tau), v_1(\tau))d\tau)ds \\ \geq \int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, I_{0+}^{\alpha}u_2(\tau), v_2(\tau))d\tau)ds,\ i = 1, 2, \cdots, N,\end{array}$$

which yield that

$$A(u_1, v_1)(t) \geq A(u_2, v_2)(t), \quad B(u_1, v_1)(t) \geq B(u_2, v_2)(t),$$

that is

$$A(u_1, v_1) \geq A(u_2, v_2), \quad B(u_1, v_1) \geq B(u_2, v_2).$$

Secondly, we prove that assumption (i) of Lemma 2.1 holds.
For any $\lambda \in (0, 1)$, and $u, v \in P$, considering hypothesis (H_4), one has

$$\begin{array}{l}\int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, \lambda I_{0+}^{\alpha}u(\tau), \frac{1}{\lambda}v(\tau))d\tau)ds \\ \geq \int_0^1 G(t,s)\phi_q(\int_0^s (\psi(\lambda))^{p-1} g_i(\tau, I_{0+}^{\alpha}u(\tau), v(\tau))d\tau)ds \\ = \psi(\lambda)\int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, I_{0+}^{\alpha}u(\tau), v(\tau))d\tau)ds,\ i = 1, 2, \cdots, N,\end{array} \tag{12}$$

which yields that

$$A(\lambda u, \lambda^{-1}v)(t) \geq \psi(\lambda)A(u, v)(t),$$

it means that

$$A(\lambda u, \lambda^{-1}v) \geq \psi(\lambda)A(u, v) \text{ for } \lambda \in (0, 1),\ u, v \in P.$$

Thirdly, we prove that assumption (ii) of Lemma 2.1 holds.
For any $\lambda \in (0, 1)$, and $u, v \in P$, considering hypothesis (H_4), we get

$$\begin{array}{l}\int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, \lambda I_{0+}^{\alpha}u(\tau), \frac{1}{\lambda}v(\tau))d\tau)ds \\ \geq \int_0^1 G(t,s)\phi_q(\int_0^s \lambda^{p-1} f_i(\tau, I_{0+}^{\alpha}u(\tau), v(\tau))d\tau)ds \\ = \lambda\int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, I_{0+}^{\alpha}u(\tau), v(\tau))d\tau)ds,\ i = 1, 2, \cdots, N,\end{array} \tag{13}$$

which yields that

$$B(\lambda u, \lambda^{-1}v)(t) \geq \lambda B(u, v)(t),$$

that is,

$$B(\lambda u, \lambda^{-1}v) \geq \lambda B(u, v) \text{ for } \lambda \in (0, 1),\ u, v \in P.$$

Next, we prove that assumption (iii) of Lemma 2.1 holds.
In fact, since

$$I_{0+}^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds = \frac{B(\alpha,\alpha)}{\Gamma(\alpha)}t^{2\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}.$$

From Lemma 2.4 and hypothesis (H_3), for any $t \in [0, 1],\ i = 1, 2, \cdots, N$, we have

$$\begin{array}{l}\int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, I_{0+}^{\alpha}h(\tau), h(\tau))d\tau)ds \\ = \int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\tau^{2\alpha-1}, \tau^{\alpha-1})d\tau)ds \\ \leq \int_0^1 G(t,s)\phi_q(\int_0^1 g_i(\tau, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)d\tau)ds \\ \leq h(t)d\phi_q(\int_0^1 g_i(\tau, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)d\tau).\end{array} \tag{14}$$

On the other hand, from Lemma 2.4 and hypothesis (H_3), for any $t \in [0,1]$, one has

$$\begin{aligned}
&\int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, I_{0^+}^{\alpha}h(\tau), h(\tau))d\tau)ds \\
&= \int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, \tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}\tau^{2\alpha-1}, \tau^{\alpha-1})d\tau)ds \\
&\geq \int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau, 0, 1)d\tau)ds \\
&\geq h(t)c\int_0^1[\int_0^1 g(\tau,s)dH(\tau)\phi_q(\int_0^s g_i(\tau,0,1)d\tau)]ds, \ i = 1, 2, \cdots, N.
\end{aligned} \tag{15}$$

