SOME APPLICATIONS OF THE MAXIMAL PRIMARY COMPONENTS OF SUBMODULES

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W. Heinzer, L.J. Ratliff Jr. And K. Shah have found in [3] *and* [4] *a close connection between ideal covers and the maximal embedded components of a non-open ideal in a local Noetherian ring.*

The aim of the paper is to obtain the analogous results concerning submodule covers for a submodule of a strong Laskerian module over a local ring (not necessarily Noetherian), which has a strongly primary component.

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1. Introduction

In [7] L.J. Ratliff Jr. And D.E. Rush have introduced the notion of ideal covers in a local Noetherian ring. After introducing and studying in [2] the concept of a maximal embedded component of a non-open ideal in a local Noetherian ring, W. Heinzer, L.J. Ratliff Jr. and K. Shah have found in [3] and [4] a close connection between ideal covers and maximal embedded components..

In [5] the author have introduced the analogue for maximal embedded components of submodules and has obtained generalizations of the results of [2] and [3] for submodules of a strong Laskerian module over a quasi-local ring.

The purpose of this paper is to obtain the analogous results of [3] and [4] concerning covers of ideals for a submodule of a module over a local ring (not necessarily Noetherian) that admits a strongly primary component.

The notation and the terminology are mainly as in [1].

Throughout the paper we denote by R a commutative ring with identity and by E an R-module. By *Ass* (*E/F*) we denote the set of all associated prime divisors of the submodule F (in the weak Bourbaki sense), i.e. $P \in Ass(E/F) \Leftrightarrow P$ is a minimal prime divisor of $F{:}xR$ for some $x \in E \setminus F$.

In Section [2] we introduce the notion of a strongly primary component and give the basic results we need later. In Section [3] the main results are (3.3) and (3.4), which describe and give the number of covers of a strongly primary component of a submodule as well as the relationship between the submodule covers and their

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strongly primary components. Two characterizations of a maximal strongly primary component are given in (3.8).

2. Strongly Primary Components

Definition 2.1**.** Let P be a prime ideal of R and G a P-primary submodule of E.

G is a P-primary component of the submodule F of E if $P \in Ass(E/F)$ and if there exists a submodule V of E such that $P \notin Ass(E/V)$ and $F = G \cap V$.

If, in addition, G is of finite exponent (i.e. there exists $n \in \mathbb{N}$ such that $P^n E \subseteq G$) then G is called *a strongly P-primary component* of *F*.

Remark 2.2. If F is a P-primary submodule of E, then F is a P-primary component of itself, since $F = F \cap E$ and $P \notin Ass(E/E) = Ass(0) = \varnothing$.

Moreover, each P-primary component of F coincides with F.

Indeed, let G be a P-primary component of F. Then $F = G \cap V$ and $P \notin Ass(E/V)$.

If $G \subseteq V$ then it holds that G=F.

If $G \not\subset V$ then there exists $x \in G$ and $x \notin V$, so $x \notin F$ .Then $V : xR = F : xR \subseteq P$ since F is P-primary. So $P \in Ass(E/V)$, a contradiction.

Therefore it holds that if F is a P-primary submodule of E, then the unique P-primary component of it is F.

Definition 2.3. Let P be a prime ideal in R and F a submodule of E, which has a strongly P-primary component.

A *maximal strongly P-primary component* of F is a strongly P-primary component of F, that is not properly contained in any other strongly P-primary component of F.

Remark 2.4. Each strongly P-primary component of F is contained in a maximal strongly P-primary component.

Indeed, let $F = Q \cap V$, where Q is a strongly P-primary component of F and $P \notin Ass(E/V)$. Let $\mathfrak{A} = \{G$; G is a strongly P-primary component of $F\}$. Since $\mathfrak{A}$ is not empty ($Q \in \mathfrak{A}$) and $\mathfrak{A}$ is inductive, by Zorn's lemma $\mathfrak{A}$ has a maximal element, that is a maximal strongly P-primary component of F.

Remark 2.5. Let us remark the fact, that this maximal strongly P-primary component depends on V.

Thus, in $\mathbb{Q}[X,Y], (X)$ is a maximal strongly (X) - primary component of the ideal (X^2, XY) and $(X^2, XY) = (X) \cap (X^2, XY, Y^n)$ for each $n \in \mathbb{N}, n \geq 2$. Therefore $V = (X^2, XY, Y^n)$ is not unique.

