RAKOTCH TYPE EXTENSION OF DARBO'S FIXED POINT THEOREM AND AN APPLICATION

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In this paper, we present a new extension of Darbo's fixed point theorem inspired by Rakotch's contraction. We also provide the alternative version of Leray-Schauder type of our new result. In order to demonstrate the applicability of our theoretical result, we present an existence theorem based on a functional equation. Finally, we provide an illustration of our existence theorem.

Keywords: fixed point, measure of noncompactness, functional equation.
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1. Introduction

The theory of integral equations has recently improved significantly as a result of various tools from both nonlinear functional analysis and topological fixed point theory. One of the most important of these tools is the measure of noncompactness (in short MNC), which was first defined by Kuratowski [10] and later used frequently by various authors. In this context, on the basis of MNC, Darbo [6] introduced the class of k-set contractive operators, which includes compact operators, and presented a fixed point theorem that generalized both the famous Schauder fixed point theorem and a version of Banach contraction theorem. Darbo's fixed point theorem was later generalized as theoretical in various ways and used to obtained existence theorems for many functional equations. For details, we refer to [1, 2, 3, 4, 7, 8, 9, 12] and the references therein.

In this paper, inspired by Rakotch type contraction stated in the following theorem, we present a new extension of Darbo's fixed point theorem. The alternative version of Leray-Schauder type of our new result is also provided. Finally, to indicate the applicability of our theoretical result, we present an existence theorem based on a functional equation.

Theorem 1.1. [11] *Let $(\mathcal{M}, \rho)$ be a complete metric space and $\Gamma : \mathcal{M} \to \mathcal{M}$ be a Rakotch type contraction, that is, there exists a function $L : (0, \infty) \to [0, 1)$ with*

$$\sup \{L(t) : 0 < a \leq t \leq b\} < 1$$

such that

$$\rho(\Gamma\xi, \Gamma\zeta) \leq L(\rho(\xi, \zeta))\rho(\xi, \zeta)$$

for each $\xi, \zeta \in \mathcal{M}$. Then Γ has a unique fixed point.

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2. **Preliminaries**

In this section, first we recall the basic notions and main characteristics of the MNC. For more informations about the MNC we refer to [4].

Definition 2.1. *Let $(\mathcal{M}, \rho)$ be a complete metric space and $\mathcal{B}(\mathcal{M})$ be the family of all bounded subsets of $\mathcal{M}$. A nonnegative real valued mapping μ defined on $\mathcal{B}(\mathcal{M})$ is called MNC on $\mathcal{M}$ if it satisfied the following axioms: for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{B}(\mathcal{M})$*

(μ_a) $\mu(\Omega) = 0$ *if and only if Ω is precompact set,*
(μ_b) $\mu(\Omega) = \mu(\overline{\Omega})$,
(μ_c) $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$.

Remark 2.1. *Let μ be a MNC of a complete metric space $\mathcal{M}$, then the following properties hold: for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{B}(\mathcal{M})$*

(μ_1) *If $\Omega_1 \subseteq \Omega_2$, then $\mu(\Omega_1) \leq \mu(\Omega_2)$,*
(μ_2) $\mu(\Omega_1 \cap \Omega_2) \leq \min\{\mu(\Omega_1), \mu(\Omega_2)\}$,
(μ_3) *If Ω is a finite set, then $\mu(\Omega) = 0$,*
(μ_4) *Let $\{\Omega_n\}$ is a decreasing sequence in $\mathcal{B}(\mathcal{M})$ which all terms are nonempty and closed. If $\mu(\Omega_n) \to 0$ as $n \to \infty$, then $\bigcap\limits_{n=1}^{\infty} \Omega_n$ is nonempty and compact.*

Besides, if $\mathcal{M}$ is a Banach space, then the function μ has additional properties, some of which are given below: for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{B}(\mathcal{M})$

(μ_5) $\mu(\lambda\Omega) = |\lambda| \mu(\Omega)$, *for any number λ,*
(μ_6) $\mu(\Omega_1 + \Omega_2) \leq \max\{\mu(\Omega_1), \mu(\Omega_2)\}$,
(μ_7) $\mu(\xi_0 + \Omega) = \mu(\Omega)$ *for any $\xi_0 \in \mathcal{M}$,*
(μ_8) $\mu(co\Omega) = \mu(\Omega)$, *where $co\Omega$ is the convex hull of Ω.*

The famous Schauder fixed point theorem is as follows: For brevity, in the rest of this paper $\mathcal{BC}(\mathcal{M})$ stands for the class of all nonempty, closed, convex and bounded subsets of Banach space $\mathcal{M}$.