For $i = 1, 2, \cdots, N$, let

$$m_i = d\phi_q(\int_0^1 g_i(\tau, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)d\tau), \tag{16}$$

$$n_i = c\int_0^1[\int_0^1 g(\tau,s)dH(\tau)\phi_q(\int_0^s g_i(\tau,0,1)d\tau)]ds. \tag{17}$$

It follows from (H_3) and (H_5), one has

$$\int_0^1 g_i(s, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)ds \geq \int_0^1 g_i(s,0,1)ds \geq \delta\int_0^1 f_i(s,0,1)ds,$$

condition $f_i(t,0,1) \neq 0$ implies that $\int_0^1 g_i(s, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)ds > 0, \ \int_0^1 g_i(s,0,1)ds > 0.$
Thus,

$$m_i > 0, \quad n_i > 0, \quad i = 1, 2, \cdots, N. \tag{18}$$

We write $n = \min\{n_i, \ i = 1, 2, \cdots, N\}$ and $m = \max\{m_i, \ i = 1, 2, \cdots, N\}$, then $n > 0$ and $m > 0$. Therefore,

$$nh(t) \leq A(h,h) \leq mh(t), \tag{19}$$

this is equal to

$$A(h,h) \in P_h. \tag{20}$$

Similarly, for any $t \in [0,1]$, one has

$$\begin{aligned}
&\int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, I_{0^+}^{\alpha}h(\tau), h(\tau))d\tau)ds \\
&= \int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, \tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}\tau^{2\alpha-1}, \tau^{\alpha-1})d\tau)ds \\
&\leq \int_0^1 G(t,s)\phi_q(\int_0^1 f_i(\tau, \tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)d\tau)ds \\
&\leq h(t)d\phi_q(\int_0^1 f_i(\tau, \tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)d\tau).
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, I_{0^+}^{\alpha}h(\tau), h(\tau))d\tau)ds \\
&= \int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, \tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}\tau^{2\alpha-1}, \tau^{\alpha-1})d\tau)ds \\
&\geq \int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau, 0, 1)d\tau)ds \\
&\geq h(t)c\int_0^1[\int_0^1 g(\tau,s)dH(\tau)\phi_q(\int_0^s f_i(\tau,0,1)d\tau)]ds, \ i = 1, 2, \cdots, N.
\end{aligned} \tag{21}$$

For $i = 1, 2, \cdots, N$, let

$$\overline{m_i} = d\phi_q(\int_0^1 f_i(\tau, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)d\tau), \tag{22}$$

$$\overline{n_i} = c\int_0^1[\int_0^1 g(\tau,s)dH(\tau)\phi_q(\int_0^s f_i(\tau,0,1)d\tau)]ds. \tag{23}$$

It follows from (H_3), one has

$$\int_0^1 f_i(s, \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}, 0)ds \geq \int_0^1 f_i(s,0,1)ds,$$

condition $f_i(t,0,1)\neq 0$ implies that $\int_0^1 f_i(s,\frac{\Gamma(\alpha)}{\Gamma(2\alpha)},0)ds>0,\ \int_0^1 f_i(s,0,1)ds>0$.
Thus,

$$\overline{m_i}>0,\quad \overline{n_i}>0,\quad i=1,2,\cdots,N. \tag{24}$$

We write $\overline{n}=\min\{\overline{n_i},\ i=1,2,\cdots,N\}$ and $\overline{m}=\max\{\overline{m_i},\ i=1,2,\cdots,N\}$, then $\overline{n}>0$ and $\overline{m}>0$. Therefore,

$$\overline{n}h(t)\leq B(h,h)\leq \overline{m}h(t), \tag{25}$$

this is equal to

$$B(h,h)\in P_h. \tag{26}$$

Finally, we prove that assumption (iv) of Lemma 2.1 holds.
For $u,v\in P$ and $t\in J, i=1,2,\cdots,N$, it follows from (H_5), we have