There are situations in which V is unique.

LEMMA 2.6. *If F is a submodule of E which has a strongly P-primary component Q,* $F = Q \cap V$ *, and P is maximal in Ass(E/F), then V is unique.*

Proof. Let $n \in \mathbb{N}$ be such that $P^n E \subseteq Q$. Then $V = F : P^n$.

3.Strongly Primary Components and Submodule Covers

Throughout this section we denote by (R, M) a local ring (not necessarily Noetherian) and by *E* an *R*-module.

Definition 3.1. If $G \subseteq H$ are submodules of *E*, then *H* is *a cover* of *G* if there exists
$x \in H \setminus G$ such that $H = G + xR$ and $Mx \subset G$.

Remark 3.2. A submodule is sheltered if and only if it admits exactly one cover.
Indeed, it is clear, that if a submodule has only one cover then it is sheltered.
For the converse, let $G \subset E$ be a sheltered submodule of *E*. Then there exists $x \in E$, $x \notin G$ such that each submodule of *E*, which strictly contains *G*, contains also $G + xR$. Therefore $G + xR/G$ is a simple submodule $G + xR/G \cong R/M$ and $Mx \subset G$. So $G + xR$ is the unique cover of *G*.
From now on we consider *F* a submodule of *E*, that admits a strongly *M*-primary embedded component and in addition we assume that $V = F : M^n$ (from Lemma (2.6)) for all large $n \in \mathbb{N}$ has an irredundant primary decomposition.
The following proposition indicates us a way to construct two larger *M*-primary components of a given *M*-primary component of *F*, which are not maximal strongly *M*-primary components of *F*.

PROPOSITION 3.3. *Let Q be a strongly M-primary component of F, which is not maximal,* $F = Q \cap V$ *and* $M \notin \mathrm{Ass}(E/V)$. *Then there exists* $v \in V \setminus F$ *such that* $F + vR$ *is a cover of* F *and*

$$v \notin \bigcup \left\{ Q' \, ; Q' \text{ is a strongly M-primary component of } F \right\}$$

and there exists $x \in E$ *such that :*

(3.3.1) $Q_1 = Q + xR$ *and* $Q_2 = Q + (x + v)R$ *are strongly M-primary components of* F.

(3.3.2) Q_1 *and* Q_2 *are covers of* Q.

(3.3.3) *There are no containment relations between* Q_1 *and* Q_2.

(3.3.4) $Q_1 \cap Q_2 = Q$.

Proof. Since $M \in Ass(E/F)$ it follows that $F \subset F : M$. Let $v \in (F : M) \setminus F$. Then $vM \subset F \subset V$. Since $M \not\subset P$ for each $P \in Ass(E/V)$, it holds that $v \in V$. Therefore $v \in V \setminus F$ and $F + vR$ is a cover of *F*.

$v \notin \bigcup \{ Q' ; Q'$ is a strongly *M*-primary component of $F \}$ because otherwise there exists Q'' a strongly *M*-primary component of *F* such that $v \in Q''$. Hence $v \in Q'' \cap V = F$, which contradicts the choice of v. Since Q is not a maximal strongly *M*-primary component of *F*, there exists $Q' \supset Q$ a strongly *M*-primary component of F. We claim that then there exists $x \in Q' \setminus Q$ such that $Mx \subset Q$.
Indeed, let $y \in Q' \setminus Q$. Since *Q* is strongly *M*-primary, there exists $n \in \mathbb{N}$ such that $M^n y \subseteq Q$ and $M^{n-1} y \not\subset Q$. Therefore it exists $a \in M^{n-1}$ such that $ay \notin Q$. Considering $x := ay$ we get $x \in Q' \setminus Q$ and $Mx \subset Q$, therefore $Q_1 = Q + xR$ is a cover of *Q*.
Since $F = Q \cap V \subseteq Q_1 \cap V \subseteq Q' \cap V = F$ it follows that Q_1 is a strongly *M*-primary component of *F*.
To complete the proof of (3.3.1) we must show that $Q_2 \cap V = F$, where $Q_2 = Q + (x+v)R$.
For this, one containment is clear: $Q_2 \cap V \supseteq F$.To prove the opposite let $y \in Q_2 \cap V$. Then there exist $q \in Q$ and $d \in R$ such that $y = q + d(x+v) \in V$. Therefore $q + dx \in V$, so $q + dx \in V \cap Q_1 = F$. If $d \in M$, then $dv \in F$, so $y \in F$, as desired. If $d \notin M$; then d is a unit in R. Since $q + dx \in Q$ we get $dx \in Q$, and because d is a unit it follows that $x \in Q$, contradiction. Therefore $d \in M$, hence $Q_2 \cap V = F$, so Q_2 is a strongly *M*-primary component of *F*.

