EFFICIENCY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING WITH GENERALIZED $(\mathcal{F}, b, \phi, \rho, \theta)$-TYPE I UNIVEX n-SET FUNCTIONS

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Stancu-Minasian and Stancu [Duality for multiple objective fractional programming with generalized type-I univexity. In: A. Migdalas et al. (eds.), Optimization Theory, Decision Making and Operations Research Applications, Springer Proceedings in Mathematics & Statistics ***31****, DOI 10.1007/978-1-4614-5134-1_14, Springer Science + Business Media New York 2013, pp. 199-209.] introduced a new class of $(\mathcal{F}, b, \phi, \rho, \theta)$-type I univex n-set functions, according to a partition, where $\mathcal{F}$ is a convex function in the third argument. In this paper we discuss semiparametric sufficient efficiency conditions for a multiobjective fractional programming problem involving aforesaid class of n-set functions.*

Keywords: Multiobjective programming, n-set functions, optimality conditions, generalized convexity, fractional programming.

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1. Introduction

Consider the multiobjective fractional subset programming problem

$$\text{(P)} \qquad \min \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, ..., \frac{F_p(S)}{G_p(S)} \right)$$

$$\text{subject to } H_j(S) \leqq 0, \ j \in \underline{q}, \ S \in A^n,$$

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where A^n is the n-fold product of the σ-algebra A of subsets of a given set X, $F_i, G_i, i \in \underline{p} = \{1, 2, ..., p\}$, and H_j, $j \in \underline{q} = \{1, 2, ..., q\}$, are real-valued functions defined on A^n, and $G_i(S) > 0$, for all $i \in \underline{p}$ and $S \in A^n$ such that $H_j(S) \leqq 0$, $j \in \underline{q}$. Let $\mathbf{F} = \{S \in A^n : H_j(S) \leqq 0, \ j \in \underline{q}\}$ be the set of all feasible solutions to (P). We further assume that $\mathbf{F}$ is nonempty.

For any vectors $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m) \in \mathbf{R}^m$, we denote $x \leqq y$ iff $x_i \leqq y_i$ for each $i \in M = \{1, 2, ..., m\}$; $x \leq y$ iff $x_i \leqq y_i$ for each $i \in M$ and $x \neq y$; $x < y$ iff $x_i < y_i$ for each $i \in M$. We write that $x \in \mathbf{R}^m_+$ iff $x \geqq 0$.

In Problem (P) minimality is taken in terms of *efficient solutions* as defined below.

A feasible solution $S^0 \in \mathbf{F}$ is said to be an efficient solution to (P), if there is no other $S \in \mathbf{F}$ such that

$$\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, ..., \frac{F_p(S)}{G_p(S)}\right) \leq \left(\frac{F_1(S^0)}{G_1(S^0)}, \frac{F_2(S^0)}{G_2(S^0)}, ..., \frac{F_p(S^0)}{G_p(S^0)}\right).$$

The analysis of optimization problems with set functions has been the subject of much interest in the recent past due to many applications in electrical insulator design, optimal distribution of crops subject to rainfall in the given region, shape optimization, fluid flow, optimal plasma confinement and statistics.

The first general theory for optimizing set functions was developed by Morris [6] who defined the notions of local convexity, global convexity and first order differentiability for set functions. Also, he established optimality conditions and Lagrangian duality relations for a general nonlinear programming problem involving set functions. Further, Morris discussed some algorithms for numerical solution of these problems. The difficulty of optimization problems on a measure space, as pointed out by Morris, lies in the poorly structured feasible domain which is not convex, not open and, actually nowhere dense. The optimality and duality results of Morris were generalized by Corley [1] for problems with n-set functions.

Preda and Stancu-Minasian [7] defined classes of n-set functions called d-type-I, d-quasi type-I, d-pseudo type-I, d-quasi-pseudo type-I, d-pseudo-quasi type-I and established some optimality and Wolfe duality results for a multiobjective programming problem involving such functions.

Preda *et al.* [9] introduced the classes of n-set functions called (ρ, ρ')-V-univex type-I, (ρ, ρ')-quasi V-univex type I, (ρ, ρ')-pseudo V-univex type-I, (ρ, ρ')-quasipseudo V-univex type-I and (ρ, ρ')-pseudoquasi V-univex type-I and studied optimality conditions and generalized Mond-Weir duality for multiobjective programming problem.

