EXISTENCE RESULTS FOR A SECOND-ORDER q-DIFFERENCE EQUATION WITH ONLY INTEGRAL CONDITIONS

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In this paper, we study a boundary value problem of second-order q-difference equation with only integral conditions. By using a variety of fixed point theorems (such as Banach's contraction principle, Boyed and Wong fixed point theorem, Leray-Schauder nonlinear alternative, and Krasnoselskii's fixed point theorem), we obtain some new existence results. As applications, some examples to illustrate our results are given.

Keywords: q-difference equations, Integrals conditions, Fixed point theorem, Existence, Uniqueness

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1. Introduction

The study of q-difference equations, initiated by Jackson [11], Carmichael [8], Mason [16] and Adams [1] in the first quarter of 20th century, has been developed over years, and evolved into a multidisciplinary subject. It plays an important role in several fields of physics such as cosmic strings and black holes, conformal quantum mechanics, nuclear and high energy physics. In recent years, this topic has attracted the attention of several scholars, and a variety of new results can be found [2]-[6],[10], [14],[17]-[25]. Among these achievements, we find that boundary value problems of q-difference equations with integral conditions constitute a very interesting class of problems, and have been studied by a number of authors [5, 10, 18, 19, 21, 22, 24].

For example, in [5], Ahmad et al. studied the boundary value problem of nonlinear q-difference equation with nonlocal and integral boundary conditions given by

$$\begin{cases} D_q^2 u(t) = f(t, u(t)), \quad t \in I_q, \\ u(0) = u_0 + g(u), \quad u(1) = \alpha \int_\mu^\nu u(s) d_q s, \quad u_0 \in \mathbb{R}, \end{cases}$$

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where $f \in \mathbb{C}(I_q \times \mathbb{R}, \mathbb{R})$ such that $f(t, u(t))$ is continuous at $t = 0$, $I_q = \{q^n : n \in \mathbb{N}\} \bigcup \{0, 1\}$, $q \in (0, 1)$ is a fixed constant and $\mu, \nu \in I_q$ with $\mu < \nu$. By using Banach's contraction principle and a fixed point theorem due to O'Regan, they derived the existence of solutions.

In [19], Pongarm et al. considered sequential derivative of nonlinear q-difference equation with three-point boundary conditions,

$$\begin{cases} D_q(D_p + \lambda)u(t) = f(t, u(t)), \quad t \in I_q^T = [0, T] \cap I_q^1, \\ u(0) = 0, \quad u(T) = \alpha \displaystyle\int_0^\eta u(s) d_r s, \end{cases}$$

where $0 < p, q, r < 1, f \in \mathbb{C}(I_q^T \times \mathbb{R}, \mathbb{R}), 0 < \eta < T$, and λ, β are given constants. Existence results are proved based on Banach's contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder degree theory.

However, we note that among the existing literature [5, 10, 15, 18, 19, 21, 22, 24], no one has studied the boundary value problems of second-order q-difference equations with only integral conditions. So, in this paper, we discuss the existence results for the following second-order boundary value problem:

$$\begin{cases} D_q^2 u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \displaystyle\int_0^1 u(t) d_q t, \quad u(1) = \int_0^1 t u(t) d_q t, \end{cases} \tag{1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given function. Our results are based on Banach's contraction principle, Leray-Schauder nonlinear alternative, Boyed and Wong fixed point theorem, and Krasnoselskii's fixed point theorem. To be detailed, we first consider the related problem (3) and find out the equivalent integral equation (4), and then define an operator F by (12). We observe that problem (1) has solutions if and only if the operator F has fixed points.

It is noteworthy that, if $q \to 1$, then problem (1) recaptures the following boundary value problem:

$$\begin{cases} u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \displaystyle\int_0^1 u(t) dt, \quad u(1) = \int_0^1 t u(t) dt, \end{cases} \tag{2}$$

of which Guezane-Lakoud et al. [10] established the existence to nontrivial solution by using Banach's contraction principle and Leray-Schauder nonlinear alternative.