To prove (3.3.2) note first that in (3.3.1) we have shown that Q_1 is a cover of Q.
Let us prove now the statement for Q_2.

Since $vM \subseteq F$ and $xM \subseteq Q$, it follows that $(x+v)M \subseteq Q + F = Q$. Hence it remains to show that $x + v \notin Q$. Suppose the opposite, if $x + v \in Q \subset Q_1$, then, since $x \in Q_1$ we get that $v \in Q_1$, and this is a contradiction with the choice of v. So the proof of (3.3.2) is complete.

To prove (3.3.3) observe that if $Q_2 \subseteq Q_1$ then $x + v \in Q_1$, so it follows that $v \in Q_1$, which does not hold.. If $Q_1 \subseteq Q_2$ then $x - r(x+v) \in Q$, with $r \in R$.
If $r \in M$ then $x(1-r) \in Q + vM \subseteq Q + F = Q$ and so $x \in Q$, which contradicts the choice of x.
If $r \notin M$ it follows that $x + v \in Q_1$, so $v \in Q_1$, and this contradicts (3.3.1). Hence (3.3.3) holds.

For (3.3.4) observe that since Q_1 and Q_2 are distinct covers of Q, from the definition of a cover it follows that $Q = Q_1 \cap Q_2$. □

PROPOSITION 3.4. *Let* $Q \subset Q^{'}$ *be two strongly M-primary components of F,* $F = Q \cap V = Q^{'} \cap V$, *where* $V = F : M^n, n \in \mathbb{N}$, *let* $x \in Q^{'} \setminus Q$ *and let* v *be as in* (3.3). *Then:*

(3.4.1) $Q + xR$ *and* $Q + (x + kv)R$ *are strongly M-primary components of F for all units* k *in R.*

(3.4.2) *If* $Q + xR$ *is a cover of Q, then for each unit* k *in R,* $Q + (x + kv)R$ *is a cover of* Q*, and there are* card*(R/M) such submodules.*

Proof. For (3.4.1), since $Q \subset Q + xR \subseteq Q^{'}$ it follows that $Q + xR$ is a strongly M-primary component of F. Then it follows, as shown in the proof of (3.3.1), that $Q + (x + v)R$ and $Q + (x + kv)R$ are strongly M-primary components of F for all units k in R.

For (3.4.2), the proof that each $Q + (x + kv)R$ is a cover of Q is the same as in (3.3.2). So, to complete the proof of (3.4.2), it suffices to show that if k and $k^{'}$ are units in R such that $k + M \neq k^{'} + M$, then $Q + (x + kv)R \not\subset Q + (x + k^{'}v)R$.

For this, if $x + kv \in Q + (x + k^{'}v)R$ then $x + kv = q + r(x + k^{'}v)$, with $q \in Q$ and $r \in R$, so

$$(k - k^{'})v = q + (r - 1)(x + k^{'}v) \in Q + (x + k^{'}v)R.$$

Since $k - k^{'}$ is a unit in R, it follows that $v \in Q + (x + k^{'}v)R$, which contradicts the choice of v, since by (3.3.1) $Q + (x + k^{'}v)R$ is a strongly M- primary component of F. □

There is a very close connection between covers of submodules and strongly M-primary components of submodules. To be more specific, note first that in (3.3) it is shown the existence of $v \in V \setminus F$ such that $F + vR$ is a cover of F. Hence the submodules F of E, which have a strongly M- primary component Q, $F = Q \cap V$, $M \notin Ass(E/V)$ and V admits an irredundant primary decomposition, have covers.