Mishra [4] defined some types of generalized convexity $(\mathcal{F}, \rho, \sigma, \theta)$-$V$ type-I, $(\mathcal{F}, \rho, \sigma, \theta)$-$V$-pseudo-quasi type-I, $(\mathcal{F}, \rho, \sigma, \theta)$-$V$-quasi-pseudo type-I) and established optimality and duality results for a multiobjective programming problem involving such functions.

Mishra *et al.* [5] defined new classes of n-set functions, called (ρ, ρ', d)-strong pseudo-quasi-type-I, (ρ, ρ', d)-weak strictly pseudo-quasi-type-I, (ρ, ρ', d)-weak strictly pseudo-type-I, (ρ, ρ', d)-weak quasi-strictly-pseudo-type-I functions and for a multiobjective programming problem established optimality and Mond-Weir duality results.

Preda, Stancu-Minasian and Koller [8] presented some optimality and duality results for a multiobjective programming problem involving generalized d-type-I vector-valued n-set functions. Stancu [10] extended these results to the case of fractional programming.

Jayswal and Stancu-Minasian [2,3] introduced new classes of generalized convex n-set functions called d-weak strictly pseudo-quasi type I univex, d-strong pseudo-quasi type-I univex and d-weak strictly pseudo type-I univex functions. Sufficient optimality conditions and duality results were obtained for a multiobjective subset programming problems involving aforesaid functions.

A good account of optimality conditions and duality for programming problems involving set and n-set functions can be found in the paper of Stancu-Minasian and Preda [12] and the references therein.

Zalmai [18] introduced a new class of generalized convex n-set functions, called $(\mathcal{F}, \alpha, \rho, \theta) - V$-convex functions, and presented numerous sets of parametric and semiparametric sufficient efficiency conditions for Problem (P). The function $\mathcal{F}$ was assumed to be a sublinear function in the third argument. Stancu and Stancu-Minasian [15] considered some types of generalized convexity and discussed new global semiparametric sufficient efficiency conditions for a multiobjective fractional programming problem involving n-set functions.

Stancu-Minasian and Paraschiv [11] presented global semiparametric sufficient efficiency conditions for Problem (P) using the class of $(\mathcal{F}, b, \phi, \rho, \theta)$-univex n-set functions, defined in Zalmai [17], assuming that $\mathcal{F}$ is a convex function in the third argument. The class of convex functions is more general than the class of sublinear functions.

Stancu-Minasian and Stancu [14] presented new global semiparametric sufficient efficiency conditions for Problem (P) using the same class of functions defined by Zalmai [17], using a partition of $\underline{q}$ and assuming that $\mathcal{F}$ is a convex function in the third argument.

Stancu-Minasian and Stancu [13] introduced a new class of $(\mathcal{F}, b, \phi, \rho, \theta)$-type I univex n-set functions, according to a partition $\{I, J\}$, where $\mathcal{F}$ is a convex function in the third argument. For problem (P) a general dual model is presented and duality results are obtained using aforesaid class of functions.

In this paper we present efficiency conditions for Problem (P), using the same class of $(\mathcal{F}, b, \phi, \rho, \theta)$-type-I-univex n-set functions introduced in [13], and using a partition of $\underline{q}$. In our approach, we also suppose that $\mathcal{F}$ is a convex function in the third argument.

2. **Definitions and Preliminaries**

Let (X, A, μ) be a finite atomless measure space with $L_1(X, A, \mu)$ separable, and let d be the pseudometric on A^n defined by

$$d(R, S) = \left[\sum_{k=1}^{n} \mu^2 (R_k \Delta S_k)\right]^{1/2},$$

where $R = (R_1, \cdots, R_n)$ and $S = (S_1, \cdots, S_n) \in A^n$ and Δ denotes the symmetric difference.

Thus, (A^n, d) is a pseudometric space. For $h \in L_1(X, A, \mu)$ and $T \in A$, the integral $\int_T h \mathrm{d}\mu$ is denoted by $\langle h, \chi_T \rangle$, where $\chi_T \in L_\infty(X, A, \mu)$ is the indicator (characteristic) function of T.