The rest of this paper is organized as follows. In Section 2, we briefly discuss about the basic definitions, some properties of q-calculus and present a lemma that will be used throughout this paper. In Section 3, we give the main results. Some examples illustrating the results established in this paper are presented in the last section.

2. **Preliminaries**

The basic definitions and some properties of q-calculus [13] are as follows.

Definition 2.1. *For $0 < q < 1$, we define the q-derivative of a real valued function u as*

$$D_q u(t) = \frac{u(t) - u(qt)}{(1-q)t}, \ t \neq 0; \quad D_q u(0) = \lim_{t \to 0} D_q u(t).$$

The higher order q-derivatives are given by

$$D_q^n u(t) = D_q D_q^{n-1} u(t), \quad n \in \mathbb{N},$$

where $D_q^0 u(t) = u(t)$.

The definite q-integral of a function u defined on the interval $[0, T]$ is given by

$$I_q u(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q) q^n u(tq^n),$$

where last term is the convergent series.

If $a \in [0, T]$, then

$$\int_a^b u(s) d_q s = I_q u(b) - I_q u(a) = (1-q) \sum_{n=0}^{\infty} q^n \left[bu(bq^n) - au(aq^n) \right].$$

We note that

$$D_q I_q u(x) = u(x),$$

and if f is continuous at $x = 0$, then

$$I_q D_q u(x) = u(x) - u(0).$$

The property of the product rule and the integration by parts formula are

$$D_q(gh)(t) = (D_q g(t)) h(t) + g(qt) D_q h(t),$$

$$\int_0^x h(t) D_q g(t) d_q t = [h(t) g(t)] \big|_0^x - \int_0^x D_q h(t) g(qt) d_q t.$$

Further, reversing the order of integration is given by

$$\int_0^t \int_0^s u(r) d_q r d_q s = \int_0^t \int_{qr}^t u(r) d_q s d_q r.$$

In the limit $q \to 1$, the above results correspond to their counterparts in standard calculus.

Lemma 2.1. *Let $0 < q < 1$. Then, for any $y \in \mathbb{C}([0,1], \mathbb{R})$, the boundary value problem*

$$\begin{cases} D_q^2 u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = \int_0^1 u(t) d_q t, & u(1) = \int_0^1 t u(t) d_q t \end{cases} \tag{3}$$

is equivalent to the integral equation

$$u(t) = -\int_0^t (t - qs) y(s) d_q s + \frac{1+q}{q} \int_0^1 (1 - qs) y(s) d_q s$$

$$- \frac{1}{q(1+q+q^2)} \int_0^1 \Big\{ \left[1-(1+q)qs+q(qs)^2\right] \times \left[q^2 s + (q^3+2q^2+2q+1) - (q+q^2+q^3)t\right] y(s) \Big\} d_q s. \tag{4}$$

Proof. Rewriting the differential equation as $D_q^2 u(t) = -y(t)$, then taking double q-integral for it, we have

$$u(t) = \int_0^t \int_0^s y(v) d_q v d_q s + a_1 t + a_2. \tag{5}$$

By changing the order of q-integration, we have

$$u(t) = -\int_0^t \int_{qv}^t y(v) d_q s d_q v + a_1 t + a_2 = -\int_0^t (t-qv) y(v) d_q v + a_1 t + a_2. \tag{6}$$

Using the first integral condition, we get $a_2 = \int_0^1 u(v) d_q v$. Substituting a_2 in (6), we have

$$u(t) = -\int_0^t (t-qv) y(v) d_q v + a_1 t + \int_0^1 u(v) d_q v. \tag{7}$$

Using the second integral condition, we have

$$a_1 = \int_0^1 (1-qv) y(v) d_q v + \int_0^1 v u(v) d_q v - \int_0^1 u(v) d_q v.$$