$$\int_0^1 G(t,s)\phi_q(\int_0^s g_i(\tau,I_{0+}^{\alpha}u(\tau),v(\tau))d\tau)ds\geq \delta\int_0^1 G(t,s)\phi_q(\int_0^s f_i(\tau,I_{0+}^{\alpha}u(\tau),v(\tau))d\tau)ds, \tag{27}$$

which yields that

$$A(u,v)\geq \delta B(u,v). \tag{28}$$

By Lemma 2.1, the fractional switched system has a unique solution $u^*\in P_h$, here $h=t^{\alpha-1}, t\in[0,1]$, and for any initial $u_0,v_0\in P_h$, we can construct successively two sequences u_n and v_n by

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 G(t,s)[\phi_q(\int_0^s g_{\sigma(\tau)}(\tau,I_{0+}^{\alpha}u_n(\tau),v_n(\tau))d\tau)\\ &\quad +\phi_q(\int_0^s f_{\sigma(\tau)}(\tau,I_{0+}^{\alpha}u_n(\tau),v_n(\tau))d\tau)]ds,\ n=0,1,2,\cdots,\\ v_{n+1}(t) &= \int_0^1 G(t,s)[\phi_q(\int_0^s g_{\sigma(\tau)}(\tau,I_{0+}^{\alpha}v_n(\tau),u_n(\tau))d\tau)\\ &\quad +\phi_q(\int_0^s f_{\sigma(\tau)}(\tau,I_{0+}^{\alpha}v_n(\tau),u_n(\tau))d\tau)]ds,\ n=0,1,2,\cdots, \end{aligned}$$

and the iterative sequence $u_n(t),v_n(t)$ converges uniformly to u^* as $n\to\infty$. □

4. **Example**

Example 4.1. *Consider the following fractional switched system:*

$$\{D_{0+}^{\frac{5}{2}}u(t)+\int_0^t g_{\sigma(s)}(s,I_{0+}^{\alpha}u(s),u(s))ds\}'+f_{\sigma(t)}(t,I_{0+}^{\alpha}u(t),u(t))=0,\qquad t\in J=[0,1], \tag{29}$$

$$D_{0+}^{\frac{5}{2}}u(0)=0,\ u(0)=u'(0)=0,\qquad u(1)=\int_0^1 u(s)dH(s), \tag{30}$$

where $p=2,\ n=3,\ \alpha=\frac{5}{2},\ \sigma:\ J\to M=\{1,2\}$ *is a finite switching signal and*

$$g_1(t,u,v)=2+(1+t)\sqrt{u}+(v+1)^{-\frac{1}{2}};$$

$$g_2(t,u,v)=1+(2+sint)\sqrt[3]{u}+(v+2)^{-\frac{1}{6}};$$

$$f_1(t,u,v)=\frac{u}{(2+t^4)(1+u)}+(v+1)^{-\frac{1}{2}};$$

$$f_2(t,u,v)=\frac{u}{2+u}+(v+2)^{-\frac{1}{6}}.$$

Let

$$H(t)=\begin{cases} 0, & t\in[0,\frac{1}{4}),\\ 3, & t\in[\frac{1}{4},\frac{3}{4}),\\ \frac{7}{2}, & t\in[\frac{3}{4},1]. \end{cases}$$

In the following, we prove that all the conditions of Theorem 3.1 hold.
(1) $\int_0^1 u(s)dH(s) = 3u(\frac{1}{4}) + \frac{1}{2}u(\frac{3}{4})$ *and* $\Delta_1 = 1 - \int_0^1 s^{\frac{3}{2}}dH(s) = 1 - 3(\frac{1}{4})^{\frac{3}{2}} - \frac{1}{2}(\frac{3}{4})^{\frac{3}{2}} \approx 0.3002 > 0.$
(2) It is obvious that $f_i, g_i : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *are continuous,* $i = 1,2,$ *and* $f_1(t,0,1) = 1 \neq 0,\ f_2(t,0,1) = 1 \neq 0;$
(3) $f_i, g_i, i = 1,2$ *are increasing with respect to the first argument and decreasing with respect to the second argument;*
(4) on the other hand, for $\lambda \in (0,1),\ t \in [0,1],\ x,y \in [0,+\infty),$ *choosing* $\psi(\lambda) = \lambda^{\frac{1}{2}} \in (\lambda,1),$ *we have*