On the other hand, if G is a submodule of E, which is not strongly M- primary but which admits a finite irredundant primary decomposition, then G has the properties of F, i.e. G has a strongly M- primary component $Q^{'}$, $G = Q^{'} \cap V^{'}$ and $M \notin Ass(E/V^{'})$.

To see this, let L be a cover of G, $L = G + xR$ and $xM \subset G$. Then $ML \subseteq G \subset L$, so $GR_P = LR_P$ for each $P \in Spec(R) \setminus \{M\}$. Therefore, from $G \subset L$ it follows that G has a strongly M-primary component. Consider now

$\mathfrak{M} = \{H \subset E$ submodule ; $H \supset F$ and H is a strongly M -primary component of $F\}$

By (2.6) it follows that for each $H \in \mathfrak{M}$ we have $F = H \cap V$, where $V = F : M^n$, $n \in \mathbb{N}$.

The next result shows that if $G \supset F$ is a submodule of E, then $G \in \mathfrak{M}$ if and only if there exists a one-to-one correspondence between the covers of F and the covers of G that are not in $\mathfrak{M}$.

THEOREM 3.5. *Let $G \subset E$ be a submodule such that $G \supset F$. Then $G \in \mathfrak{M}$ if and only if there exists a one-to-one correspondence between the covers $F + wR$ of F and the covers $G + xR$ of G such that $G + xR \notin \mathfrak{M}$.*

This correspondence is given by $F + wR$ corresponds to $G + wR$ and $G + xR$ corresponds to $F + wR$, where $w = g + x \in (F : M) \setminus F$, for some $g \in G$.

Proof . Assume first that $G \in \mathfrak{M}$ and let $F + wR$ be a cover of F. Then $w \in (F : M) \setminus F \subseteq V \setminus F$, so $w \in (G + wR) \cap V$, hence $G + wR \notin \mathfrak{M}$. Since $G \in \mathfrak{M}$ it follows that $w \notin G$. Further $wM \subseteq F \subseteq G$, therefore $G + wR$ is a cover of G.

Consider now $G + xR$ a cover of G such that $G + xR \notin \mathfrak{M}$.Then $F \subset (G + xR) \cap V$, implying the existence of $y \in ((G + xR) \cap V) \setminus F$ such that

$$yM \subseteq ((G + xR) \cap V)M \subseteq (G + xM) \cap V \subseteq G \cap V = F .$$

Then $G \subset G + yR$, since $y \notin G$ (otherwise $y \in G \cap V = F$, which contradicts the choice of y) and $G + yR \subseteq G + xR$, so $G + yR = G + xR$. Therefore $y = g + xr$ for a unit $r \in R$ (otherwise $y \in G$). Hence $r^{-1}y = r^{-1}g + x$. Taking $w = r^{-1}y$ we get that $F + wR$ is a cover of F since $w \notin F$ (otherwise $y \in F$) and $wM \subset F$ because $yM \subset F$.

Observe that from this argument it follows that distinct covers of G which are not in $\mathfrak{M}$ determine distinct covers of F.

Let $F + wR$ and $F + uR$ be two distinct covers of F and suppose that $G + wR = G + uR$. Then $u = g + rw$ for some unit $r \in R$ (since according to the first part of the proof if $G \in \mathfrak{M}$, $F + wR$ and $F + uR$ are covers of F, then $G + wR$ and $G + uR$ are covers of G), so $u - rw \in G \cap (F : M) \subseteq Q \cap V = F$, where Q is a maximal strongly M -primary component of F that contains G. Therefore $F + wR = F + uR$, which contradicts the hypothesis. Hence $G + wR \neq G + uR$ and the one-to-one correspondence readily follows from this result and (3.4).

Conversely, assume that $G \notin \mathfrak{M}$. Then $F \subset G \cap V$. Consider $x \in (G \cap V) \setminus F$. Then as before, there exists $y \in F + xR$ such that $F + yR$ is a cover of F. But $y \in F + xR \subseteq G$, so there does not exist such a one-to-one correspondence. □

Remark 3.6. From (3.5) it follows that if Q is a strongly M-primary submodule of E that contains F, then Q is a maximal strongly M-primary component of F if and only if the covers of Q are the distinct submodules $Q+wR$, where the submodules $F+wR$ are distinct covers of F.