Substituting a_1 in (7), we obtain

$$u(t) = -\int_0^t (t-qs) y(s) d_q s + t\int_0^1 (1-qs) y(s) d_q s + t\int_0^1 s u(s) d_q s + (1-t)\int_0^1 u(s) d_q s. \tag{8}$$

Integrating (8) over $[0,1]$, it yields

$$\int_0^1 u(s) d_q s = -\int_0^1 \left[1-(1+q)qs+q(qs)^2\right] y(s) d_q s + \int_0^1 (1-qs) y(s) d_q s + \int_0^1 s u(s) d_q s. \tag{9}$$

Substituting (9) in (8), we get

$$u(t) = -\int_0^t (t-qs) y(s) d_q s + \int_0^1 (1-qs) y(s) d_q s + \int_0^1 s u(s) d_q s - (1-t)\int_0^1 \left[1-(1+q)qs+q(qs)^2\right] y(s) d_q s. \tag{10}$$

Multiplying (10) by t, then integrating the resultant equality over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 su(s)d_qs \\
=& -\frac{1}{q(1+q+q^2)}\int_0^1 (1-qs)(1-q^2s)(1+q+q^2s)y(s)d_qs \\
&-\frac{q}{1+q+q^2}\int_0^1 \left[1-(1+q)qs+q(qs)^2\right] y(s)d_qs+\frac{1}{q}\int_0^1 (1-qs)y(s)d_qs. \qquad (11)
\end{aligned}$$

Substituting (11) in (10), it yields

$$\begin{aligned}
u(t) =& -\int_0^t (t-qs)y(s)d_qs+\frac{1+q}{q}\int_0^1 (1-qs)y(s)d_qs \\
&-\frac{1}{q(1+q+q^2)}\int_0^1 \Big\{ \left[1-(1+q)qs+q(qs)^2\right] \\
&\times \left[q^2s+(q^3+2q^2+2q+1)-(q+q^2+q^3)t\right] y(s)\Big\}d_qs.
\end{aligned}$$

This completes the proof. □

Let $\mathbb{C}=\mathbb{C}([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\{|u(t)|, t\in[0,1]\}$. Define an operator $F:\mathbb{C}\to\mathbb{C}$ by

$$\begin{aligned}
(Fu)(t) =& -\int_0^t (t-qs)f(s,u(s))d_qs+\frac{1+q}{q}\int_0^1 (1-qs)f(s,u(s))d_qs \\
&-\frac{1}{q(1+q+q^2)}\int_0^1 \Big\{ \left[1-(1+q)qs+q(qs)^2\right] \\
&\times \left[q^2s+(q^3+2q^2+2q+1)-(q+q^2+q^3)t\right] f(s,u(s))\Big\}d_qs. \qquad (12)
\end{aligned}$$

Observe that problem (1) has solutions if and only if the operator F has fixed points. For the sake of convenience, we set a constant Λ as

$$\begin{aligned}
\Lambda =& \frac{1}{q}+\frac{q}{1+q+q^2}+\frac{(1+q)(1-q^3)}{q(1+q+q^2)}+\frac{q^3(1+q)^2}{(1+q+q^2)^2} \\
&+\frac{q^4}{(1+q+q^2)(1+q+q^2+q^3)}. \qquad (13)
\end{aligned}$$

3. **Existence results**

In this section, we will introduce our main results. Our first result is based on Banach's fixed point theorem.

Theorem 3.1. *Assume that* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the conditions*

(H_1) $|f(t,u) - f(t,v)| \leq L\,|u - v|\,, \quad \forall\ t \in [0,1]$ *and* $u, v \in \mathbb{R}$,

(H_2) $L\Lambda < 1$,

where L *is a Lipschitz constant, and* Λ *is defined by* (13). *Then problem* (1) *has a unique solution.*

