U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 4, 2016 ISSN 1223-7027

# FIXED POINTS OF MULTIVALUED MAPS VIA $(G,\varphi)$-CONTRACTION

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*In this paper, we extend the notion of $(G,\varphi)$-contraction, introduced by Ozturk and Girgin, to multi-valued mappings. By using our new notion we prove a fixed point theorem for multi-valued mappings. Our results imply Nadler's theorem, and generalized version of Nadler's theorem on partial metric spaces.*

**Keywords and phrases.** Directed graph, Path, $(G,\varphi)$-contraction maps.
**2010 Mathematics Subject Classification.** 47H10, 54H25.

## 1. Introduction

Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings [2]. They proved the following result.

**Theorem 1.1** *Let $(X,\preccurlyeq)$ be a partially ordered set such that every pair $x,y\in X$ has an upper and lower bound and endowed with the complete metric $d$. Suppose $f:X\to X$ be a continuous monotone (either order preserving or order reversing) mapping satisfying the following conditions:*
*(1) There exists an $\alpha\in(0,1)$ with*

$$d(fx,fy)\le \alpha d(x,y) \text{ for each } x\preccurlyeq y.$$

*(2) There exists $x_0\in X$ with $x_0\preccurlyeq fx_0$ or $fx_0\preccurlyeq x_0$.*
*Then $f$ has a unique fixed point $x^*\in X$ and for each $x\in X$, $\lim_{n\to\infty} f^n(x)=x^*$.*

Afterward, different authors considered the problem of existence of a fixed point for contraction mappings in partially ordered metric spaces. Jachymski [3]

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combined graph theory and fixed point theory and gave some fixed point results, which is a nice generalization of [2]. Some other results for existence of fixed point for single-valued and multi-valued operators in metric spaces with a graph are given by Bojor [4, 5], Beg *et al.* [6], Nicolae *et al.* [8], Aleomraninejad *et al.* [9], Samreen *et al.* [10, 11], Kamran *et al.* [12], and Vetro and Vetro [13].

Very recently, Ozturk and Girgin [14] introduced the notion of $(G,\varphi)$-contractions for single-valued mappings and proved some fixed point theorem. In this paper, we extend $(G,\varphi)$ notion to multivalued mappings and extend some results of [14] to multivalued mappings.

## 2. Preliminaries

In this section, we present some notions which are helpful for the understanding of the
paper.

Let $(X,d)$ be a metric space. We denote by $CB(X)$ the class of nonempty closed and bounded subsets of $X$. For $A,B\in CB(X)$, $H:CB(X)\times CB(X)\to \mathbb{R}$ defined by

$$H(A,B)=\max\{\sup_{b\in B} d(b,A),\sup_{a\in A} d(a,B)\}.$$

is a metric on $CB(X)$. It is called Hausdorff metric generated by the metric $d$.

A directed graph $(V,E)$ consists of a set of vertices $V$, and a set of directed edges $E$. The elements of $E$ are ordered pairs of vertices. A directed graph, can have loops and permits two edges joining the same vertices. More than one edges going in the same direction between the same vertices are called parallel edges, which are not allowed in our results.

Let $G=(V,E)$ be a graph. A path in $G$ from a vertex $x$ to a vertex $y$ of length $N\,(N\in\{0,1,2,\cdots\})$ is a sequence $(x_j)_{j=0}^{N}$ of $N+1$ distinct vertices such that $x_0=x$, $x_N=y$ and $(x_{j-1},x_j)\in E$ $forall\, i=1,2,\cdots,N$. A graph $G=(V,E)$ is said to be connected if there is a path between any two vertices. We say that a directed graph $G$ is weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ is the graph obtained from $G$ by neglecting the directions of the edges of $G$.

For a graph G such that E(G) is symmetric and x is a vertex in G, the subgraph $G_x$ consisting of all edges and vertices which are contained in some path beginning at x is called component of G containing x

Let $G=(V,E)$ be a disconnected graph then different paths in $G$ are known as its components. If $x$ be any vertex in $G$, then be a component of $G$ consisting of all edges and vertices which are included in some path starting at $x$.

In this case $V(G_x)=[x]_G$ ., where $[x]_G$ is the equivalence class of a relation R defined on V(G) by the rule: y R z if there is a path in G from y to z.

