BICOMPLEX MATRIX TRANSFORMATIONS BETWEEN c_0 AND c IN BICOMPLEX SETTING

Birsen Sağır[1], Nilay Değirmen[2], Cenap Duyar[3]

In this paper, we give the characterizations of bicomplex matrix transformations between $c_0(\mathbb{BC})$ and $c(\mathbb{BC})$ extending some results in complex versions of them. Also, we state and prove their bicomplex counterparts utilizing Silverman-Toeplitz theorem and Kojima-Schur theorem.

Keywords: Matrix transformations, bicomplex numbers, sequence spaces, $\mathbb{BC}-$linear operator.

MSC2010: 40A05, 46A45, 46B45, 40C05.

1. Introduction

Bicomplex numbers are mentioned for the first time in the history of mathematics in [23]. An extensive review of the bicomplex space and related context is given in G.B Price's book [20]. Alpay et al [4] has developed the bicomplex version of functional analysis with complex scalars and it was the next significant push in subsequent studies on theory of functions with bicomplex variables. Bicomplex numbers has new applications with their use in fields such as neural networks [3], intelligent radio access networks [24], electromagnetic wave propagation [21], integral transforms and fractional calculus [1]. So, researchers working on bicomplex analysis reveal the importance of these numbers in real-world problems. The other recent notable applications can be found in [13, 8, 2, 11, 19, 14, 12, 10].

Now, we introduce a basic review of bicomplex numbers. Further, we refer to the books [20], [4] and [15] for more comprehensive knowledge.

The set of bicomplex numbers $\mathbb{BC}$ consists of the elements of the form $z_1 + jz_2$ where $z_1, z_2 \in \mathbb{C}$, $j^2 = -1$ and $ij = ji$. Also, it forms an algebra and $\mathbb{BC}-$module with respect to the standard operations and also, it has two distinguished zero divisors $e_1 = \frac{1+ij}{2}, e_2 = \frac{1-ij}{2}$. If $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + iz_2$, the idempotent representation of $z = z_1 + jz_2$ is uniquely written as $z = \beta_1 e_1 + \beta_2 e_2$.

[1]Professor, Ondokuz Mayıs University, Department of Mathematics, Samsun, Turkey, e-mail: `bduyar@omu.edu.tr`

[2]Associate Professor, Ondokuz Mayıs University, Department of Mathematics, Samsun, Turkey

[3]Professor, Ondokuz Mayıs University, Department of Mathematics, Samsun, Turkey

For two bicomplex numbers $z=\beta_1e_1+\beta_2e_2$ and $w=\gamma_1e_1+\gamma_2e_2$ we have the followings:

$$z\pm w=(\beta_1\pm\gamma_1)\,e_1+(\beta_2\pm\gamma_2)\,e_2,\ \ zw=(\beta_1\gamma_1)\,e_1+(\beta_2\gamma_2)\,e_2.$$

A map $\|.\|_{\mathbb{BC}}:\mathbb{BC}\to\mathbb{R}^+\cup\{0\}$, $\|z\|_{\mathbb{BC}}=\|z_1+jz_2\|_{\mathbb{BC}}=\sqrt{|z_1|^2+|z_2|^2}$ is a real valued norm on $\mathbb{BC}$, and also it satisfies the following properties:

(i) $\|zw\|_{\mathbb{BC}}\leq\sqrt{2}\,\|z\|_{\mathbb{BC}}\|w\|_{\mathbb{BC}}$ and $\|z\pm w\|_{\mathbb{BC}}\leq\|z\|_{\mathbb{BC}}+\|w\|_{\mathbb{BC}}$.

(ii) If $z=\beta_1e_1+\ \beta_2e_2$, then $\|z\|_{\mathbb{BC}}=\frac{1}{\sqrt{2}}\sqrt{|\beta_1|^2+|\beta_2|^2}$.

A sequence (z_n) in $\mathbb{BC}$ converges to $z_0\in\mathbb{BC}$ with respect to the norm $\|.\|_{\mathbb{BC}}$ if for every $\varepsilon>0$ there is a natural number n_0 such that $\|z_n-z_0\|_{\mathbb{BC}}<\varepsilon$ for all $n\geq n_0$. In this paper, we denote this convergence by $\lim\limits_{n\to\infty}z_n=z_0$.

Every $\mathbb{BC}-$module X is written in the idempotent decomposition $X=e_1X_1+e_2X_2$ or equivalents $X=e_1X+e_2X$, where $X_1:=e_1X$ and $X_2:=e_2X$. Assume that X_1 and X_2 are normed spaces with respective norms $\|.\|_1,\|.\|_2$. For any $x=x_1e_1+x_2e_2\in X$, the function $\|.\|_X$ defined as $\|x\|_X:=\frac{1}{\sqrt{2}}\sqrt{\|x_1\|_1^2+\|x_2\|_2^2}$ is a norm on X, the so-called Euclidean-type norm in X and $\|\zeta x\|_X\leq\sqrt{2}\,\|\zeta\|_{\mathbb{BC}}\|x\|_X$ for any $\zeta\in\mathbb{BC}$ and for any $x\in X$.

For two $\mathbb{BC}-$modules X and Y, a map $T:X\to Y$ is said to be a $\mathbb{BC}-$linear operator if $T(\zeta x+y)=\zeta T(x)+T(y)$ holds for any $x,y\in X$ and $\zeta\in\mathbb{BC}$ [4].

Every $\mathbb{BC}-$linear operator T on X is written in the idempotent decomposition $T=e_1T_1+e_2T_2$ where $X=e_1X_1+e_2X_2$ is idempotent decomposition of X and the linear operator T_l maps X_l to itself as $x\to e_lT(e_lx)$ for $l=1,2$. Also, $\mathbb{BC}-$linear operator T on X is bounded if and only if T_1 and T_2 are both bounded [9].

In [22], which is our first article on bicomplex sequence spaces, we set up the spaces $c_0(\mathbb{BC})$ and $c(\mathbb{BC})$ and obtained that they are Banach spaces according to the norm $\|.\|_{\infty,\mathbb{BC}}$ defined as $\|z\|_{\infty,\mathbb{BC}}=\sup\limits_{k\in\mathbb{N}}\|z_k\|_{\mathbb{BC}}$ for every $z=(z_k)\in c_0(\mathbb{BC})$ (or $c(\mathbb{BC})$).