Definition 2.1. *(Morris[6]) A function $F : A \to \mathbf{R}$ is said to be differentiable at $S^* \in A$ if there exist $DF(S^*) \in L_1(X, A, \mu)$, called the derivative of F at S^*, and $V_F : A \times A \to \mathbf{R}$ such that*

$$F(S) = F(S^*) + \langle DF(S^*), \chi_S - \chi_{S^*} \rangle + V_F(S, S^*),$$

for each $S \in A$, where $V_F(S, S^*)$ is $o(d(S, S^*))$, that is,

$$\lim_{d(S,S^*) \to 0} \frac{V_F(S, S^*)}{d(S, S^*)} = 0.$$

Definition 2.2. *(Corley[1]) A function $G : A^n \to \mathbf{R}$ is said to have a partial derivative at $S^* = (S_1^*, ..., S_n^*) \in A^n$ with respect to its i-th argument, if the function $F(S_i) = G(S_1^*, \cdots, S_{i-1}^*, S_i, S_{i+1}^*, \cdots, S_n^*)$ has derivative $DF(S_i^*), i \in \underline{n} = \{1, 2, \ldots, n\}$.*

We define $D_i G(S^*) := DF(S_i^*)$ and write $DF(S^*) = (D_1 F(S^*), \cdots, D_n F(S^*))$.

Definition 2.3. *(Corley[1]) A function $G : A^n \to \mathbf{R}$ is said to be differentiable at S^* if there exist $DF(S^*)$ and $W_G : A^n \times A^n \to \mathbf{R}$ such that*

$$G(S) = G(S^*) + \sum_{i=1}^{n} \left\langle D_i G(S^*), \chi_{S_i} - \chi_{S_i^*} \right\rangle + W_G(S, S^*),$$

where $W_G(S, S^*)$ is $o(d(S, S^*))$ for all $S \in A^n$.

In [13], Stancu-Minasian and Stancu introduced a new class of $(\mathcal{F}, b, \phi, \rho, \theta)$-type I univex n-set functions according to a partition $\{I, J\}$. We recall here these definitions in the particular case where the set I has a single element.

In what follows we consider $\mathcal{F} : A^n \times A^n \times \mathbb{R} \to \mathbb{R}$ and $F : A^n \to \mathbb{R}$ and $G : A^n \to \mathbb{R}^q$ two differentiable functions. Let a function $b : A^n \times A^n \to \mathbb{R}_+$, a function $\theta : A^n \times A^n \to A^n \times A^n$ such that $S \neq S^* \Longrightarrow \theta(S, S^*) \neq (0, 0)$, the functions $\phi_i : \mathbb{R} \to \mathbb{R}, i \in \{0\} \cup J$ and let $\phi = (\phi_0, \phi_j), j \in J$, where J is an index set. Let ρ_0 and ρ_j be real numbers and let $\rho = (\rho_0, \rho_j), \ j \in J$,

Definition 2.4. *(Stancu-Minasian and Stancu [13]) For each $j \in J$ the pair of functions (F, G_j) is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$-pseudo quasi univex type-I at $S^* \in A^n$ if for all $S \in A^n$ the implications*

$$\mathcal{F}(S, S^*; b(S, S^*)DF(S^*)) \geqq -\rho_0 d^2(\theta(S, S^*)) \Longrightarrow \phi_0 \left(F(S) - F(S^*)\right) \geqq 0, \tag{2.1}$$

and

$$\phi_j(-G_j(S^*)) \leqq 0 \Rightarrow \mathcal{F}(S, S^*; b(S, S^*)DG_j(S^*)) \leqq -\rho_j d^2(\theta(S, S^*)), \ j \in J \tag{2.2}$$

both hold.

If the second (implied) inequality in (2.1) is strict $(S \neq S^*)$, then we say that (F, G_j) is $(\mathcal{F}, b, \phi, \rho, \theta)$-strictly pseudo quasi univex type-I at $S^* \in A^n$.

Definition 2.5. *(Stancu-Minasian and Stancu [13]) For each $j \in J$ the pair of functions (F, G_j) is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$-quasi pseudo univex type-I at $S^* \in A^n$ if for all $S \in A^n$ the implications*

$$\phi_0 \left(F(S) - F(S^*)\right) \leqq 0 \Rightarrow \mathcal{F}(S, S^*; b(S, S^*)DF(S^*)) \leqq -\rho_0 d^2(\theta(S, S^*)), \tag{2.3}$$

and

$$\mathcal{F}(S, S^*; b(S, S^*)DG_j(S^*)) \geqq -\rho_j d^2(\theta(S, S^*)) \Longrightarrow \phi_j \left(-G_j(S^*)\right) \geqq 0, j \in J \tag{2.4}$$

both hold.

If the first (implied) inequality in (2.3) is strict $(S \neq S^*)$, and the second (implied) inequality in (2.4) is strict $(S \neq S^*)$, then we say that (F, G_j) is $(\mathcal{F}, b, \phi, \rho, \theta)$-prestrictly quasi strictly pseudo univex type-I at $S^* \in A^n$.