We use the following lemmas and the condition in next section.

**Lemma 2.1** *[15] Let $(X,d)$ be a metric space and $A,B\in CB(X)$. Then, for each $q>1$ and for each $a\in A$ there exists $b\in B$ such that*

$$d(a,b)\le qH(A,B). \tag{1}$$

**Lemma 2.2** *[15] Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} H(A_n,A)=0$ for $A\in CB(X)$. If $x_n\in A_n$ and $\lim_{n\to\infty} d(x_n,x)=0$, then $x\in A$.*

**Condition A** ([3], Remark 3.1). *For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n\to x$ and $(x_n,x_{n+1})\in E$ for $n\in\mathbb{N}$, then $(x_n,x)\in E$.*

## 3. Main result

We start this section by defining a subclass of the family of mappings introduced in [14].

Let $\Phi$ be the class of nondecreasing functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfies the following conditions:

$(\varphi_1)$ for every $\{x_n\}\in\mathbb{R}^+$, $\varphi(x_n)\to 0$ if and only if $x_n\to 0$;

$(\varphi_2)$ for every $x_1,x_2\in\mathbb{R}^+$, $\varphi(x_1+x_2)\le\varphi(x_1)+\varphi(x_2)$;

$(\varphi_3)$ $\varphi(qt)\le q\varphi(t)$ for every $t>0$ (where $q>1$).

Subsequently, throughout this paper, we assume that $(X,d)$ be a metric space, $G$ is a directed graph with $V=X$, $\Delta=\{(x,x):x\in X\}\subseteq E$ and $G$ has no parallel edges.

**Definition 3.1** *A mapping $F:X\to CB(X)$ is said to be a $(G,\varphi)$-contraction if following conditions hold:*

*(i) there exists $\alpha\in(0,1)$ and $\varphi\in\Phi$ such that*

$$\varphi(H(Fx,Fy))\le\alpha\varphi(d(x,y)),\ for all\ (x,y)\in E; \tag{2}$$

*(ii) for $(x,y)\in E$, if $u\in Fx$ and $v\in Fy$ are such that*

$$\varphi(d(u,v))\le\sqrt{\alpha}\varphi(d(x,y)), \tag{3}$$

*then $(u,v)\in E$.*

Now we state and proof our main results.

**Theorem 3.2** *Let $(X,d)$ be a complete metric space endowed with the graph $G$ such that Condition A holds, $F:X\to CB(X)$ be a $(G,\varphi)$-contraction and $X_F=\{x\in X:(x,u)\in E\ for some\ u\in Fx\}$. Then the following statements hold:*

*(i) For any* $x \in X_F$ , $F|_{[x]_{\tilde{G}}}$ *has a fixed point;*

*(ii) If* $X_F \neq \varnothing$ *and* $G$ *is weakly connected, then* $F$ *has a fixed point in* $X$ *;*

*(iii) If* $Y = \bigcup \{[x]_{\tilde{G}} : x \in X_F\}$, *then* $F|_Y$ *has a fixed point in* $Y$ *;*

*(iv) If* $F \subset E$ *then* $F$ *has a fixed point;*

*(v)* $FixF \neq \varnothing$ *if and only if* $X_F \neq \varnothing$.

*Proof.*

(i) Let $x_0$ be any arbitrary point in $X_F$, then there exists $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$. Since $F$ is a $(G,\varphi)$-contraction, from (2), we have

$$\varphi[H(Fx_0, Fx_1)] \leq \alpha\varphi[d(x_0, x_1)].$$

Let $q = \frac{1}{\sqrt{\alpha}} > 1$ by using lemma (2.1), we have $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq \frac{1}{\sqrt{\alpha}} H(Fx_0, Fx_1).$$

By applying $\varphi$, we get

$$\begin{aligned}\varphi[d(x_1, x_2)] &\leq \varphi[\frac{1}{\sqrt{\alpha}} H(Fx_0, Fx_1)], \\ &\leq \frac{1}{\sqrt{\alpha}} \varphi[H(Fx_0, Fx_1)], \\ &\leq \frac{1}{\sqrt{\alpha}} \alpha\varphi[d(x_0, x_1)] \\ &. \\ &= \sqrt{\alpha}\varphi[d(x_0, x_1)\end{aligned} \tag{4}$$