The theory of sequence spaces has always been of great interest in the study on summability which has applications in many different fields such as functional anaylsis, numerical analysis, approximation theory, the theory of orthogonal series. The theory of matrix transformations is also one of the main topics studied in the theory of sequence spaces. Special theorems and results in summability theory motivated the authors to study matrix transformations. We refer to works and books [16, 25, 18, 7, 6, 5] on characterizations of matrix transformations between some complex sequence spaces.

The following theorems given in [16] serve as a motivation of our main results.

Theorem 1.1. *Let the following properties be satisfied for the matrix* $A=(a_{nk})$:

i) $a_{nk} \to 0$ *($n \to \infty$, k fixed).*

ii) $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty.$

Then, $A \in B(c_0, c_0)$ *and* $\|A\| = M.$

Theorem 1.2. *Let* $A \in B(c_0, c_0)$. *Then, the bounded linear transformation* A *assigns a matrix* (a_{nk}) *such that* $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ *for every* $x = (x_k) \in c_0$. *In addition, the following conditions are satisfied:*

i) $a_{nk} \to 0$ *($n \to \infty$, k fixed).*

ii) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty.$

Lemma 1.1. *If* $\sum_{k=1}^{\infty} a_k x_k < \infty$ *for any* $x = (x_k) \in c$, *then* $\sum_{k=1}^{\infty} |a_k| < \infty.$

Theorem 1.3. *(Silverman-Toeplitz Theorem)* $A \in (c, c; p)$ *if and only if*

i) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty.$

ii) $a_{nk} \to 0$ *($n \to \infty$, k fixed).*

iii) $\sum_{k=0}^{\infty} a_{nk} \to 1$ *($n \to \infty$).*

Theorem 1.4. *(Kojima-Schur theorem)* $A \in (c, c)$ *if and only if*

i) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty.$

ii) $\lim_{n\to\infty} \sum_{k=m}^{\infty} a_{nk} = a_m$ *for every* $m \in \mathbb{N}$.

Motivated by the importance of the applications of bicomplex numbers and matrix transformations, this article is devoted to establish the matrix transformations in bicomplex analysis. In more detail, in the present work, we evaluate the bicomplex matrix transformations between $c_0(\mathbb{BC})$ and $c(\mathbb{BC})$ using the fundamentals in complex versions of them. So, we transfer some theorems such as Silverman-Toeplitz theorem and Kojima-Schur theorem that exist in the literature to the bicomplex setting.

2. Main Results

If $z_k = z_{k,1}e_1 + z_{k,2}e_2$ for any $k \in \mathbb{N}$, where $\{z_{k,1}\}$ and $\{z_{k,2}\}$ are complex $(\mathbb{C}(i))$ sequences, then $z = \{z_k\}$ is a bicomplex sequence. Therefore given any complex sequence space $(X(\mathbb{C}), \|.\|_X)$ we can always define a bicomplex version $\left(X(\mathbb{BC}), \|.\|_{X(\mathbb{BC})}\right)$ comprising of all sequences of the type $\{z_k\} = \{z_{k,1}e_1 + z_{k,2}e_2\}$, where $\{z_{k,1}\}$ and $\{z_{k,2}\}$ are in $\left(X(\mathbb{C}), \|.\|_{X(\mathbb{C})}\right)$ and

$$\|z\|_{X(\mathbb{BC})} = \frac{1}{\sqrt{2}} \left(\|\{z_{k,1}\}\|^2_{X(\mathbb{C})} + \|\{z_{k,2}\}\|^2_{X(\mathbb{C})} \right)^{\frac{1}{2}}.$$

The addition and scalar multiplication on $\left(X(\mathbb{BC}), \|.\|_{X(\mathbb{BC})}\right)$ is defined as follows:

$$\begin{aligned} z+w &= \{x_{k,1}e_1 + x_{k,2}e_2\} + \{w_{k,1}e_1 + w_{k,2}e_2\} = \{z_{k,1} + w_{k,1}\} e_1 + \{z_{k,2} + w_{k,2}\} e_2, \\ \alpha z &= (\alpha_1 e_1 + \alpha_2 e_2)\{z_{k,1}e_1 + z_{k,2}e_2\} = \{\alpha_1 z_{k,1}\} e_1 + \{\alpha_2 z_{k,2}\} e_2 \end{aligned}$$

where $z, w \in \left(X(\mathbb{BC}), \|.\|_{X(\mathbb{BC})}\right)$ and $\alpha \in \mathbb{BC}$.

Example 2.1. *The spaces c and c_0 of convergent and null complex sequences are given by*

$$\begin{aligned} c &:= \left\{x = (x_k) \in s : \lim_{k\to\infty} |x_k - l| = 0 \text{ for some } l \in \mathbb{C}\right\}, \\ c_0 &:= \left\{x = (x_k) \in s : \lim_{k\to\infty} x_k = 0\right\} \end{aligned}$$

and they are Banach spaces with respect to the norm $\|.\|_\infty$ defined as $\|x\|_\infty = \sup_k |x_k|$. Then, their corresponding bicomplex sequence space $c_0(\mathbb{BC})$ (or $c(\mathbb{BC})$) comprises of all sequences of the type $z = \{z_k\} = \{z_{k,1}e_1 + z_{k,2}e_2\}$ where $\{z_{k,1}\}, \{z_{k,2}\} \in c_0$ (or c). Also,

$$\|z\|_{\infty,\mathbb{BC}} = \|\{z_{k,1}e_1 + z_{k,2}e_2\}\|_{\infty,\mathbb{BC}} = \frac{1}{\sqrt{2}}\left(\|\{z_{k,1}\}\|_\infty^2 + \|\{z_{k,2}\}\|_\infty^2\right)^{\frac{1}{2}}.$$

We will refer to $\|.\|_{\infty,\mathbb{BC}}$ as the Euclidean-type norm on $c_0(\mathbb{BC})$ and $c(\mathbb{BC})$.