Theorem 2.1. *Let $\mathcal{M}$ be a Banach space and $\Omega \in \mathcal{BC}(\mathcal{M})$. If $\Gamma : \Omega \to \Omega$ is continuous and compact mapping, then Γ has at least a fixed point in Ω.*

Darbo [6] presented the following definition and theorem:

Definition 2.2. *Let $\Omega \neq \emptyset$ be a subset of a Banach space $\mathcal{M}$ and $\Gamma : \Omega \to \Omega$ be a mapping. Then, Γ is called a k-set contraction if, for each $\Lambda \subseteq \Omega$ with bounded, $\Gamma\Lambda$ is bounded and there exists $k \in [0, 1)$ such that*

$$\mu(\Gamma\Lambda) \leq k\mu(\Lambda). \tag{1}$$

Theorem 2.2. *Let $\mathcal{M}$ be a Banach space and $\Omega \in \mathcal{BC}(\mathcal{M})$. Then each continuous k-set contraction $\Gamma : \Omega \to \Omega$ has at least one fixed point in Ω.*

3. **Main Results**

In this section, we first give the definition of the Rakotch type μ-set contraction.

Definition 3.1. *Let Ω be nonempty subset of a Banach space $\mathcal{M}$, μ be a MNC in $\mathcal{M}$, and let $\Gamma : \Omega \to \Omega$ be a mapping. If there exists a function $L : [0, \infty) \to [0, 1)$ satisfying*

$$\sup\{L(r) : 0 < p \leq r \leq q\} < 1$$

such that

$$\mu(\Gamma\Lambda) \leq L(\mu(\Lambda))\mu(\Lambda) \tag{2}$$

for any nonempty and bounded subset Λ of Ω. Then, Γ is said to be Rakotch type μ-set contraction with respect to L.

It is obvious that every k-set contraction is also Rakotch type μ-set contraction. Hence, the below theorem is a generalization of both Theorem 2.1 and Theorem 2.2.

Theorem 3.1. *Let $\mathcal{M}$ be a Banach space and $\Omega \in \mathcal{BC}(\mathcal{M})$ and let $\Gamma : \Omega \to \Omega$ be a continuous and Rakotch type μ-set contraction mapping with respect to L. Then, Γ has a fixed point in Ω.*

Proof. Define a sequence $\{\Lambda_n\}$ such that

$$\Lambda_0 = \Omega \text{ and } \Lambda_n = \overline{co}\Gamma\Lambda_{n-1} \tag{3}$$

for all $n \in \mathbb{N}$. First prove that

$$\Lambda_{n+1} \subseteq \Lambda_n \text{ and } \Gamma\Lambda_n \subseteq \Lambda_n \tag{4}$$

for all $n \in \mathbb{N}$.

If $n = 1$, then from (3) we get

$$\Lambda_1 = \overline{co}\Gamma\Lambda_0 = \overline{co}\Gamma\Omega \subseteq \Omega = \Lambda_0.$$

Next, for $n > 1$, we assume that

$$\Lambda_n \subseteq \Lambda_{n-1}.$$

Then, $\Gamma\Lambda_n \subseteq \Gamma\Lambda_{n-1}$ and so by (3) we get

$$\Lambda_{n+1} = \overline{co}\Gamma\Lambda_n \subseteq \overline{co}\Gamma\Lambda_{n-1} = \Lambda_n \tag{5}$$

hence the first part of (4) hold. By (5) we have

$$\Gamma\Lambda_n \subseteq \overline{co}\Gamma\Lambda_n = \Lambda_{n+1} \subseteq \Lambda_n$$

hence the second part of (4) also hold.

If there exists $n_0 \in \mathbb{N}$ such that $\mu(\Lambda_{n_0}) = 0$, then Λ_{n_0} is a compact subset of $\mathcal{M}$. Also since $\Gamma\Lambda_{n_0} \subseteq \Lambda_{n_0}$ and Γ is continuous, then by Theorem 2.1, Γ has a fixed point in Λ_{n_0}.