In [5] we introduced the following definition for an M-primary submodule:

Definition 3.7. *Let Q be an M-primary submodule of E. IC(Q) denotes the set of irreducible M-primary submodules that appear in some decomposition of Q as an irredundant intersection of irreducible submodules.*

We gave in [5] the following description of this set

$IC(Q)=\{G\ ;\ G$ is an irreducible submodule of $E,\ Q\subseteq G$ and $Q:M\not\subset G\}$

The next result gives us a connection between covers and the above mentioned set *IC(Q)* as well as two characterizations of a maximal strongly *M*-primary component of *F*. The first equivalence of (3.8) is quite useful since in the case of strong Laskerian modules it localizes nicely (see (3.9)).

THEOREM 3.8. *Let Q be a strongly M-primary component of F. Then the following are equivalent:*

(3.8.1) *Q is a maximal strongly M-primary component of F.*

(3.8.2) *No submodule of IC(Q) contains F:M.*

(3.8.3) *For each* $G\in IC(Q)$ *there exists* $x\in F:M$ *such that* $G+xR$ *is the unique cover of G.*

Proof. Assume that (3.8.1) holds and let $G\in IC(Q)$. Then $Q:M\not\subset G$ by [5, 2.13] and $Q:M=Q+(F:M)$ by [5,2.6]. So, since $Q\subset G$ it follows that $F:M\not\subset G$, hence (3.8.1) implies (3.8.2).

Suppose that (3.8.2) holds and let $G\in IC(Q)$. Then there exists $x\in(F:M)\setminus G$ hence $G\subset G+xR$. Also $xM\subseteq F\subseteq G$, hence $G+xR$ is a cover of G. Therefore, since G is an irreducible submodule, it follows that $G+xR$ is the unique cover of G, proving thus that (3.8.2) implies (3.8.3). It is obvious that (3.8.2) is a consequence of (3.8.3).

To complete the proof it suffices to show that if (3.8.1) does not hold, then (3.8.2) does not. For this purpose, assume that Q is not a maximal strongly M-primary component of F. Then $Q+(F:M)\subset Q:M$ by[5,2.6]. Therefore there exists an irreducible component G of $Q+(F:M)$, that does not contain $Q:M$. Then $Q\subseteq G$ and $(Q:M)\not\subset G$, that means $G\in IC(Q)$ by [5,2.13] and $(F:M)\subseteq G$ by construction. Therefore (3.8.2) does not hold. Hence we proved that (3.8.2) implies (3.8.1). □

COROLLARY 3.9. *Let A be a commutative ring with identity, P a prime ideal of A with ht(P)*≥ 1, E' *a strong Laskerian A-module,* F' *a submodule of* E', Q' *a strong P-primary submodule of* E' *that contains* F'. *Then* Q' *is a maximal P-primary embedded component of* F' *if and only if no submodule of* $IC(Q')$ *contains* $F':P$.

Proof. The statement follows immediately from the implication (3.8.1)$\Rightarrow$(3.8.2) and [5, (3.3)]. □

PROPOSITION 3.10. *If Q is a strongly M-primary submodule of E and if* $G \in IC(Q)$, *then* $Q:M \subseteq H$, *where H is the unique cover of G. Moreover, it holds that* $H = G + (Q:M)$.

Proof. Let $G \in IC(Q)$. Then G is an irreducible submodule and therefore $H = G:M$. Now, since $Q \subseteq G$, it follows that $Q:M \subseteq G:M = H$. But by [5,(2.13)] $(Q:M) \not\subset G$, therefore $H = G + (Q:M)$. □

COROLLARY 3.11. *If Q is a maximal strongly M-primary component of F and* $G \in IC(Q)$, *then* $F:M$ *is a cover of* $G \cap (F:M)$.

Proof. (3.8.1) $\Rightarrow$(3.8.2) shows that $F:M \not\subset G$ and (3.10) shows that $G+(Q:M)$ is the unique cover of G. Hence from $F:M \subseteq Q:M$ we conclude that $G+(F:M) = G+(Q:M)$. The conclusion of the corollary follows now from

$$(F:M)/(G \cap (F:M)) \cong (G+(F:M))/G$$ □

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