Lemma 2.1. *$c_0(\mathbb{BC})$ (or $c(\mathbb{BC})$) equipped with $\|.\|_{\infty,\mathbb{BC}}$ is a Banach space over $\mathbb{BC}$.*

Proof. It is clear from the definition of addition and scalar multiplication that $c_0(\mathbb{BC})$ (or $c(\mathbb{BC})$) is a module over $\mathbb{BC}$. It is also easy to show that $\|.\|_{\infty,\mathbb{BC}}$ defines a norm on $c_0(\mathbb{BC})$ (or $c(\mathbb{BC})$). Now it only remains to show that $c_0(\mathbb{BC})$ is also complete with respect to $\|.\|_{\infty,\mathbb{BC}}$. For that let $\left\{z_n^k\right\}_{k=1}^\infty$ be a Cauchy sequence $c_0(\mathbb{BC})$. For $\{z_n = z_{n,1}e_1 + z_{n,2}e_2\}_{n=1}^\infty$ we have

$$\|\{z_n\}\|_{\infty,\mathbb{BC}} = \frac{1}{\sqrt{2}}\left(\|\{z_{n,1}\}\|_\infty^2 + \|\{z_{n,2}\}\|_\infty^2\right)^{\frac{1}{2}}. \tag{1}$$

Therefore one has $\|\{z_{n,i}\}\|_\infty \leq \sqrt{2}\,\|\{z_n\}\|_{\infty,\mathbb{BC}}$ for $i = 1, 2$. Now for $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|\{z_n^s - z_n^m\}\|_{\infty,\mathbb{BC}} < \frac{\varepsilon}{\sqrt{2}} \tag{2}$$

for all $s, m \geq n_0$. Thus (1) and (2) yield that $\left\{z_{n,i}^k\right\}_{k=1}^\infty$ is a Cauchy sequence in $(c_0, \|.\|_\infty)$ for $i = 1, 2$. In view of the completeness of the $(c_0, \|.\|_\infty)$, we have that $\left\{z_{n,i}^k\right\}_{k=1}^\infty$ converges to some $\{z_{n,i}\} \in c_0$. Therefore for $i = 1, 2$

there exists $N_i \in \mathbb{N}$ such that $\left\| \left\{ z_{n,i}^s - z_{n,i} \right\} \right\|_\infty < \sqrt{\varepsilon}$ for all $s \geq N_i$. Now for $N = \max\{N_1, N_2\}$ and $s \geq N$ one has

$$\|\{z_n^s - z_n\}\|_{\infty,\mathbb{BC}}^2 = \frac{1}{2}\left(\left\| \left\{ z_{n,i}^s - z_{n,i} \right\} \right\|_\infty^2 + \left\| \left\{ z_{n,i}^s - z_{n,i} \right\} \right\|_\infty^2 \right) < \frac{1}{2}(\varepsilon + \varepsilon) = \varepsilon.$$

Thereby showing that $\left\{ z_n^k \right\}_{k=1}^\infty$ is a convergent sequence in $c_0(\mathbb{BC})$. So, $c_0(\mathbb{BC})$ equipped with $\|.\|_{\infty,\mathbb{BC}}$ is a Banach space over $\mathbb{BC}$. The proof is completed. $\square$

Suppose $A' = (a'_{nk})$ is an infinite matrix of bicomplex numbers a'_{nk}, where $k, n \in \mathbb{N}$. Since $a'_{nk} \in \mathbb{BC}$, it is uniquely written in the form $a'_{nk} = a_{nk}^1 e_1 + a_{nk}^2 e_2$ where (a_{nk}^1) and (a_{nk}^2) denote infinite matrices with complex terms. For $\zeta = \{\zeta_k\} = \{\zeta_k^1 e_1 + \zeta_k^2 e_2\} \in s(\mathbb{BC})$, we obtain the sequence $A'\zeta$, the $A'-$transform of ζ, by the usual matrix product

$$A'\zeta = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1k} & \cdots \\ a'_{21} & a'_{22} & \cdots & a'_{2k} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nk} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \\ \vdots \end{bmatrix} = \begin{bmatrix} a'_{11}\zeta_1 + a'_{12}\zeta_2 + ... + a'_{1k}\zeta_k + ... \\ a'_{21}\zeta_1 + a'_{22}\zeta_2 + ... + a'_{2k}\zeta_k + ... \\ \vdots \\ a'_{n1}\zeta_1 + a'_{n2}\zeta_2 + ... + a'_{nk}\zeta_k + ... \\ \vdots \end{bmatrix}.$$

Hence, in this way, we transform the $\mathbb{BC}$-sequence ζ into the $\mathbb{BC}$-sequence $A'\zeta = \{(A'\zeta)_n\}$ with $(A'\zeta)_n = \sum_k a'_{nk}\zeta_k = A_n^1(\zeta) e_1 + A_n^2(\zeta) e_2$ where $A_n^1(\zeta) = \sum_{k=1}^\infty a_{nk}^1 \zeta_{k,1}$ and $A_n^2(\zeta) = \sum_{k=1}^\infty a_{nk}^2 \zeta_{k,2}$. For each n, the existence of the sum of $A'_n(x) = a'_{n1}\zeta_1 + a'_{n2}\zeta_2 + ... + a'_{nk}\zeta_k + ...$ is accepted. If idempotent representation is used, we get

$$\begin{aligned} A'_n(\zeta) &= \sum_{k=1}^\infty \left(a_{nk}^1 e_1 + a_{nk}^2 e_2 \right) (\zeta_{k,1} e_1 + \zeta_{k,2} e_2) = \sum_{k=1}^\infty \left(a_{nk}^1 \zeta_{k,1} e_1 + a_{nk}^2 \zeta_{k,2} e_2 \right) \\ &= \left(\sum_{k=1}^\infty a_{nk}^1 \zeta_{k,1} \right) e_1 + \left(\sum_{k=1}^\infty a_{nk}^2 \zeta_{k,2} \right) e_2 \end{aligned}$$

and so

$$A'_n(\zeta) = \left(\sum_{k=1}^\infty a_{nk}^1 \zeta_{k,1} \right) e_1 + \left(\sum_{k=1}^\infty a_{nk}^2 \zeta_{k,2} \right) e_2. \tag{3}$$

Then, (3) can be also written $A'_n(\zeta) = A_n^1(\zeta) e_1 + A_n^2(\zeta) e_2$ where $A_n^1(\zeta) = \sum_{k=1}^\infty a_{nk}^1 \zeta_{k,1}$ and $A_n^2(x) = \sum_{k=1}^\infty a_{nk}^2 \zeta_{k,2}$.