Definition 2.6. *(Stancu-Minasian and Stancu [13]) For each $j \in J$ the pair of functions (F, G_j) is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$-quasi quasi univex type-I at $S^* \in A^n$ if for all $S \in A^n$ the implications*

$$\phi_0 \left(F(S) - F(S^*)\right) \leqq 0 \Rightarrow \mathcal{F}(S, S^*; b(S, S^*)DF(S^*)) \leqq -\rho_0 d^2(\theta(S, S^*)), \tag{2.5}$$

and

$$\phi_j(-G_j(S^*)) \leqq 0 \Rightarrow \mathcal{F}(S, S^*; b(S, S^*)DG_j(S^*)) \leqq -\rho_j d^2(\theta(S, S^*)), \ j \in J \tag{2.6}$$

both hold.

If the first (implied) inequality in (2.5) is strict $(S \neq S^*)$, then we say that (F, G_j) is $(\mathcal{F}, b, \phi, \rho, \theta)$-prestrictly quasi quasi univex type-I at $S^* \in A^n$.

Definition 2.7. *For each* $j \in J$ *the pair of functions* (F, G_j) *is said to be* $(\mathcal{F}, b, \phi, \rho, \theta)$*-pseudo pseudo univex type-I at* $S^* \in A^n$ *if for all* $S \in A^n$ *the implications*

$$\mathcal{F}(S, S^*; b(S, S^*)DF(S^*)) \geqq -\rho_0 d^2(\theta(S, S^*)) \Longrightarrow \phi_0 \left(F(S) - F(S^*)\right) \geqq 0, \tag{2.7}$$

and

$$\mathcal{F}(S, S^*; b(S, S^*)DG_j(S^*)) \geqq -\rho_j d^2(\theta(S, S^*)) \Longrightarrow \phi_j \left(-G_j(S^*)\right) \geqq 0, j \in J \tag{2.8}$$

both hold.

If the second (implied) inequality in (2.7) and (2.8) is strict $(S \neq S^*)$, then we say that (F, G_j) is $(\mathcal{F}, b, \phi, \rho, \theta)$-strictly pseudo strictly pseudo univex type-I at $S^* \in A^n$.

3. **Generalized sufficient efficiency criteria**

For Problem (P), Zalmai ([18], Theorem 2.1) gave necessary conditions of efficiency. In [17], Zalmai presented for Problem (P) numerous sets of global semiparametric sufficient efficiency conditions under generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions, assuming that $\mathcal{F}$ is a sublinear function.

In this section we formulate and discuss several families of generalized sufficiency results for (P) with the help of a partitioning scheme on the index set $\underline{q}$ and assuming that $\mathcal{F}$ is a convex function in the third argument.

Let $\{J_0, J_1, ..., J_m\}$ be a partition of the index set $\underline{q}$; thus $J_r \subset \underline{q}$ for each $r \in \{0, 1, 2, ..., m\}$, $J_r \cap J_s = \emptyset$ for each $r \neq s$, and $\bigcup_{r=0}^{m} J_r = \underline{q}$. In addition, we shall make use of the functions $\Omega(\cdot, S, u, v) : A^n \to \mathbf{R}$ and $\Lambda_t(\cdot, v)$: $A^n \to \mathbf{R}$ defined for fixed S, u, v by

$$\Omega(T, S, u, v) = \sum_{i=1}^{p} u_i \left[G_i(S)F_i(T) - F_i(S)G_i(T)\right] + \sum_{j \in J_0} v_j H_j(T),$$

$$\Lambda_t(T, v) = \sum_{j \in J_t} v_j H_j(T), \ t \in \underline{m} \cup \{0\}.$$

Let $m = |\underline{m}|$, $m_1 = |\underline{m}_1|$, $m_2 = |\underline{m}_2|$, where $\{\underline{m}_1, \underline{m}_2\}$ is a partition of $\underline{m} = \{1, 2 \ldots, m\}$.

Let $S^* \in \mathbf{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U = \left\{u \in \mathbf{R}^p : u > 0, \sum\limits_{i=1}^{p} u_i = 1\right\}$ and

$v^* \in \mathbf{R}^q_+$ such that

$$\mathcal{F}\left(S, S^*; b(S, S^*)\left\{\sum_{i=1}^{p} u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \right.\right.$$
$$\left.\left. + \sum_{j=1}^{q} v_j^* DH_j(S^*)\right\}\right) \geqq 0, S \in \mathbf{F}, \tag{3.1}$$

$$v_j^* H_j(S^*) = 0, \; j \in \underline{q}, \tag{3.2}$$

where $\mathcal{F}(S, S^*; \cdot) : L_1^n(X, A, \mu) \to \mathbf{R}$ is a convex function.