Now assume $\mu(\Lambda_n) > 0$ for all $n \in \mathbb{N}$. Then from (4), we have

$$\mu(\Lambda_{n+1}) \leq \mu(\Lambda_n),$$

that is, $\{\mu(\Lambda_n)\}$ is a nonincreasing sequence and bounded below. Hence there exist $\delta \geq 0$ such that

$$\lim_{n\to\infty} \mu(\Lambda_n) = \delta^+.$$

Assume that $\delta > 0$ and set

$$\lambda = \sup\{L(r) : 0 < \delta \leq r \leq \mu(\Lambda_0)\}.$$

Then, observing that

$$0 < \delta \leq \mu(\Lambda_n) \leq \mu(\Lambda_0)$$

for all $n \in \mathbb{N}$, we have

$$L(\mu(\Lambda_n)) \leq \lambda$$

for all $n \in \mathbb{N}$. Hence we have

$$\begin{aligned}
\mu(\Lambda_n) &= \mu(\overline{co}\Gamma\Lambda_{n-1}) \\
&= \mu(\Gamma\Lambda_{n-1}) \\
&\leq L(\mu(\Lambda_{n-1}))\mu(\Lambda_{n-1}) \\
&\leq \lambda\mu(\Lambda_{n-1}) \\
&= \lambda\mu(\overline{co}\Gamma\Lambda_{n-2}) \\
&\leq \lambda^2\mu(\Lambda_{n-2}) \\
&\ \ \vdots \\
&\leq \lambda^n\mu(\Lambda_0).
\end{aligned}$$

Taking as $n \to \infty$, we get $\lim_{n\to\infty} \mu(\Lambda_n) = 0$, which contradict to $\delta > 0$. Hence $\delta = 0$ and so $\mu(\Lambda_n) \to 0$ as $n \to \infty$. Thus, $\Lambda_\infty = \bigcap_{n=1}^{\infty} \Lambda_n$ is nonempty and compact subset of Ω. Also, since $\Lambda_\infty \subseteq \Lambda_n$ for all $n \in \mathbb{N}$, then we have $\Gamma\Lambda_\infty \subseteq \Lambda_\infty$ and so by Theorem 2.1, Γ has a fixed point in Λ_∞. □

Now, we are in position to establish an alternative version of Leray-Schauder type of our main theorem.

Theorem 3.2. *Let $\mathcal{M}$ be a Banach space and $\Omega \in \mathcal{BC}(\mathcal{M})$, U an open subset of Ω and $\xi_0 \in U$. Suppose $\Gamma : \overline{U} \to \Omega$ be a continuous and Rakotch type μ-set contraction mapping with a nondecreasing function L. Then, either*

(i) Γ has a fixed point in $\overline{U}$, or

(ii) there exists $\xi \in \partial U$ and $\lambda \in (0,1)$ such that $\xi = \lambda\Gamma\xi + (1-\lambda)\xi_0$.

Proof. Assume that (ii) does not hold and Γ has no fixed point in ∂U. Then

$$\xi \neq \lambda\Gamma\xi + (1-\lambda)\xi_0$$

for $\xi \in \partial U$ and $\lambda \in [0,1]$. Consider the set

$$K = \left\{\xi \in \overline{U} : \xi = \lambda\Gamma\xi + (1-\lambda)\xi_0 \text{ for some } \lambda \in [0,1]\right\}.$$

Since $\xi_0 \in K$, then K is nonempty. Also K is closed because of the continuity of Γ. Further we have $K \cap \partial U = \emptyset$. Thus there exists a continuous function $\lambda : \overline{U} \to [0,1]$ such that $\lambda(K) = 1$ and $\lambda(\partial U) = 0$. Now define a map $\Psi : \Omega \to \Omega$ as

$$\Psi\xi = \begin{cases} \lambda(\xi)\Gamma\xi + (1-\lambda(\xi))\xi_0 & , \quad \xi \in \overline{U} \\ \\ \xi_0 & , \quad \xi \in \Omega\backslash\overline{U} \end{cases}.$$

Then Ψ is continuous. In addition Ψ is a Rakotch type μ-set contraction. Indeed, let $\Lambda \subseteq \Omega$ be any set. Then we have

$$\Psi(\Lambda) \subseteq \overline{co}(\Gamma(\Lambda \cap \overline{U}) \cup \{\xi_0\})$$

and hence

$$\begin{aligned} \mu(\Psi(\Lambda)) &\leq \mu(\overline{co}(\Gamma(\Lambda \cap \overline{U}) \cup \{\xi_0\})) \\ &= \mu(\Gamma(\Lambda \cap \overline{U})) \\ &\leq L(\mu(\Lambda \cap \overline{U}))\mu(\Lambda \cap \overline{U}) \\ &\leq L(\mu(\Lambda)\mu(\Lambda). \end{aligned}$$