Let $X(\mathbb{BC})$ and $Y(\mathbb{BC})$ be any two bicomplex sequence spaces. If $A\zeta$ exists and is in $Y(\mathbb{BC})$ for every $\mathbb{BC}-$sequence $\zeta = (\zeta_k) \in X(\mathbb{BC})$, then we say that A' defines a $\mathbb{BC}-$matrix mapping from $X(\mathbb{BC})$ into $Y(\mathbb{BC})$, and we denote it by writting $A' : X(\mathbb{BC}) \to Y(\mathbb{BC})$. By $(X(\mathbb{BC}), Y(\mathbb{BC}))$, we denote the class of all $\mathbb{BC}-$matrices A' such that $A' : X(\mathbb{BC}) \to Y(\mathbb{BC})$. Thus, $A' \in (X(\mathbb{BC}), Y(\mathbb{BC}))$ if and only if the series on the right hand side of (3) converges for each $n \in \mathbb{N}$ and every $\zeta \in X(\mathbb{BC})$, and we have $A'\zeta = \{(A'\zeta)_n\}_{n\in\mathbb{N}} \in Y(\mathbb{BC})$ for all $\zeta \in X(\mathbb{BC})$.

Since $A_1 = (a^1_{nk})$ and $A_2 = (a^2_{nk})$ are linear, A' is $\mathbb{BC}-$linear as follows:

$$\begin{aligned} A'(\lambda\zeta+\eta) &= \sum_{k=1}^{\infty} a'_{nk}(\lambda\zeta+\eta) \\ &= \sum_{k=1}^{\infty}\left[a^1_{nk}e_1 + a^2_{nk}e_2\right]\left[(\lambda\zeta_{k,1}+\eta_{k,1})e_1 + (\lambda\zeta_{k,2}+\eta_{k,2})e_2\right] \\ &= \sum_{k=1}^{\infty}\left[a^1_{nk}(\lambda\zeta_{k,1}+\eta_{k,1})e_1 + a^2_{nk}(\lambda\zeta_{k,2}+\eta_{k,2})e_2\right] \\ &= \left[\lambda\sum_{k=1}^{\infty}a^1_{nk}\zeta_{k,1} + \sum_{k=1}^{\infty}a^1_{nk}\eta_{k,1}\right]e_1 + \left[\lambda\sum_{k=1}^{\infty}a^2_{nk}\zeta_{k,2} + \sum_{k=1}^{\infty}a^2_{nk}\eta_{k,2}\right]e_2 \\ &= \lambda A'(\zeta) + A'(\eta) \end{aligned}$$

for all $\zeta, \eta \in X(\mathbb{BC}), \lambda \in \mathbb{BC}$.

In this section, we give the characterizations of some bicomplex matrix classes. We begin with stating the necessary and sufficient condition on an infinite $\mathbb{BC}-$matrix belonging to the class $(c_0(\mathbb{BC}), c_0(\mathbb{BC}))$. For this, the known fundamental theorems for c and c_0 and their results will be used.

First of all, let's give the following two theorems, which we can call the existence theorems:

Theorem 2.1. *Let the following properties be satisfied for the bicomplex matrix* $A' = (a'_{nk})$:

i) $a'_{nk} \to 0$ *($n \to \infty$, k fixed).*

ii) $\|M'\|_{\mathbb{BC}} = \sup_n \sum_{k=1}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty$.

Then, A' *defines a* $\mathbb{BC}-$*linear bounded operator on* $c_0(\mathbb{BC})$ *into itself and* $|||A'||| = \|M'\|_{\mathbb{BC}}$.

Proof. Since $a'_{nk} \to 0$ ($n \to \infty$, k fixed), for given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|a'_{nk} - 0\|_{\mathbb{BC}} = \|a'_{nk}\|_{\mathbb{BC}} = \frac{1}{\sqrt{2}}\sqrt{|a^1_{nk}|^2 + |a^2_{nk}|^2} < \varepsilon$$

for all $n > n_0$.Then we obtain that for given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|a^1_{nk}| < \sqrt{2}\varepsilon, |a^2_{nk}| < \sqrt{2}\varepsilon$ for all $n > n_0$. This implies that

$$a^1_{nk} \to 0 \text{ and } a^2_{nk} \to 0 \ (n \to \infty, k \text{ fixed}). \tag{4}$$

Also, since $\sup_n \sum_{k=1}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty$ and

$$\sum_{k=1}^{\infty}\|a'_{nk}\|_{\mathbb{BC}} = \sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}\sqrt{|a^1_{nk}|^2 + |a^2_{nk}|^2}, \tag{5}$$

we have $M_1 = \frac{1}{\sqrt{2}}\sup\limits_n \sum\limits_{k=1}^{\infty} |a_{nk}^1| < \infty$ and $M_2 = \frac{1}{\sqrt{2}}\sup\limits_n \sum\limits_{k=1}^{\infty} |a_{nk}^2| < \infty$. Thus, by (4), (5) and Theorem 1.1 we deduce that $A^1, A^2 \in B(c_0, c_0)$ and $\|A^1\| = M_1, \|A^2\| = M_2$ since $A' = A^1 e_1 + A^2 e_2$ where $A^1 = (A_n^1(\zeta))$, $A^2 = (A_n^2(\zeta))$. Since $A^1, A^2 \in B(c_0, c_0)$, we write $\|A^1(\zeta_1)\|_\infty \leq K_1 \|\zeta_1\|_\infty$ and $\|A^2(\zeta_1)\|_\infty \leq K_2 \|\zeta_2\|_\infty$ where $\zeta_1 = (\zeta_{k,1}), \zeta_1 = (\zeta_{k,1}) \in c_0$. Hence,

$$\begin{aligned} \left\|A^1(\zeta)\right\|_{\infty,\mathbb{BC}} &= \frac{1}{\sqrt{2}}\sqrt{\|A^1(\zeta_1)\|_\infty^2 + \|A^2(\zeta_2)\|_\infty^2} \\ &\leq K.\frac{1}{\sqrt{2}}\sqrt{\|\zeta_1\|_\infty^2 + \|\zeta_2\|_\infty^2} = K\|\zeta\|_{\infty,\mathbb{BC}} \end{aligned}$$

for $K = \max\{K_1, K_2\}$. Finally, A' is bounded. It is clear that A' is $\mathbb{BC}-$linear. So, A' defines a $\mathbb{BC}-$linear bounded operator on $c_0(\mathbb{BC})$ into itself.

On the other hand, if we take $M' = M_1 e_1 + M_2 e_2$, we get

$$|||A'||| = \frac{1}{\sqrt{2}}\sqrt{\|A^1\|^2 + \|A^2\|^2} = \frac{1}{\sqrt{2}}\sqrt{M_1^2 + M_2^2} = \|M'\|_{\mathbb{BC}}.$$

This completes the proof. □

Theorem 2.1 shows us that a $\mathbb{BC}-$matrix of a certain type describes a bounded linear transformation from $c_0(\mathbb{BC})$ into itself. Now let's show the converse.