Theorem 3.1. *Let $S^* \in \mathbf{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and there exist $u^* \in U$ and $v^* \in \mathbf{R}^q_+$ such that (3.1) and (3.2) hold. Furthermore, we assume that any one of the following sets of hypotheses is satisfied:*

(a) (i) For each $t = 0, 1, \ldots, m$, $(2\Omega(\cdot, S^*, u^*, v^*);\; 2m\Lambda_t(\cdot, v^*))$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-strictly pseudo quasi univex type-I at the point S^*, ϕ_0 is increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;

(ii) $\rho_0 + \frac{1}{m}\sum_{t=1}^{m} \rho_t \geqq 0;$

(b) (i) For each $t = 0, 1, \ldots, m$, $(2\Omega(\cdot, S^*, u^*, v^*);\; 2m\Lambda_t(\cdot, v^*))$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi strictly pseudo univex type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;

(ii)$\rho_0 + \frac{1}{m}\sum_{t=1}^{m} \rho_t \geqq 0;$

(c) (i) For each $t = 0, 1, \ldots, m$, $(2\Omega(\cdot, S^*, u^*, v^*); 2m\Lambda_t(\cdot, v^*))$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi quasi univex-type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;

(ii) $\rho_0 + \frac{1}{m}\sum_{t=1}^{m} \rho_t > 0;$

(d) (i) For each $t \in \{0\} \cup \underline{m}_1$, $(3\Omega(\cdot, S^*, u^*, v^*); 3m_1\Lambda_t(\cdot, v^*))$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi strictly pseudo univex-type-I at the point S^*, and for each $t \in \{0\} \cup \underline{m}_2$, $(3\Omega(\cdot, S^*, u^*, v^*); 3m_2\Lambda_t(\cdot, v^*))$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$- prestrictly quasi quasi univex-type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$, for each $t = 0, 1, \ldots, m$, where $\{\underline{m}_1, \underline{m}_2\}$ is a partition of $\underline{m}$, $\underline{m}_1 \neq \emptyset$, $m_1 = |\underline{m}_1|$ and $\underline{m}_2 \neq \emptyset$, $m_2 = |\underline{m}_2|$;

(ii) $\rho_0 + \frac{1}{m_1}\sum_{t\in\underline{m}_1} \rho_t + \frac{1}{m_2}\sum_{t\in\underline{m}_2} \rho_t \geqq 0;$ *where* $m_1 + m_2 = m;$

Then S^* is an efficient solution to (P).

Proof. (a) Suppose to the contrary that S^* is not an efficient solution to (P). Then there exists $\bar{S} \in \mathbf{F}$ such that $G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S}) \leqq 0$, $i \in \underline{p}$, and $G_l(S^*)F_l(\bar{S}) - F_l(S^*)G_l(\bar{S}) < 0$ for some $l \in \underline{p}$.

Since $u^* > 0$, these inequalities yield

$$\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S})\right] < 0.$$

Because $v_j^* H_j(\bar{S}) \leqq 0$ for each $j \in \underline{q}$, $\bar{S}, S^* \in \mathbf{F}$, it follows from these inequalities and (3.2) that

$$2\Omega(\bar{S}, S^*, u^*, v^*) = 2\left[\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S})\right] + \right.$$
$$\left. + \sum_{j\in J_0} v_j^* H_j(\bar{S})\right] < 0 =$$

$$= 2\left[\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(S^*) - F_i(S^*)G_i(S^*)\right] + \sum_{j\in J_0} v_j^* H_j(S^*)\right] =$$

$$= 2\Omega(S^*, S^*, u^*, v^*).$$

From the properties of ϕ_0 (ϕ_0 is increasing and $\phi_0\,(0) = 0$), we see that

$$\phi_0\left(2\Omega(\bar{S}, S^*, u^*, v^*) - 2\Omega(S^*, S^*, u^*, v^*)\right) \leqq 0. \tag{3.3}$$