Consequently $\Psi : \Omega \to \Omega$ is continuous and Rakotch type μ-set contraction mapping. Therefore by Theorem 3.1, there exists $\eta \in \Omega$ such that $\eta = \Psi\eta$. Notice that $\eta \in \overline{U}$ since $\xi_0 \in U$. Hence

$$\eta = \lambda(\eta)\Gamma\eta + (1-\lambda(\eta))\xi_0$$

and so $\eta \in K$. Consequently $\lambda(\eta) = 1$ which implies $\eta = \Gamma\eta$. □

4. Application to a Functional Equation

Here and subsequently, we will study in the space $C[0,1]$ containing of all continuous real valued functions defined on the interval $[0,1]$. For the sake of simplicity, we set $I = [0,1]$ and $C(I) = C[0,1]$. It is well-known that the space $C(I)$ equipped with the standard norm

$$\|\xi\| = \max\{|\xi(t)| : t \in I\}$$

is a Banach space. Now, we remember the definition of a MNC in $C(I)$, presented and investigated in [5], that will be applied in the following. Let $\Upsilon \neq \emptyset$ be a bounded subset of $C(I)$. For $\varepsilon > 0$ and $\xi \in \Upsilon$, let $\omega(\xi, \varepsilon)$ be the modulus of continuity of ξ which is defined by

$$\omega(\xi, \varepsilon) := \sup\{|\xi(t) - \xi(s)| : t,\ s \in I,\ \ |t-s| \leq \varepsilon\}.$$

Further, let us put

$$\begin{aligned} \omega(\Upsilon, \varepsilon) &= \sup\{\omega(\xi, \varepsilon) : \xi \in \Upsilon\}, \\ \omega_0(\Upsilon) &= \lim_{\varepsilon \to 0} \omega(\Upsilon, \varepsilon). \end{aligned} \tag{6}$$

It is known that the function ω_0 is a MNC in the space $C(I)$ (cf. [?]). Now, we are interested with the following functional equation

$$\xi(t) = F\left(a(t), \xi(t), \Lambda\xi(t) \int_0^t v(t, \tau, \xi(\tau))d\tau\right),\ t \in I. \tag{7}$$

The function ξ is an unknown while a, v functions and the operator Λ appearing in this equation are known. Here we will examine this equation under the following assumptions:

(i) $a : I \to I$ and $v : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous.

(ii) There exists an increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that, the inequality
$$|v(t, \tau, \xi)| \leq g(|\xi|),$$
holds for all $t, \tau \in I$ and $\xi \in \mathbb{R}$.

(iii) The operator Λ maps continuously the space $C(I)$ into itself. Also there exists a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|\Lambda\xi\| \leq \phi(\|\xi\|)$ for any $\xi \in C(I)$.

(iv) The function $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is continuous and there exist positive constants k_1, k_2 and k_3 with $k_2 < 1$ such that
$$|F(\alpha_1, \alpha_2, \alpha_3) - F(\beta_1, \beta_2, \beta_3)| \leq k_1 |\alpha_1 - \beta_1| + k_2 |\alpha_2 - \beta_2| + k_3 |\alpha_3 - \beta_3|, \tag{8}$$
and $F_K = \sup\{|F(a(t), 0, 0)| : t \in I\}$.

(v) The inequalitiy
$$k_2 r + k_3 \phi(r) g(r) + F_K \leq r \tag{9}$$
has a positive solution r_0.

(vi) For any $\Upsilon \in C(I)$, we have $\omega_0(\Lambda\Upsilon) \leq L_1(\omega_0(\Upsilon))\omega_0(\Upsilon)$, where the function $L_1 : [0, \infty) \to [0, \frac{1-k_2}{k_3 g(r_0)})$ satisfies that
$$\sup\{L_1(r) : 0 < p \leq r \leq q\} < \frac{1-k_2}{k_3 g(r_0)}.$$

Now, we are ready to present the following theorem:

Theorem 4.1. *The equation (7) has at least one positive solution in $C(I)$ under the assumptions (i)-(vi).*

Proof. Define a mapping Γ on the space $C(I)$ having the form

$$\Gamma\xi(t) = F\left(a(t), \xi(t), \Lambda\xi(t) \int_0^t v(t, \tau, \xi(\tau))d\tau\right).$$

Based on the assumptions (i), (iii) and (iv) we infer that the function $\Gamma\xi$ is continuous.