Theorem 2.2. *Let A' be a $\mathbb{BC}-$linear bounded operator on $c_0(\mathbb{BC})$ into itself. Then, A' determines a $\mathbb{BC}-$matrix (a'_{nk}) such that*

$$A'_n(\zeta) = \sum_{k=1}^{\infty} a'_{nk}\zeta_k \text{ for every } \zeta = (\zeta_k) \in c_0(\mathbb{BC}) \tag{6}$$

holds and such that

i) $a'_{nk} \to 0$ *($n \to \infty$, k fixed).*

ii) $\|A'\| = \sup\limits_n \sum\limits_{k=1}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty$.

Proof. The $\mathbb{BC}-$sequence (e'_k) defined as $e'_k = (0, 0, ..., 0, e_1 + e_2, 0, ...) = e_k^1 e_1 + e_k^2 e_2$ is a basis for $c_0(\mathbb{BC})$ where $e_k^1 = e_k^2 = \left(0, 0, ..., 0, \underset{k-\text{th term}}{1}, 0, ...\right)$, $(e_k^1), (e_k^2)$ are a basis for c_0. So, we can write

$$\zeta = \sum_{k=1}^{\infty} \zeta_k e'_k = \left(\sum_{k=1}^{\infty} \zeta_{k,1} e_k^1\right) e_1 + \left(\sum_{k=1}^{\infty} \zeta_{k,2} e_k^2\right) e_2.$$

Since A' is a $\mathbb{BC}-$bounded linear transformation, we have $A'(\zeta) = \sum\limits_{k=1}^{\infty} \zeta_k A'(e'_k)$. On the other hand, since $(e'_k) \in c_0(\mathbb{BC})$ for every $k \in \mathbb{N}$, by hypothesis

$A'(e'_k) = (a'_{1,k}, a'_{2,k}, ..., a'_{n,k}, ...) \in c_0(\mathbb{BC})$ is obtained. This yields the following statement:

$$A'(\zeta) = \sum_{k=1}^{\infty} \zeta_k (a'_{n,k}).$$

accordingly, the general term of this new sequence is $A'_n(\zeta) = \sum_{k=1}^{\infty} a'_{n,k}\zeta_k$in other notation, as $A'_n(\zeta) = \sum_{k=1}^{\infty} a'_{nk}\zeta_k$. This proves (6).

Let us now show that conditions i) and ii) are satisfied. By hypothesis that
$A\zeta \in c_0(\mathbb{BC})$ whenever $\zeta \in c_0(\mathbb{BC})$, we deduce that $A'(e'_k) \in c_0(\mathbb{BC})$, $k = 1, 2, ...$ implies $a'_{nk} \to 0$ $(n \to \infty)$, $k = 1, 2, ...$ It remains to show that $\|A'\| = \sup_n \sum_{k=1}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty$. $\|a'_{nk}\|_{\mathbb{BC}} = \frac{1}{\sqrt{2}}\sqrt{|a^1_{nk}|^2 + |a^2_{nk}|^2}$ where $a'_{nk} = a^1_{nk}e_1 + a^2_{nk}e_2$, the statement $a'_{nk} \to 0$ $(n \to \infty$, k fixed) implies that $a^1_{nk} \to 0$ and $a^2_{nk} \to 0$ $(n \to \infty$, k fixed). Since A' is a $\mathbb{BC}-$linear bounded operator on $c_0(\mathbb{BC})$ into itself and

$$|||A'||| = \frac{1}{\sqrt{2}}\sqrt{\|A^1\|^2 + \|A^2\|^2} < \infty,$$

we derive that

$$\|A^1\| = M_1 = \sup_n \sum_{k=1}^{\infty} |a^1_{nk}| < \infty, \quad \|A^2\| = M_2 = \sup_n \sum_{k=1}^{\infty} |a^2_{nk}| < \infty,$$

by Theorem 1.2 and so $A^1, A^2 \in B(c_0, c_0)$. Therefore, we obtain that

$$\begin{aligned} \sup_n \sum_{k=1}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} &= \frac{1}{\sqrt{2}} \sup_n \sum_{k=1}^{\infty} \sqrt{|a^1_{nk}|^2 + |a^2_{nk}|^2} \\ &\leq \frac{1}{\sqrt{2}} \sup_n \sum_{k=1}^{\infty} \left(|a^1_{nk}| + |a^2_{nk}|\right) < \infty. \end{aligned}$$

Also, for $M' = M_1e_1 + M_2e_2$ we get

$$|||A'||| = \frac{1}{\sqrt{2}}\sqrt{\|A^1\|^2 + \|A^2\|^2} \leq \|A^1\| + \|A^2\| = M_1 + M_2.$$

This completes the proof. □

Lemma 2.2. *If* $\sum_{k=1}^{\infty} a_k\zeta_k < \infty$ *for any* $\zeta = (\zeta_k) \in c(\mathbb{BC})$, *then* $\sum_{k=1}^{\infty} \|a_k\|_{\mathbb{BC}} < \infty$.

Proof. Let $\zeta = (\zeta_k) \in c(\mathbb{BC})$ and $a_k \in \mathbb{BC}$. Then, we can write $\zeta_k = \zeta_{k,1}e_1 + \zeta_{k,2}e_2$, $a_k = a_{k,1}e_1 + a_{k,2}e_2$, and so $a_k\zeta_k = a_{k,1}\zeta_{k,1}e_1 + a_{k,2}\zeta_{k,2}e_2$. If

$\sum\limits_{k=1}^{\infty} a_k\zeta_k < \infty$, then we have

$$S_n = \left(\sum_{k=1}^{n} a_{k,1}\zeta_{k,1}\right) e_1 + \left(\sum_{k=1}^{n} a_{k,2}\zeta_{k,2}\right) e_2 = S_{n,1}e_1 + S_{n,2}e_2$$

where $S_n = \sum\limits_{k=1}^{n} a_k\zeta_k$, $S_{n,1} = \sum\limits_{k=1}^{n} a_{k,1}\zeta_{k,1}$, $S_{n,2} = \sum\limits_{k=1}^{n} a_{k,2}\zeta_{k,2}$. Since $\sum\limits_{k=1}^{\infty} a_k\zeta_k < \infty$, there exists $S = S_1e_1 + S_2e_2 \in \mathbb{BC}$ such that $S_n \to S$. We note that

$$\begin{aligned} \|S_n - S\|_{\mathbb{BC}} &= \|(S_{n,1} - S_1)\, e_1 + (S_{n,2} - S_2)\, e_2\|_{\mathbb{BC}} \\ &= \frac{1}{\sqrt{2}}\sqrt{|S_{n,1} - S_1|^2 + |S_{n,2} - S_2|^2} < \varepsilon. \end{aligned}$$