Proof. We transform problem (1) into a fixed point problem $u = Fu$, where $F : \mathbb{C} \to \mathbb{C}$ is defined by (12). Assume that $\sup\limits_{t\in[0,1]} |f(t,0)| = M$, and choose a constant R satisfying

$$R \geq \frac{M\Lambda}{1 - L\Lambda}. \tag{14}$$

First, we will show that $FB_R \subset B_R$, where $B_R = \{u \in \mathbb{C} : \|u\| \leq R\}$. For any $u \in B_R$, we have

$$\begin{aligned}
&\|Fu\| \\
\leq &\sup_{t\in[0,1]} \Bigg| \int_0^t (t - qs)(L\|u\| + M) d_q s + \frac{1+q}{q} \int_0^1 (1 - qs)(L\|u\| + M) d_q s \\
&+ \frac{1}{q(1+q+q^2)} \int_0^1 \Big\{ \left[1 - (1+q)qs + q(qs)^2\right] \\
&\times \left[q^2 s + (q^3 + 2q^2 + 2q + 1) - (q + q^2 + q^3)t\right] (L\|u\| + M) \Big\} d_q s \Bigg| \\
\leq &\left(\frac{1}{q} + \frac{q}{1+q+q^2} + \frac{(1+q)(1-q^3)}{q(1+q+q^2)} + \frac{q^3(1+q)^2}{(1+q+q^2)^2} \right. \\
&\left. + \frac{q^4}{(1+q+q^2)(1+q+q^2+q^3)} \right) \times (LR + M) \\
= &(LR + M)\Lambda \leq R.
\end{aligned}$$

Therefore $FB_R \subset B_R$.

Next, we will show that F is a contraction. For any $u, v \in \mathbb{C}$ and for each $t \in [0,1]$, we have

$$\begin{aligned}
\|Fu - Fv\| \leq &\sup_{t\in[0,1]} \Bigg\{ L\|u - v\| \int_0^t (t - qs) d_q s + L\|u - v\| \frac{1+q}{q} \int_0^1 (1 - qs) d_q s \\
&+ L\|u - v\| \frac{1}{q(1+q+q^2)} \int_0^1 \Bigg[\left[1 - (1+q)qs + q(qs)^2\right] \\
&\times \left[q^2 s + (q^3 + 2q^2 + 2q + 1) - (q + q^2 + q^3)t\right] \Bigg] d_q s \Bigg\} \\
\leq &L\Lambda \|u - v\|.
\end{aligned}$$

As $L\Lambda < 1$, F is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof. □

Remark 3.1. *In Theorem 3.1, if* $q \to 1$, *then* $\Lambda = \frac{67}{36}$. *Problem* (1) *reduces to Problem* (2) *and Problem* (2) *has a unique solution.*

Next, we prove the existence of solutions of problem (1) by using the following Leray-Schauder nonlinear alternative:

Theorem 3.2. (*Nonlinear Alternative for Single Valued Maps*) [9] *Let* $\mathbb{E}$ *be a Banach space,* $\mathbb{C}$ *be a closed convex subset of* $\mathbb{E}$, $\mathbb{U}$ *be an open subset of* $\mathbb{C}$, *and* $0 \in \mathbb{U}$. *Suppose that* $F : \overline{\mathbb{U}} \to \mathbb{C}$ *is a continuous, compact* (*that is,* $F(\overline{\mathbb{U}})$ *is a relatively compact subset of* $\mathbb{C}$) *map. Then, either*

(1) F *has a fixed point in* $\overline{\mathbb{U}}$ *or*

(2) *there is a* $u \in \partial\mathbb{U}$ (*the boundary of* $\mathbb{U}$ *in* $\mathbb{C}$) *and* $\lambda \in (0,1)$ *with* $u = \lambda F(u)$.

Theorem 3.3. *Assume that:*

(H_3) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,1],\mathbb{R}^+)$ *such that* $|f(t,u)| \le p(t)\psi(\|u\|)$, *for each* $(t,u) \in [0,1] \times \mathbb{R}$;

(H_4) *there exists a constant* $M > 0$ *such that* $\dfrac{M}{\psi(\|u\|)\|p\|_{L^1}\Lambda} > 1$,

where $\|p\|_{L^1} = \int_0^1 p(s)d_qs \neq 0$.