Furthermore, keeping the assumptions (ii), (iii) and (iv) in mind we get, for all $t \in I$ and $\xi \in C(I)$,

$$\begin{aligned} |\Gamma\xi(t)| &\leq \left| F\left(a(t), \xi(t), \Lambda\xi(t) \int_0^t v(t, \tau, \xi(\tau))d\tau\right) - F(a(t), 0, 0)\right| + |F(a(t), 0, 0)| \\ &\leq k_2 |\xi(t)| + k_3 \left|\Lambda\xi(t) \int_0^t v(t, \tau, \xi(\tau))d\tau\right| + F_K \end{aligned}$$

$$\leq k_2\,|\xi(t)| + k_3\,|\Lambda\xi(t)| \int_0^t |v(t,\tau,\xi(\tau))|\,d\tau + F_K$$
$$\leq k_2\,\|\xi\| + k_3\phi(\|\xi\|)g(\|\xi\|) + F_K.$$

Then, we have

$$\|\Gamma\xi\| \leq k_2\,\|\xi\| + k_3\phi(\|\xi\|)g(\|\xi\|) + F_K. \tag{10}$$

By virtue of the assumption (v), we know that there exists $r_0 > 0$ such that

$$k_2 r_0 + k_3\phi(r_0)g(r_0) + F_K \leq r_0.$$

Hence for $\xi \in B_{r_0} = \{\xi \in C(I) : \|\xi\| \leq r_0\}$, we have

$$\begin{aligned}\|\Gamma\xi\| &\leq k_2\,\|\xi\| + k_3\phi(\|\xi\|)g(\|\xi\|) + F_K\\ &\leq k_2 r_0 + k_3\phi(r_0)g(r_0) + F_K\\ &\leq r_0.\end{aligned}$$

Therefore the operator Γ maps B_{r_0} into itself. Put

$$B_{r_0}^{+} = \{\xi \in B_{r_0} : \xi(t) \geq 0 \text{ for } t \in I\}.$$

Obviously, the set $B_{r_0}^{+}$ is nonempty, closed, bounded, and convex. By assumption (iv), we conclude that Γ transforms the set $B_{r_0}^{+}$ into itself.

Now we show that Γ is continuous on $B_{r_0}^{+}$. Let $\varepsilon > 0$, $\xi, \zeta \in B_{r_0}^{+}$ with $\|\xi - \zeta\| \leq \varepsilon$. Then, for a fixed $t \in I$, we have

$$\begin{aligned}|\Gamma\xi(t) - \Gamma\zeta(t)| &= \left|F\left(a(t), \xi(t), \Lambda\xi(t)\int_0^t v(t,\tau,\xi(\tau))d\tau\right)\right.\\
&\quad \left.-F\left(a(t), \zeta(t), \Lambda\zeta(t)\int_0^t v(t,\tau,\zeta(\tau))d\tau\right)\right|\\
&\leq k_2|\xi(t) - \zeta(t)|\\
&\quad + k_3\left|\Lambda\xi(t)\int_0^t v(t,\tau,\xi(\tau))d\tau - \Lambda\zeta(t)\int_0^t v(t,\tau,\zeta(\tau))d\tau\right|\\
&\leq k_2|\xi(t) - \zeta(t)| + k_3\left|\Lambda\xi(t)\int_0^t v(t,\tau,\xi(\tau))d\tau - \Lambda\zeta(t)\int_0^t v(t,\tau,\xi(\tau))d\tau\right|\\
&\quad + k_3\left|\Lambda\zeta(t)\int_0^t v(t,\tau,\xi(\tau))d\tau - \Lambda\zeta(t)\int_0^t v(t,\tau,\zeta(\tau))d\tau\right|\\
&\leq k_2|\xi(t) - \zeta(t)| + k_3\,|\Lambda\xi(t) - \Lambda\zeta(t)| \int_0^t |v(t,\tau,\xi(\tau))|\,d\tau\\
&\quad + k_3\,|\Lambda\zeta(t)| \int_0^t |v(t,\tau,\xi(\tau)) - v(t,\tau,\zeta(\tau))|\,d\tau\\
&\leq k_2|\xi(t) - \zeta(t)| + k_3\,|\Lambda\xi(t) - \Lambda\zeta(t)|\,g(r_0) + k_3\phi(r_0)\int_0^t \beta(\varepsilon, r_0)d\tau,\end{aligned}$$