From (3.2), it follows that $2m\Lambda_t(S^*, v^*) = 0$, for each $t = 1, 2, \ldots, m$. It follows that $-2m\Lambda_t(S^*, v^*) = 0$, $t = 1, 2, \ldots, m$, and from the properties of ϕ_t, $t = 1, \ldots, m$, ($\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$) we have $\phi_t\left(-2m\Lambda_t(S^*, v^*)\right) = 0, t = 1, 2, \ldots, m$ i.e. $\phi_t\left(-2m\Lambda_t(S^*, v^*)\right) \geqq 0, t = 1, 2, \ldots, m$, and

$$\phi_t\left(-2m\Lambda_t(S^*, v^*)\right) \leqq 0, t = 1, 2, \ldots, m. \tag{3.4}$$

From (3.3) and (3.4) and assumption (i), we deduce that

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2\left\{\sum_{i=1}^{p} u_i^* \left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \right.\right.$$
$$\left.\left. + \sum_{j\in J_0} v_j^* DH_j(S^*)\right\}\right) < -\rho_0 d^2(\theta(\bar{S}, S^*)), \tag{3.5}$$

and

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2m\sum_{j\in J_t} v_j^* DH_j\,(S^*)\right) \leqq -\rho_t d^2(\theta(\bar{S}, S^*)) \text{ for each } t = 1, 2, \ldots, m. \tag{3.6}$$

The convexity of $\mathcal{F}\left(\bar{S}, S^*; \cdot\right)$ and (3.6) imply that

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2\sum_{t=1}^{m}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) =$$

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)\sum_{t=1}^{m}\frac{1}{m}2m\sum_{j\in J_t}v_j^* DH_j(S^*)\right) \leqq$$
$$\leqq \frac{1}{m}\sum_{t=1}^{m}\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2m\sum_{j\in J_t}v_j^* DH_j(S^*)\right) \leqq$$
$$\leqq -\frac{1}{m}\sum_{t=1}^{m}\rho_t\ d^2(\theta(\bar{S}, S^*)),$$

i.e.,

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2\sum_{t=1}^{m}\sum_{j\in J_t}v_j^* DH_j(S^*)\right) \leqq -\frac{1}{m}\sum_{t=1}^{m}\rho_t d^2(\theta(\bar{S}, S^*)). \tag{3.7}$$

Now, using (3.1), the convexity of $\mathcal{F}\left(\bar{S}, S^*; \cdot\right)$, (3.5) and (3.7), we obtain

$$0 \leqq \mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)\left\{\sum_{i=1}^{p}2u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \sum_{j\in J_0}2v_j^* DH_j(S^*)\right\}\right) + \mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2\sum_{t=1}^{m}\sum_{j\in J_t}v_j^* DH_j(S^*)\right) <$$
$$< -\rho_0 d^2(\theta(\bar{S}, S^*)) - \frac{1}{m}\sum_{t=1}^{m}\rho_t d^2(\theta(\bar{S}, S^*)),$$

which contradicts (a) (ii).

Therefore, we conclude that S^* is an efficient solution to (P).

(b) The proof is similar to that of part (a) in which inequality (3.3) is replaced by

$$\phi_0\left(2\Omega(\bar{S}, S^*, u^*, v^*) - 2\Omega(S^*, S^*, u^*, v^*)\right) < 0, \tag{3.8}$$

inequality (3.5) is replaced by inequality

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2\left\{\sum_{i=1}^{p}u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \sum_{j\in J_0}v_j^* DH_j(S^*)\right\}\right) \leqq -\rho_0 d^2(\theta(\bar{S}, S^*)), \tag{3.9}$$

and inequality (3.6) is replaced by inequality

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)2m\sum_{j\in J_t}v_j^* DH_j(S^*)\right) < -\rho_t d^2(\theta(\bar{S}, S^*)),\ \text{for } t = 1, \dots, m. \tag{3.10}$$

(c) The proof is similar to that of part (a).

(d) Proceeding as in the cases (a)-(c) we have

$$\phi_0\left(3\Omega(\bar{S}, S^*, u^*, v^*) - 3\Omega(S^*, S^*, u^*, v^*)\right) < 0. \tag{3.11}$$