$$\begin{aligned} g_1(t,\lambda u,\lambda^{-1}v) &= 2 + (1+t)\sqrt{\lambda u} + (\lambda^{-1}v)^{-\frac{1}{2}} \\ &\geq \lambda^{\frac{1}{2}}(2 + (1+t)\sqrt{u} + v^{-\frac{1}{2}}) = \psi(\lambda)g_1(t,u,v). \end{aligned}$$

$$\begin{aligned} g_2(t,\lambda u,\lambda^{-1}v) &= 1 + (2+sint)\sqrt[3]{\lambda u} + (\lambda^{-1}v)^{-\frac{1}{6}} \\ &\geq \lambda^{\frac{1}{3}}(1 + (2+sint)\sqrt[3]{u} + v^{-\frac{1}{6}}) \\ &\geq \lambda^{\frac{1}{2}}(1 + (2+sint)\sqrt[3]{u} + v^{-\frac{1}{6}}) = \psi(\lambda)g_2(t,u,v). \end{aligned}$$

Similarly, for all $\lambda \in (0,1),\ t \in [0,1],\ x,y \in [0,+\infty),$ *one has*

$$\begin{aligned} f_1(t,\lambda u,\lambda^{-1}v) &= \tfrac{\lambda u}{(2+t^4)(1+\lambda u)} + (\lambda^{-1}v)^{-\frac{1}{2}} > \lambda(\tfrac{u}{(2+t^4)(1+\lambda u)} + v^{-\frac{1}{2}}) \\ &> \lambda(\tfrac{u}{(2+t^4)(1+u)} + v^{-\frac{1}{2}}) = \lambda f_1(t,u,v). \end{aligned}$$

$$\begin{aligned} f_2(t,\lambda u,\lambda^{-1}v) &= \tfrac{\lambda u}{2+\lambda u} + (\lambda^{-1}v)^{-\frac{1}{6}} > \lambda(\tfrac{u}{2+\lambda u} + v^{-\frac{1}{6}}) \\ &> \lambda(\tfrac{u}{2+u} + v^{-\frac{1}{6}}) = \lambda f_2(t,u,v). \end{aligned}$$

(5) Taking $\delta_0 = 1,$ *for all* $t \in [0,1],\ x,y \in [0,+\infty),$ *we have*

$$\begin{aligned} g_1(t,u,v) &= 2 + (1+t)\sqrt{u} + v^{-\frac{1}{2}} \geq 2 + \sqrt{u} + v^{-\frac{1}{2}} \\ &\geq \tfrac{u}{(2+t^4)(1+u)} + v^{-\frac{1}{2}} = f_1(t,u,v). \end{aligned}$$

$$\begin{aligned} g_2(t,u,v) &= 1 + (2+sint)\sqrt[3]{u} + v^{-\frac{1}{6}} \geq 1 + \sqrt[3]{u} + v^{-\frac{1}{6}} \\ &\geq \tfrac{u}{2+u} + v^{-\frac{1}{6}} = f_2(t,u,v). \end{aligned}$$

Thus, all the conditions of Theorem 3.1 hold. Hence, we conclude that (29)(30) has one and only one positive solution $u^* \in P_h,$ *here* $h(t) = t^{\frac{3}{2}},\ t \in [0,1].$

5. **Conclusion**

This work focuses on developing a new approach to prove the existence of positive solutions for the fractional switched system with boundary condition of Riemann-Stieltjes integral. A notable technical challenge arises due to the hybrid equation, making the analysis inherently more complex and innovative. Another highlight of this work is the construction of the fractional switched equation (1), we add the fractional integral of the unknown function $u(t)$ to the nonlinear term g and f in (1). In addition, this work also highlights the iterative schemes, which can approximate the solutions.

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