Then problem (1) *has at least one solution.*

Proof. We define $F : \mathbb{C} \to \mathbb{C}$ as in (12). The proof consists of several steps.

(1) F maps bounded sets into bounded sets in $\mathbb{C}([0,1],\mathbb{R})$.

Let $B_K = \{u \in \mathbb{C}([0,1],\mathbb{R}) : \|u\| \le K\}$ be a bounded set in $\mathbb{C}([0,1],\mathbb{R})$ and $u \in B_K$. Then we have

$$
\begin{aligned}
|Fu(t)| \le & \int_0^t (t-qs)\,|f(s,u(s))|\,d_qs + \frac{1+q}{q}\int_0^1 (1-qs)\,|f(s,u(s))|\,d_qs \\
& + \frac{1}{q(1+q+q^2)}\int_0^1 \Big\{ \left[1-(1+q)qs+q(qs)^2\right] \\
& \times \left[q^2s+(q^3+2q^2+2q+1)-(q+q^2+q^3)t\right]|f(s,u(s))| \Big\} d_qs \\
\le & \psi(\|u\|)\|p\|_{L^1}\int_0^t (t-qs)d_qs + \psi(\|u\|)\|p\|_{L^1}\left(\frac{1+q}{q}\right)\int_0^1 (1-qs)d_qs \\
& + \frac{\psi(\|u\|)\|p\|_{L^1}}{q(1+q+q^2)}\int_0^1 \Big\{ \left[1-(1+q)qs+q(qs)^2\right] \\
& \times \left[q^2s+(q^3+2q^2+2q+1)-(q+q^2+q^3)t\right] \Big\} d_qs \\
= & \psi(\|u\|)\|p\|_{L^1}\Lambda.
\end{aligned}
$$

Thus

$$\|Fu\| \le \psi(K)\|p\|_{L^1}\Lambda.$$

(2) F maps bounded sets into equicontinuous sets of $\mathbb{C}([0,1],\mathbb{R})$.

Let $r_1, r_2 \in [0,1], r_1 < r_2$ and B_K be a bounded set of $\mathbb{C}([0,1],\mathbb{R})$ as before, then for $u \in B_k$ we have

$$\begin{aligned}
&|Fu(r_2) - Fu(r_1)| \\
\leq& \int_0^{r_1} |r_2 - r_1| \, p(s)\psi(K) d_q s + \int_{r_1}^{r_2} (r_2 - qs)\psi(K) d_q s \\
&+ \frac{1}{q(1+q+q^2)} \int_0^1 \Big\{ \left[1 - (1+q)qs + q(qs)^2\right] \\
&\times (q+q^2+q^3) \, |r_2 - r_1| \, p(s)\psi(K) \Big\} d_q s.
\end{aligned}$$

As $r_2 - r_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $u \in B_K$. Thus F is equicontinuous. As F satisfies the above assumptions, it follows by the Arzela-Ascoli theorem that $F : \mathbb{C}([0,1],\mathbb{R}) \to \mathbb{C}([0,1],\mathbb{R})$ is completely continuous.

(3) Let $\lambda \in (0,1)$ and let $u = \lambda F u$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
&|u(t)| = |\lambda F u(t)| \\
\leq& \int_0^t (t - qs) \, |f(s, u(s))| \, d_q s + \frac{1+q}{q} \int_0^1 (1 - qs) \, |f(s, u(s)| \, d_q s \\
&+ \frac{1}{q(1+q+q^2)} \int_0^1 \Big\{ \left[1 - (1+q)qs + q(qs)^2\right] \\
&\times \left[q^2 s + (q^3 + 2q^2 + 2q + 1) - (q + q^2 + q^3)t\right] |f(s, u(s))| \Big\} d_q s \\
\leq& \psi(\|u\|)\|p\|_{L^1}\Lambda,
\end{aligned}$$

and consequently

$$\frac{\|u\|}{\psi(\|u\|)\|p\|_{L^1}\Lambda.} \leq 1.$$

In view of (H_4), there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in \mathbb{C}([0,1],\mathbb{R}) : \|u\| < M\}.$$