From (3.2), it follows that $3m_1\Lambda_t(S^*, v^*) = 0$, for all $t \in \underline{m}_1$ and $3m_2\Lambda_t(S^*, v^*) = 0$, for all $t \in \underline{m}_2$. It follows that $-3m_1\Lambda_t(S^*, v^*) = 0$, $t = 1, 2, \ldots, m$, and for all $t \in \underline{m}_1$ and $-3m_2\Lambda_t(S^*, v^*) = 0$, for all $t \in \underline{m}_2$. From the properties of ϕ_t, $t \in \underline{m}_1 \cup \underline{m}_2$, we have $\phi_t\left(-3m_1\Lambda_t(S^*, v^*)\right) = 0, t \in \underline{m}_1$ and $\phi_t\left(-3m_2\Lambda_t(S^*, v^*)\right) = 0, t \in \underline{m}_2$ i.e. $\phi_t\left(-3m_1\Lambda_t(S^*, v^*)\right) \geqq 0, t \in \underline{m}_1$, and

$$\phi_t\left(-3m_1\Lambda_t(S^*, v^*)\right) \leqq 0, t \in m_1, \tag{3.12}$$

and $\phi_t\left(-3m_2\Lambda_t\left(S^*, v^*\right)\right) \geqq 0, t \in \underline{m}_2$ and

$$\phi_t\left(-3m_2\Lambda_t\left(S^*, v^*\right)\right) \leqq 0, t \in \underline{m}_2. \tag{3.13}$$

From (3.11), (3.12) and (3.13) and assumption (d) (i), we deduce that

$$\begin{aligned}\mathcal{F}\Bigg(&\bar{S}, S^*; b(\bar{S}, S^*)3\left\{\sum_{i=1}^{p} u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] +\right.\\ &\left.+\sum_{j\in J_0} v_j^* DH_j(S^*)\right\}\Bigg) \leqq -\rho_0 d^2(\theta(\bar{S}, S^*)),\end{aligned} \tag{3.14}$$

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)3m_1\sum_{j\in J_t} v_j^* DH_j(S^*)\right) < -\rho_t d^2(\theta(\bar{S}, S^*)), \tag{3.15}$$

and

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)3m_2\sum_{j\in J_t} v_j^* DH_j(S^*)\right) \leqq -\rho_t d^2(\theta(\bar{S}, S^*)). \tag{3.16}$$

Now proceeding as in the proof of parts a)-b) inequalities (3.15) and (3.16) imply that,

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)3\sum_{t\in\underline{m}_1}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) < -\frac{1}{m_1}\sum_{t\in\underline{m}_1}\rho_t d^2(\theta(\bar{S}, S^*)) \tag{3.17}$$

and

$$\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)3\sum_{t\in\underline{m}_2}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) \leqq -\frac{1}{m_2}\sum_{t\in\underline{m}_2}\rho_t d^2(\theta(\bar{S}, S^*)). \tag{3.18}$$

Using (3.1), the convexity of $\mathcal{F}\left(\bar{S}, S^*; \cdot\right)$, (3.14), (3.17) and (3.18) we obtain

$$\begin{aligned}0 \leqq\ & \mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)\left\{\textstyle\sum_{i=1}^{p} 3u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] +\right.\right.\\ &\left.\left.+\sum_{j\in J_0} 3v_j^* DH_j(S^*)\right\}\right) + \mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)3\sum_{t\in\underline{m}_1}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) +\\ &+\mathcal{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)3\sum_{t\in\underline{m}_2}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) <\end{aligned}$$

$$< -\rho_0 d^2(\theta(\bar{S}, S^*)) - \frac{1}{m_1} \sum_{t \in \underline{m}_1} \rho_t d^2(\theta(\bar{S}, S^*)) - \frac{1}{m_2} \sum_{t \in \underline{m}_2} \rho_t d^2(\theta(\bar{S}, S^*))$$

which contradicts the inequality (d) (ii). □

Note that Theorem 3.1. contains a number of special cases that can be easily identified by appropriate choices of the partitioning sets $J_0, J_1, \ldots, J_m$.

Corollary 3.1. *Let $S^* \in \mathbf{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and there exist $u^* \in U$ and $v^* \in \mathbf{R}^q_+$ such that (3.1) and (3.2) hold. Furthermore, we assume that any one of the following sets of hypotheses is satisfied:*

(a) (i) For each $t = 0, 1, \ldots, m$, $\left(2 \sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right];\ 2mv_t^* H_t(\cdot)\right)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-strictly pseudo quasi univex type-I at the point S^*, ϕ_0 is increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;

(ii) $\rho_0 + \frac{1}{m} \sum_{t=1}^{m} \rho_t \geqq 0$;

(b) (i) For each $t = 0, 1, \ldots, m$, $\left(2 \sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right];\ 2mv_t^* H_t(\cdot)\right)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$- prestrictly quasi strictly pseudo univex type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;

(ii)$\rho_0 + \frac{1}{m} \sum_{t=1}^{m} \rho_t \geqq 0$;