where $\beta(\varepsilon, r_0)$ is defined as $\beta(\varepsilon, r_0) = \sup\{|v(t,\tau,\xi(\tau)) - v(t,\tau,\zeta(\tau))| : t,\tau \in I,\ \xi,\zeta \in B_{r_0}^{+},\ \|\xi - \zeta\| \leq \varepsilon\}$.

Next, we get $\|\Gamma\xi - \Gamma\zeta\| \leq k_2\,\|\xi - \zeta\| + k_3\,\|\Lambda\xi - \Lambda\zeta\|\,g(r_0) + k_3\phi(r_0)\beta(\varepsilon, r_0)$.

By the continuity of the function v on the set $I \times I \times [0, r_0]$ and the continuity of Λ, we deduce that Γ is continuous on the space $B_{r_0}^{+}$.

Next, take a nonempty subset Υ of $B_{r_0}^{+}$ and a number $\varepsilon > 0$. Then, in view of our assumptions, for $\xi \in \Upsilon$ and $t, s \in I$ with $0 \leq t - s \leq \varepsilon$, we obtain

$$|\Gamma\xi(t) - \Gamma\xi(s)| = \left|F\left(a(t), \xi(t), \Lambda\xi(t)\int_0^t v(t,\tau,\xi(\tau))d\tau\right)\right.$$

$$
\begin{aligned}
& -F\left(a(s), \xi(s), \Lambda\xi(s) \int_0^s v(s,\tau,\xi(\tau))d\tau\right)\Bigg| \\
& \le k_1 |a(t) - a(s)| + k_2 |\xi(t) - \xi(s)| \\
& + k_3 \left|\Lambda\xi(t) \int_0^t v(t,\tau,\xi(\tau))d\tau - \Lambda\xi(s) \int_0^s v(s,\tau,\xi(\tau))d\tau\right| \\
& \le k_1\omega(a,\varepsilon) + k_2\omega(\xi,\varepsilon) + k_3 \left|\Lambda\xi(t) \int_0^t v(t,\tau,\xi(\tau))d\tau - \Lambda\xi(s) \int_0^t v(t,\tau,\xi(\tau))d\tau\right| \\
& + k_3 \left|\Lambda\xi(s) \int_0^t v(t,\tau,\xi(\tau))d\tau - \Lambda\xi(s) \int_0^s v(s,\tau,\xi(\tau))d\tau\right| \\
& \le k_1\omega(a,\varepsilon) + k_2\omega(\xi,\varepsilon) + k_3 |\Lambda\xi(t) - \Lambda\xi(s)| \int_0^t |v(t,\tau,\xi(\tau))| d\tau \\
& + k_3 |\Lambda\xi(s)| \left|\int_0^t v(t,\tau,\xi(\tau))d\tau - \int_0^s v(s,\tau,\xi(\tau))d\tau\right| \\
& \le k_1\omega(a,\varepsilon) + k_2\omega(\xi,\varepsilon) + k_3 |\Lambda\xi(t) - \Lambda\xi(s)| \int_0^t |v(t,\tau,\xi(\tau))| d\tau \\
& + k_3 |\Lambda\xi(s)| \int_0^s |v(t,\tau,\xi(\tau)) - v(s,\tau,\xi(\tau))| d\tau \\
& + k_3 |\Lambda\xi(s)| \int_s^t |v(t,\tau,\xi(\tau))| d\tau \\
& \le k_1\omega(a,\varepsilon) + k_2\omega(\xi,\varepsilon) + k_3\omega(\Lambda\xi,\varepsilon)g(\|\xi\|) \\
& + k_3\phi(\|\xi\|)\gamma_{r_0}(\varepsilon) + k_3\phi(\|\xi\|)g(\|\xi\|)\varepsilon, \\
& \le k_1\omega(a,\varepsilon) + k_2\omega(\xi,\varepsilon) + k_3\omega(\Lambda\xi,\varepsilon)g(r_0) + k_3\phi(r_0)\gamma_{r_0}(\varepsilon) + k_3\phi(r_0)g(r_0)\varepsilon,
\end{aligned}
$$

where $\gamma_{r_0}(\varepsilon) = \sup\{|v(t,\tau,\xi) - v(s,\tau,\xi)| : t,s \in I, |t-s| \le \varepsilon,\ \xi \in [0,r_0]\}$.