Note that the operator $F : \overline{U} \to \mathbb{C}([0,1],\mathbb{R})$ is continuous and completely continuous (which is well known to be compact restricted to bounded sets). From the choice of U, there is no $u \in \partial U$ such that $u = \lambda F u$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point $u \in \overline{\mathbb{U}}$, which is a solution of problem (1). This completes the proof. □

Remark 3.2. *In Theorem 3.3, if* $q \to 1$*, then* $\Lambda = \frac{67}{36}$ *and* M *satisfies* $\dfrac{36M}{67\psi(\|u\|)\|p\|_{L^1}} > 1$*, where* $\|p\|_{L^1} = \displaystyle\int_0^1 p(s)ds \neq 0$*. Problem* (1) *reduces to Problem* (2) *and Problem* (2) *has at least one solution.*

The third result is based on Boyed and Wong fixed point theorem below.

Definition 3.1. [22] *Let* $\mathbb{E}$ *be a Banach space and let* $A : \mathbb{E} \to \mathbb{E}$ *be a mapping.* A *is said to be a nonlinear contraction if there exists a continuous nondecreasing function* $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\Psi(0) = 0$ *and* $\Psi(\rho) < \rho$ *for all* $\rho > 0$ *with the following property:*

$$\|Ax - Ay\| \leq \Psi(\|x - y\|), \quad \forall x, y \in \mathbb{E}.$$

Lemma 3.1. (*Boyed and Wong*) [7] *Let* $\mathbb{E}$ *be a Banach space and let* $A : \mathbb{E} \to \mathbb{E}$ *be a nonlinear contraction. Then,* A *has a unique fixed point in* $\mathbb{E}$*.*

Theorem 3.4. *Suppose that*

(H_5) *there exists a continuous function* $h : [0, 1] \to \mathbb{R}^+$ *such that*

$$|f(t, x) - f(t, y)| \leq h(t)\frac{|x - y|}{G + |x - y|}$$

for all $t \in [0, 1]$ *and* $x, y \geq 0$*, where*

$$G = \frac{1 + 2q}{q}\int_0^1 (1 - qs)h(s)d_qs + \frac{1}{q(1 + q + q^2)}\int_0^1 \Big\{ \big[1 - (1 + q)qs + q(qs)^2\big] \times \big[q^2s + (q^3 + 2q^2 + 2q + 1)\big]\, h(s)\Big\} d_qs.$$

Then, problem (1) *has a unique solution.*

Proof. Let the operator $F : \mathbb{C} \to \mathbb{C}$ be defined as in (12). We define a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\rho) = \frac{G\rho}{G + \rho}, \quad \forall \rho \geq 0,$$

such that $\Psi(0) = 0$ and $\Psi(\rho) < \rho$, for all $\rho > 0$. Let $u, v \in \mathbb{C}$. Then, we get

$$|f(s, u(s)) - f(s, v(s))| \leq h(s)\frac{|u - v|}{G + |u - v|}.$$

Thus

$$\begin{aligned}|Fu(t) - Fv(t)| \leq &\Big\{ \int_0^1 (1 - qs)h(s)d_qs + \frac{1 + q}{q}\int_0^1 (1 - qs)h(s)d_qs \\ &+ \frac{1}{q(1 + q + q^2)}\int_0^1 \Big[\big[1 - (1 + q)qs + q(qs)^2\big] \\ &\times \big[q^2s + (q^3 + 2q^2 + 2q + 1)\big]\, h(s)\Big] d_qs \Big\} \times \frac{\|u - v\|}{G + \|u - v\|}\end{aligned}$$

$$=\frac{G\|u-v\|}{G+\|u-v\|}, \quad \forall t \in [0,1].$$

This implies that $\|Fu-Fv\| \le \Psi(\|u-v\|)$. Hence, F is a nonlinear contraction. Therefore, by Lemma 3.1, the operator F has a unique fixed point in $\mathbb{C}$, which is a unique solution of problem (1). □