(c) (i) For each $t = 0, 1, \ldots, m$, $\left(2 \sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right]; 2mv_t^* H_t(\cdot)\right)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi quasi univex-type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;

(ii) $\rho_0 + \frac{1}{m} \sum_{t=1}^{m} \rho_t > 0$;

(d) (i) For each $t \in \{0\} \cup \underline{m}_1$, $\left(3 \sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right]; 3m_1 v_t^* H_t(\cdot)\right)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi strictly pseudo univex-type-I at the point S^*, and for each $t \in \{0\} \cup \underline{m}_2$, $\left(3 \sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right]; 3m_2 v_t^* H_t(\cdot)\right)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi quasi univex-type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$, for each $t = 0, 1, \ldots, m$, where $\{\underline{m}_1, \underline{m}_2\}$ is a partition of $\underline{m}$, $\underline{m}_1 \neq \emptyset$, $m_1 = |\underline{m}_1|$ and $\underline{m}_2 \neq \emptyset$, $m_2 = |\underline{m}_2|$;

(ii) $\rho_0 + \frac{1}{m_1} \sum_{t \in \underline{m}_1} \rho_t + \frac{1}{m_2} \sum_{t \in \underline{m}_2} \rho_t \geqq 0$; where $m_1 + m_2 = m$;

Then S^* is an efficient solution to (P).

Proof. Let $m = q, J_0 = \emptyset$ and $J_t = \{t\}, t = 1, 2, \ldots, q$ in Theorem 3.1. □

Corollary 3.2. *Let $S^* \in \mathbf{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and there exist $u^* \in U$ and $v^* \in \mathbf{R}^q_+$ such that (3.1) and (3.2) hold. Furthermore, we assume that any one of the following sets of hypotheses is satisfied:*

(a)(i) For each $t = 0, 1, \ldots, m$, $\left(2\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right];\ 2m\Lambda_t(\cdot, v^*)\right)$ *is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-strictly pseudo quasi univex type-I at the point S^*, ϕ_0 is increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;*

$$\textit{(ii)}\ \rho_0 + \frac{1}{m}\sum_{t=1}^{m} \rho_t \geqq 0;$$

(b) (i) For each $t = 0, 1, \ldots, m$, $\left(2\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right];\ 2m\Lambda_t(\cdot, v)\right)$ *is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi strictly pseudo univex type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;*

$$\textit{(ii)}\rho_0 + \frac{1}{m}\sum_{t=1}^{m} \rho_t \geqq 0;$$

(c) (i) For each $t = 0, 1, \ldots, m$, $\left(2\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right]; 2m\Lambda_t(\cdot, v)\right)$ *is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi quasi univex-type-I at the point S^*, ϕ_0 is strictly increasing and $\phi_t(0) = 0$ for each $t = 0, 1, \ldots, m$;*

$$\textit{(ii)}\ \rho_0 + \frac{1}{m}\sum_{t=1}^{m} \rho_t > 0;$$

(d) (i) For each $t \in \{0\} \cup \underline{m}_1$, $\left(3\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right]; 3m_1\Lambda_t(\cdot, v)\right)$ *is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi strictly pseudo univex-type-I at the point S^*, and for each $t \in \{0\} \cup \underline{m}_2$,* $\left(3\sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(\cdot) - F_i(S^*)G_i(\cdot)\right]; 3m_2\Lambda_t(\cdot, v)\right)$ *is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-prestrictly quasi quasi univex-type-I at the point S^*, ,ϕ_0 is strictly increasing and $\phi_t(0) = 0$, for each $t = 0, 1, \ldots, m$, where $\{\underline{m}_1, \underline{m}_2\}$ is a partition of $\underline{m}$, $\underline{m}_1 \neq \emptyset$, $m_1 = |\underline{m}_1|$ and $\underline{m}_2 \neq \emptyset$, $m_2 = |\underline{m}_2|$;*

$$\textit{(ii)}\ \rho_0 + \frac{1}{m_1}\sum_{t\in\underline{m}_1} \rho_t + \frac{1}{m_2}\sum_{t\in\underline{m}_2} \rho_t \geqq 0;\ \textit{where}\ m_1 + m_2 = m;$$

Then S^* is an efficient solution to (P).

Proof. Let $J_0 = \emptyset$ in Theorem 3.1. □

4. **Conclusions**

In this paper we have obtained sufficient efficiency conditions for a multiobjective fractional programming problem involving $(\mathcal{F}, b, \phi, \rho, \theta)$-type I-univex n-set functions, where $\mathcal{F}$ is a convex function in the third argument.

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