This means that $S_{n,1} \to S_1$ and $S_{n,2} \to S_2$. With the help of Lemma 1.1 we see that $\sum\limits_{k=1}^{\infty} |a_{k,1}| < \infty$ and $\sum\limits_{k=1}^{\infty} |a_{k,2}| < \infty$. In the light of these, we write

$$\begin{aligned} \sum_{k=1}^{\infty} \|a_k\|_{\mathbb{BC}} &= \frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\sqrt{|a_{k,1}|^2 + |a_{k,2}|^2} \\ &\leq \frac{1}{\sqrt{2}}\sum_{k=1}^{\infty} (|a_{k,1}| + |a_{k,2}|) \\ &= \frac{1}{\sqrt{2}}\left(\sum_{k=1}^{\infty} |a_{k,1}| + \sum_{k=1}^{\infty} |a_{k,2}|\right) < \infty. \end{aligned}$$

□

Definition 2.1. *By* $(X(\mathbb{BC}), Y(\mathbb{BC}))$ *we shall denote the set of all matrices* A' *which map* $X(\mathbb{BC})$ *into* $Y(\mathbb{BC})$. *By* $(X(\mathbb{BC}), Y(\mathbb{BC}); p)$ *we denote that subset of* $(X(\mathbb{BC}), Y(\mathbb{BC}))$ *for which limits or sums are preserved. For example,*
$A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$ *means that* $A'_n(\zeta)$ *exists for each* n *whenever* $\zeta \in c(\mathbb{BC})$ *and* $A'_n(\zeta) \to t$ $(n \to \infty)$ *whenever* $\zeta_k \to t$ $(k \to \infty)$.

Now we give some basic results.

Theorem 2.3. *(*$\mathbb{BC}$*−Silverman-Toeplitz Theorem)* $A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$ *if and only if*

i) $\sup\limits_{n}\sum\limits_{k=0}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty$.
ii) $a'_{nk} \to 0$ *(*$n \to \infty$, k *fixed).*
iii) $\sum\limits_{k=0}^{\infty} a'_{nk} \to 1$ *(*$n \to \infty$*).*

Proof. Let's assume that i), ii) and iii) are satisfied. We want to show that $A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$. Take an arbitrary $\zeta = (\zeta_k) \in c(\mathbb{BC})$. Let $\zeta_k \to t$ $(k \to \infty)$. Then, in the view of i) we write

$$\sum_{k=0}^{\infty} a'_{nk}\zeta_k = \sum_{k=0}^{\infty} a'_{nk}(\zeta_k - t) + t\sum_{k=0}^{\infty} a'_{nk}.$$

By iii), we have $\lim_{n\to\infty} t\sum_{k=0}^{\infty} a'_{nk} = t$. Also, since $\eta = (\eta_k) \in c_0(\mathbb{BC})$ where $\eta_k = \zeta_k - t$, it is clear that $\sum_{k=0}^{\infty} a'_{nk}(\zeta_k - t) \in c_0(\mathbb{BC})$ by i), ii) and Theorem 2.2. This implies that

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a'_{nk}(\zeta_k - t) = 0$$

and equivalently, $\lim_{n\to\infty}\sum_{k=0}^{\infty} a'_{nk}\zeta_k = t$.So, it results that $A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$.

Conversely, let $A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$. Then, since $\zeta = e'_k = (\zeta_k)$, $\zeta = (0, 0, ..., 0, e_1, 0, ...) \in c_0(\mathbb{BC}) \subset c(\mathbb{BC})$ with $k \in \mathbb{N}$ arbitrary and fixed, $A'_n(\zeta) = \sum_{k=0}^{\infty} a'_{nk}\zeta_k$ exists for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} A'_n(\zeta) = \lim_{n\to\infty}\zeta_n = 0$ holds. This produces

$$\lim_{n\to\infty} A'_n(\zeta) = \lim_{n\to\infty}\sum_{k=0}^{\infty} a'_{nk}\zeta_k = \lim_{n\to\infty} a'_{nk} = 0 \ (k \text{ fixed}),$$

which proves ii).

Let $\zeta = e = (e_1, e_1, ...)$. It follows that $\zeta = e \in c(\mathbb{BC})$ and $\zeta_n \to 1$ $(n \to \infty)$. Since $A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$, one can easily see that

$$1 = \lim_{n\to\infty} A'_n(\zeta) = \lim_{n\to\infty}\sum_{k=0}^{\infty} a'_{nk}\zeta_k = \lim_{n\to\infty}\sum_{k=0}^{\infty} a'_{nk},$$

which gives iii).

Again let $\zeta = (\zeta_k) \in c(\mathbb{BC})$. So, we say that $(\zeta_{k,1}), (\zeta_{k,2}) \in c$ where $\zeta = \zeta_{k,1}e_1 + \zeta_{k,2}e_2$. Because of $A' \in (c(\mathbb{BC}), c(\mathbb{BC}); p)$, we have

$$A'_n(\zeta) = \sum_{k=0}^{\infty} a'_{nk}\zeta_k = \left(\sum_{k=0}^{\infty} a^1_{nk}\zeta_{k,1}\right)e_1 + \left(\sum_{k=0}^{\infty} a^2_{nk}\zeta_{k,2}\right)e_2 \in c_0(\mathbb{BC})$$

where $a'_{nk} = a^1_{nk}e_1 + a^2_{nk}e_2$.