Remark 3.3. *In Theorem 3.4, if $q \to 1$, then $G = \frac{1}{3}\int_0^1 (1-s)(s^2+5s+3)h(s)ds$. Problem* (1) *reduces to Problem* (2) *and Problem* (2) *has a unique solution.*

As the fourth result, we prove the existence of solutions of (1) by using Krasnoselskii's fixed point theorem below.

Theorem 3.5. [12] *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that:*
(1) *$Ax + By \in K$ whenever $x, y \in K$,*
(2) *A is compact and continuous,*
(3) *B is a contraction mapping.*
Then, there exists $z \in K$ such that $z = Az + Bz$.

Theorem 3.6. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (H_1) and the following assumption holds:*
(H_6) $|f(t,u)| \le \mu(t)$, $\forall (t,u) \in [0,1] \times \mathbb{R}$, and $\mu \in \mathbb{L}^1([0,1], \mathbb{R}^+)$.
If

$$L\left\{\frac{1}{q}+\frac{(1+q)(1-q^3)}{q(1+q+q^2)}+\frac{q}{(1+q)(1+q+q^2)}+\frac{q^3(1+q)^2}{(1+q+q^2)^2}+\frac{q^4}{(1+q+q^2)(1+q+q^2+q^3)}\right\}<1, \tag{15}$$

then problem (1) *has at least one solution on* $[0,1]$.

Proof. Setting $\max_{t\in[0,1]} |\mu(t)| = \|\mu\|$ and choosing a constant $R \ge \|\mu\|\Lambda$, where Λ is given by (13), and define $B_R = \{u \in \mathbb{C} : \|\mu\| \le R\}$.

We define the operators F_1 and F_2 on the ball B_R as

$$(F_1u)(t) = -\int_0^t (t-qs)f(s,u(s))d_qs$$

$$\begin{aligned}(F_2u)(t) =& \frac{1+q}{q}\int_0^1 (1-qs)f(s,u(s))d_qs \\ &- \frac{1}{q(1+q+q^2)}\int_0^1 \Big\{ \left[1-(1+q)qs+q(qs)^2\right] \\ &\times \left[q^2s+(q^3+2q^2+2q+1)-(q+q^2+q^3)t\right] f(s,u(s))\Big\}d_qs.\end{aligned}$$

For $u, v \in B_R$, we have

$$\|F_1u + F_2v\| \leq \|\mu\|\Lambda \leq R.$$

Therefore, $F_1u + F_2v \in B_R$. In view of condition (15), it follows that F_2 is a contraction mapping.

Next, we will show that F_1 is compact and continuous. The continuity of f together with the assumption (H_6) implies that the operator F_1 is continuous and uniformly bounded on B_R. We define $\sup_{(t,u)\in[0,1]\times B_R} |f(t, u)| = f_{max} < \infty$. Then, for $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$ and $u \in B_R$, we have

$$\begin{aligned}
|F_1u(t_2) - F_1u(t_1)| &= \left| -\int_0^{t_2} (t_2 - qs)f(s, u(s))d_qs + \int_0^{t_1} (t_1 - qs)f(s, u(s))d_qs \right| \\
&= \left| \int_0^{t_2} (t_2 - qs)f(s, u(s))d_qs - \int_0^{t_1} (t_1 - qs)f(s, u(s))d_qs \right| \\
&= \left| \int_0^{t_1} (t_2 - t_1)f(s, u(s))d_qs + \int_{t_1}^{t_2} (t_2 - qs)f(s, u(s))d_qs \right| \\
&\leq f_{max} \left(\int_0^{t_1} (t_2 - t_1)d_qs + \int_{t_1}^{t_2} |t_2 - qs|\, d_qs \right).
\end{aligned}$$