Hence, we have the estimate

$$
\omega(\Gamma\xi,\varepsilon) \le k_1\omega(a,\varepsilon) + k_2\omega(\xi,\varepsilon) + k_3 g(r_0)\omega(\Lambda\xi,\varepsilon) + k_3\phi(r_0)\gamma_{r_0}(\varepsilon) + k_3\phi(r_0)g(r_0)\varepsilon.
$$

Notice, taking into account the uniform continuity of v on the set $I \times I \times [0,r_0]$ we have that $\gamma_{r_0}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, by assumptions (vi) we have

$$
\begin{aligned}
\omega_0(\Gamma\Upsilon) &\le k_2\omega_0(\Upsilon) + k_3 g(r_0)\omega_0(\Lambda\Upsilon) \le k_2\omega_0(\Upsilon) + k_3 g(r_0)L_1(\omega_0(\Upsilon))\omega_0(\Upsilon) \\
&= [k_2 + k_3 g(r_0)L_1(\omega_0(\Upsilon))]\,\omega_0(\Upsilon) = L(\omega_0(\Upsilon))\omega_0(\Upsilon),
\end{aligned}
$$

where $L(t) = k_2 + k_3 g(r_0)L_1(t)$. Note that, from (vi), the function L maps $[0,\infty)$ to $[0,1)$ and also have the following property $\sup\{L(r) : 0 < p \le r \le q\} < 1$. Hence, we conclude that by Theorem 3.1, (7) has at least one positive solution in $C(I)$. □

Finally, we present an illustrative example for Theorem 4.1.

Example 4.1. *Consider the following functional equation for $t \in [0,1]$,*

$$
\xi(t) = \frac{1}{5} + \frac{1}{3}|\xi(t)| \int_0^t \frac{(t+\tau)\xi^2(\tau)}{1+\xi^2(\tau)} d\tau. \tag{11}
$$

It is easily seen that (11) is a special case of (7) with

$$
a(t) = \frac{1}{5},\ v(t,\tau,\xi) = \frac{(t+\tau)\xi^2}{1+\xi^2},\ \Lambda\xi(t) = \frac{1}{3}|\xi(t)|,\ F(u,v,w) = u + w.
$$

Then, all assumptions of Theorem 4.1 are satisfied. Indeed,

(i) *It is obvious that the functions a and v are continuous.*

(ii) *Taking into account the function* $g:\mathbb{R}_+\to\mathbb{R}_+$ *defined by* $g(t)=\dfrac{2t^2}{1+t^2}$ *which is increasing, we have* $g(|\xi|)\geq|v(t,\tau,\xi)|$, *for all* $\xi\in\mathbb{R}$ *and* $t,\tau\in[0,1]$.

(iii) *It is clear that* Λ *transforms continuously the space* $C[0,1]$. *Also for the function* $\phi:\mathbb{R}_+\to\mathbb{R}_+$, $\phi(t)=\dfrac{t}{3}$, *we have* $\phi(||\xi||)\geq||\Lambda\xi||$, *for any* $\xi\in C[0,1]$.

(iv) *The continuity of* F *is obvious. Also, we have the inequality (8) for* $k_1=1,k_2=\dfrac{1}{3},k_3=1$ *and* $F_K=\sup\{|F(a(t),0,0)|,t\in[0,1]\}=\dfrac{1}{5}$.

(v) *The inequality* $\frac{r}{3}+\frac{r}{3}\frac{2r^2}{1+r^2}+\frac{1}{5}\leq r$ *has a positive solution* $r_0=1$.

(vi) *The condition (vi) is satisfied with the constant function* $L_1(t)=\dfrac{1}{3}$.

As a consequence, these above facts lead to a positive solution for the functional integral equation (11) *in* $C[0,1]$.

5. Conclusions

In this article, we introduced and studied a new version of Darbo's fixed point theorem with the help of Rakotch type contraction. Also the alternative of Leray-Schauder type of our result is also given. We have established an existence theorem based on a functional equation to demonstrate the application of our theoretical finding. Finally, we show how our existence theorem works by an example.

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