Suppose that $\lim_{n\to\infty} A'_n(\zeta) = t$ and $t = t_1e_1 + t_2e_2$. Thus, we get $t_1 = \lim_{n\to\infty} A^1_n(\zeta_{k,1})$, $t_2 = \lim_{n\to\infty} A^2_n(\zeta_{k,2})$ and so $A^1, A^2 \in (c, c; p)$. From Theorem 1.3 we obtain that $\sup_n \sum_{k=0}^{\infty} |a^1_{nk}| < \infty$, $\sup_n \sum_{k=0}^{\infty} |a^2_{nk}| < \infty$.

$$\begin{aligned}\sup_n \sum_{k=0}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} &= \frac{1}{\sqrt{2}} \sup_n \sum_{k=0}^{\infty} \sqrt{|a^1_{nk}|^2 + |a^2_{nk}|^2} \\ &\le \frac{1}{\sqrt{2}} \left(\sup_n \sum_{k=0}^{\infty} |a^1_{nk}| + \sup_n \sum_{k=0}^{\infty} |a^2_{nk}| \right) < \infty,\end{aligned}$$

which is exactly iii). The proof is completed. □

Remark 2.1. *In Theorem 2.1 and Theorem 2.3, $\mathbb{BC}-$matrices that transform $\mathbb{BC}-$convergent sequences into $\mathbb{BC}-$convergent sequences are characterized.*

Definition 2.2. *A $\mathbb{BC}-$matrix satisfying the conditions of Theorem 2.3 is called a $\mathbb{BC}-$Toeplitz matrix or $\mathbb{BC}-$regular matrix.*

Definition 2.3. *Let a $\mathbb{BC}-$infinite matrix $A' = (a'_{nk})$ be given. If $a'_{nk} = 0$ for $k > n$, then the $\mathbb{BC}-$matrix A' is called $\mathbb{BC}-$triangular matrix.*

Corollary 2.1. *If a $\mathbb{BC}-$triangular matrix $A' = (a'_{nk})$ satisfies the condition $\lim_{n\to\infty} a'_{n,n-k} = 0$ for every fixed $k \in \mathbb{N}$ with $0 \le k \le n$ in addition to the conditions of Theorem 2.3, then*

$$\xi_n := \sum_{k=0}^{n} a'_{nk} z_k w_{n-k} \to zw \ (n \to \infty)$$

where $\|z_n - z\|_{\mathbb{BC}} \to 0, \|w_n - w\|_{\mathbb{BC}} \to 0$ ($n \to \infty$).

Proof. Let the $\mathbb{BC}-$triangular matrix $A' = (a'_{nk})$ satisfy the given conditions and let $z_n \to z, w_n \to w$ $(n \to \infty)$. Therefore, we obtain

$$\begin{aligned}\xi_n \ :&= \sum_{k=0}^{n} a'_{nk} w_{n-k} (z_k - z) + \sum_{k=0}^{n} a'_{nk} w_{n-k} z \qquad (7)\\ &= \sum_{k=0}^{n} a'_{nk} w_{n-k} (z_k - z) + z \sum_{k=0}^{n} a'_{n,n-k} w_k.\end{aligned}$$

Put $b'_{nk} := a'_{nk} w_{n-k}$. Clearly, the matrix $B' = (b'_{nk})$ satisfies the conditions of Theorem 2.1. Also, since $(z_n - z) \in c_0(\mathbb{BC})$, taking into account Theorem 2.1 we obtain that

$$\sum_{k=0}^{n} a'_{nk} w_{n-k} (z_k - z) \to 0 (n \to \infty). \qquad (8)$$

The matrix $C' = (c'_{nk})$ obtained by taking $c'_{nk} := a'_{n,n-k}$ satisfies the conditions of Theorem 2.3. Indeed, the following statements explain this claim:

$$\begin{aligned} \sup_n \sum_{k=0}^{n} \|c'_{nk}\|_{\mathbb{BC}} &= \sup_n \sum_{k=0}^{n} \left\|a'_{n,n-k}\right\|_{\mathbb{BC}} = \sup_n \sum_{k=0}^{n} \|a'_{nk}\|_{\mathbb{BC}} < \infty, \\ c'_{nk} &= a'_{n,n-k} \to 0 \ (n \to \infty, k \text{ fixed}), \\ \sum_{k=0}^{n} c'_{nk} &= \sum_{k=0}^{n} a'_{n,n-k} = \sum_{k=0}^{n} a'_{nk} \to 1 \ (n \to \infty). \end{aligned}$$

So, using Theorem 2.3, we get

$$\sum_{k=0}^{n} c'_{nk} w_k = \sum_{k=0}^{n} a'_{n,n-k} w_k \to w \ (n \to \infty). \tag{9}$$

Substituting (8) and (9) in (7), $\xi_n \to zw \ (n \to \infty)$is obtained. So, the proof ends. □

Now let's give $\mathbb{BC}-$Kojima-Schur theorem, which is a generalization of $\mathbb{BC}-$Silverman-Toeplitz theorem.

Theorem 2.4. *($\mathbb{BC}-$Kojima-Schur theorem) $A' \in (c(\mathbb{BC}), c(\mathbb{BC}))$ if and only if*

i) $\sup_n \sum_{k=0}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty.$

ii) $\lim_{n\to\infty} \sum_{k=p}^{\infty} a'_{nk} = a_p$ *for every* $p \in \mathbb{N}$.

Proof. Let $A' \in (c(\mathbb{BC}), c(\mathbb{BC}))$. The necessity of i) is proved as in Theorem 2.3. Therefore, we only prove the necessity of ii). The sequence $\zeta = (\zeta_k)$ defined as

$$\zeta = \zeta_k := \begin{cases} 0, & 0 \le k < p \\ e_1, & k \ge p \end{cases} = (0, 0, ..., 0, e_1, e_1, ...)$$

is in $c(\mathbb{BC})$. Since $A' \in (c(\mathbb{BC}), c(\mathbb{BC}))$, we have $A'_n(\zeta) = \sum_{k=p}^{\infty} a'_{nk}$ and the limit

$$\lim_{n\to\infty} A'_n(\zeta) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a'_{nk}\zeta_k = \lim_{n\to\infty} \sum_{k=p}^{\infty} a'_{nk}$$

exists. If we denote this limit depending on p with a_p, we get $\lim_{n\to\infty} \sum_{k=p}^{\infty} a'_{nk} = a_p$.