Actually, as $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to be zero. So F_1 is relatively compact on B_R. Hence, by the Arzela-Ascoli Theorem, F_1 is compact on B_R. Thus all the assumption of Theorem 3.5 are satisfied and the conclusion of Theorem 3.5 implies that problem (1) has at least one solution on $[0, 1]$. This completes the proof. □

Remark 3.4. *In Theorem 3.6, if $q \to 1$, then*

$$\begin{aligned}
&\frac{1}{q} + \frac{(1 + q)(1 - q^3)}{q(1 + q + q^2)} + \frac{q}{(1 + q)(1 + q + q^2)} + \frac{q^3(1 + q)^2}{(1 + q + q^2)^2} \\
&+ \frac{q^4}{(1 + q + q^2)(1 + q + q^2 + q^3)} = \frac{61}{36}.
\end{aligned}$$

Problem (1) *reduces to Problem* (2) *and Problem* (2) *has at least one solution.*

4. **Examples**

Example 4.1. *Consider the following boundary value problem:*

$$\begin{cases} D^2_{\frac{1}{2}} u(t) + \dfrac{2}{5}\dfrac{e^{-\sin^2 t}}{1 + e^{\cos^2 t}} u(t) = 0, \quad 0 < t < 1, \\ u(0) = \displaystyle\int_0^1 u(t)d_qt, \quad u(1) = \int_0^1 tu(t)d_qt. \end{cases} \tag{16}$$

Here, $f(t,u(t)) = \frac{2}{5}\frac{e^{-\sin^2 t}}{1+e^{\cos^2 t}}, q = \frac{1}{2}$. *We find that*

$$\begin{aligned}\Lambda =& \frac{1}{q} + \frac{q}{1+q+q^2} + \frac{(1+q)(1-q^3)}{q(1+q+q^2)} + \frac{q^3(1+q)^2}{(1+q+q^2)^2} \\ &+ \frac{q^4}{(1+q+q^2)(1+q+q^2+q^3)} \\ \approx& 3.8965\end{aligned}$$

Since, $|f(t,u) - f(t,v)| \leq \frac{1}{5}|u-v|$, *then* (H_1) *is satisfied with* $L = \frac{1}{5}$. *We find that* $L\Lambda \approx 0.7793 < 1$. *Hence, by Theorem 3.1, problem* (16) *has a unique solution on* $[0,1]$.

Example 4.2. *Consider the following boundary value problem:*

$$\begin{cases} D^2_{\frac{1}{2}} u(t) + \dfrac{(t+1)\,|u|}{2+|u|} + \dfrac{1}{2} = 0, \quad 0 < t < 1 \\ u(0) = \displaystyle\int_0^1 u(t)d_qt, \quad u(1) = \displaystyle\int_0^1 tu(t)d_qt. \end{cases} \tag{17}$$

Here, $f(t,u(t)) = \frac{(t+1)\,|u|}{2+|u|} + \frac{1}{2}, q = \frac{1}{2}$. *Choosing* $h(t) = t+1$, *we find that*

$$\begin{aligned}G =& \frac{(1+2q)(2+q+q^2) + q^5 + 2q^2 + 2q + 1}{q(1+q)(1+q+q^2)} \\ &+ \frac{q^5 + q^4 - 2q^3 - 4q^2 - 2q - 1}{(1+q+q^2)^2} + \frac{q^5 + 3q^4 + 2q^3 + q^2}{(1+q+q^2)(1+q+q^2+q^3)} \\ &+ \frac{q^4}{(1+q+q^2)(1+q+q^2+q^3+q^4)} + \frac{q^3 + 2q^2 + 2q + 1}{q(1+q+q^2)} \\ \approx& 7.9158\end{aligned}$$

Here, $|f(t,u) - f(t,v)| \leq \frac{(t+1)\,|u-v|}{7.9158 + |u-v|}$. *Therefore, by Theorem 3.4, problem* (17) *has a unique solution on* $[0,1]$.

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