On the contrary, assume that i) and ii) are satisfied. Let $\zeta = (\zeta_k) = \zeta_{k,1}e_1 + \zeta_{k,2}e_2 \in c(\mathbb{BC})$ and $\zeta_k \to a = a_1e_1 + a_2e_2$. We know that

$$A'_n(\zeta) = \left(\sum_{k=0}^{\infty} a^1_{nk}\zeta_{k,1}\right) e_1 + \left(\sum_{k=0}^{\infty} a^2_{nk}\zeta_{k,2}\right) e_2$$

where $a'_{nk} = a^1_{nk}e_1 + a^2_{nk}e_2$. Also, since $\zeta = (\zeta_k) \in c(\mathbb{BC})$, we say that $(\zeta_{k,1}), (\zeta_{k,2}) \in c$. The statement $\sup_n \sum_{k=0}^{\infty} \|a'_{nk}\|_{\mathbb{BC}} < \infty$ implies that $\sup_n \sum_{k=0}^{\infty} |a^1_{nk}| < \infty$, $\sup_n \sum_{k=0}^{\infty} |a^2_{nk}| < \infty$. Similarly, $\lim_{n\to\infty} \sum_{k=p}^{\infty} a'_{nk} = a'_p$ implies that $\lim_{n\to\infty} \sum_{k=p}^{\infty} a^1_{nk} = a^1_p$, $\lim_{n\to\infty} \sum_{k=p}^{\infty} a^2_{nk} = a^2_p$. By Theorem 1.4, one can deduce that $A^1_n(\zeta) = \sum_{k=0}^{\infty} a^1_{nk}\zeta_{k,1}, A^2_n(\zeta) = \sum_{k=0}^{\infty} a^2_{nk}\zeta_{k,2} \in c$ and so $A' \in (c(\mathbb{BC}), c(\mathbb{BC}))$. Thus, the proof comes to the end. □

Definition 2.4. *A $\mathbb{BC}-$matrix satisfying the conditions of Theorem 2.4 is called a $K - \mathbb{BC}-$matrix.*

Remark 2.2. *A $K - \mathbb{BC}-$matrix may not be $\mathbb{BC}-$regular.*

3. Concluding Remarks

In this paper, the characterizations of bicomplex matrix transformations between $c_0(\mathbb{BC})$ and $c(\mathbb{BC})$ are given. This characterization is based on the definition of a $\mathbb{BC}-$linear bounded operator, and converts matrix transformations between complex sequence spaces c and c_0 into bicomplex ones. We believe that these new matrix transformations can be a powerful tool for summability theory in bicomplex analysis. Hereupon, a subject of future research might be Tauberian theorems for Cesaro and Abel summability methods of single sequences.

REFERENCES

[1] *R. Agarwal, U. P. Sharma*, Bicomplex Mittag-Leffler function and applications in integral transform and fractional calculus, In Mathematical and Computational Intelligence to Socio-scientific Analytics and Applications, 157-167, Lecture Notes in Networks and Systems, volume 518, Springer, Singapore, 2023.

[2] *R. Agarwal, U. P. Sharma, R. P. Agarwal, D. L. Suthar, S. D. Purohit*, Bicomplex Landau and Ikehara theorems for the Dirichlet series, J. Math., 2022, Article ID 4528209, 8 pages. doi:10.1155/2022/4528209.

[3] *D. Alpay, K. Diki, M. Vajiac*, A note on the complex and bicomplex valued neural networks, Appl. Math. Comput., **445** (2023), 127864.

[4] *D. Alpay, M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa*, Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis, Springer Science & Business Media, 2014.

[5] *J. Banaś, M. Mursaleen*, Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations, New Delhi: Springer, 2014.

[6] *F. Başar*, Summability Theory and Its Applications, Bentham Science Publishers, 2012.

[7] *F. Başar, E. Malkowsky, B. Altay*, Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1-summable and bounded sequences, Publ. Math. Debrecen, **73** (2008), No.1-2, 193-213.

[8] *A. Chandola, R. M. Pandey, K. S. Nisar*, On the new bicomplex generalization of Hurwitz–Lerch zeta function with properties and applications. Analysis.**43** (2023), No. 2, 71-88.

[9] *F. Colombo, I. Sabadini, D. C. Struppa*, Bicomplex holomorphic functional calculus, Math. Nachr., **287** (2014), No. 10, 1093-1105.

[10] *N. Değirmen, B. Sağır*, On bicomplex $\mathbb{BC}-$modules $l_p^k(\mathbb{BC})$ and some of their geometric properties, GeorgianMath.J., **30** (2023), No. 1, 65-79.

[11] *J. O. González-Cervantes, J. Bory-Reyes*, A bicomplex $(\vartheta,\varphi)-$weighted fractional Borel-Pompeiu type formula, J. Math. Anal. Appl., **520** (2023), No. 2, 126923.

[12] *P. N. Koumantos*, On the mean ergodic theorem in bicomplex Banach modules, Adv. Appl. Clifford Algebr., 33:14 (2023).

[13] *R. Kumar, K. Singh, H. Saini, S. Kumar*, Bicomplex weighted Hardy spaces and bicomplex C^*-algebras, Adv. Appl. Clifford Algebr.,**26** (2016), 217-235.

[14] *Z. Li, B. Dai*, The Schwarz lemma in bicomplex analysis, Math. Methods Appl. Sci., **46** (2023), No. 8, 9351-9361.

[15] *M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa, A. Vajiac*, Bicomplex holomorphic functions: the algebra, geometry and analysis of bicomplex numbers, Birkhäuser, 2015.

[16] *I. J. Maddox*, Elements of Functional Analysis, CUP Archive, 1988.

[17] *E. Malkowsky, V. Rakočević, S. Živković* , Matrix transformations between the sequence space bv^p and certain BK spaces. Bulletin (Académie serbe des sciences et des arts. Classe des sciences mathématiques et naturelles. Sciences mathématiques), 33-46, 2002.

[18] *E. Malkowsky, E. Savas*, Matrix transformations between sequence spaces of generalized weighted means, Appl. Math. Comput., **147** (2004), No. 2, 333-345.

[19] *G. Mani, A. J. Gnanaprakasam, O. Ege, N. Fatima, N. Mlaiki*, Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued $B-$metric space, Symmetry, **15** (2023), No. 2, 297.

[20] *G. B. Price*, An Introduction to Multicomplex Spaces and Functions, Marcel Dekker Inc, New York, 1991.

[21] *T. Reum, H. Toepfer*, Investigation of electromagnetic wave propagation in the bicomplex 3D-FEM using a wavenumber Whitney Hodge operator, COMPEL-The international journal for computation and mathematics in electrical and electronic engineering, **41** (2022), No. 3, 996-1010.

[22] *N. Sager, B. Sağır* , On completeness of some bicomplex sequence spaces, Palest. J. Math., **9** (2020), No. 2, 891-902.

[23] *C. Segre*, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, Math. Ann., **40** (1892), No. 3, 413-467.

[24] *Z. Valkova-Jarvis, V. Poulkov, V. Stoynov, D. Mihaylova, G. Iliev*, A method for the design of bicomplex orthogonal DSP algorithms for applications in intelligent radio access networks, Symmetry, **14** (2022), 613.

[25] *A. Wilansky*, Summability through Functional Analysis. Elsevier, 2000.