From (3) and (4), we get $(x_1, x_2) \in E$. Now by using (2), we have

$$\varphi[H(Fx_1, Fx_2)] \leq \alpha\varphi[d(x_1, x_2)].$$

Again for $q = \frac{1}{\sqrt{\alpha}} > 1$ by using lemma (2.1), we have $x_3 \in Fx_2$ such that

$$d(x_2, x_3) \leq \frac{1}{\sqrt{\alpha}} H(Fx_1, Fx_2).$$

By applying $\varphi$, we get

$$\begin{aligned}\varphi[d(x_2, x_3)] &\leq \varphi[\frac{1}{\sqrt{\alpha}} H(Fx_1, Fx_2)] \\ &\leq \frac{1}{\sqrt{\alpha}} \varphi[H(Fx_1, Fx_2)]\end{aligned}$$

$$\leq \frac{1}{\sqrt{\alpha}}\alpha\varphi[d(x_1,x_2)] \tag{5}$$
$$\leq \frac{1}{\sqrt{\alpha}}\alpha\sqrt{\alpha}\varphi[d(x_0,x_1)] = \alpha\varphi[d(x_0,x_1)].$$

Thus, from (3) and (5), we get $(x_2,x_3)\in E$. Continuing in similar manner, we obtain a sequence $\{x_n\}\subset X$ such that $x_{n+1}\in Fx_n$ with $(x_n,x_{n+1})\in E$ for each $n\in\{0\}\cup\mathbb{N}$ and

$$\varphi[d(x_n,x_{n+1})]\leq(\sqrt{\alpha})^n\varphi[d(x_0,x_1)]\ for each\, n\in\mathbb{N}.$$
(6)

By using triangular inequality, $(\varphi_2)$ and (6), we have

$$\begin{aligned}\varphi[d(x_n,x_{n+m})] &\leq \varphi[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+m-1},x_{n+m})]\\ &\leq \varphi[d(x_n,x_{n+1})]+\varphi[d(x_{n+1},x_{n+2})]+\cdots+\varphi[d(x_{n+m-1},x_{n+m})]\\ &\leq (\sqrt{\alpha})^n\varphi[d(x_0,x_1)]+(\sqrt{\alpha})^{n+1}\varphi[d(x_0,x_1)]+\cdots+\\ &\quad (\sqrt{\alpha})^{n+m-1}\varphi[d(x_0,x_1)]\\ &= (\sqrt{\alpha})^n[1+\sqrt{\alpha}+(\sqrt{\alpha})^2+\cdots+(\sqrt{\alpha})^{m-1}]\varphi[d(x_0,x_1)]\\ &< \frac{(\sqrt{\alpha})^n}{1-(\sqrt{\alpha})}\varphi[d(x_0,x_1)]\end{aligned}$$

Letting $n\to\infty$ in above inequality, we have

$$\lim_{n\to\infty}\varphi[d(x_n,x_{n+m})]=0. \tag{7}$$

By using $(\varphi_1)$, from (7), we have

$$\lim_{n\to\infty}d(x_n,x_{n+m})=0.$$

Thus $\{x_n\}$ is a Cauchy sequence in $X$. As $X$ is complete, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Now by using Condition A, we have $(x_n,x^*)\in E$. Since $F$ is a $(G,\varphi)$-contraction, so we have

$$\varphi[H(Fx_n,Fx^*)]\leq\alpha\varphi[d(x_n,x^*)]\to 0\ as\ n\to\infty.$$

Thus, we have

$$H(Fx_n,Fx^*)\to 0\ as\ n\to\infty.$$

Since $x_{n+1}\in Fx_n$ and $x_n\to x^*$, then by lemma (2.2), $x^*\in Fx^*$. On the other hand, since $(x_n,x^*)\in E$ for each $n\in\mathbb{N}$. Thus, we conclude that $(x_0,x_1...,x_n,x^*)$ is a path in $G$ and also in $\tilde{G}$ from $x_0$ to $x^*$. Therefore, $x^*\in[x_0]_{\tilde{G}}$.

(ii) Since $X_F \neq \emptyset$, then there exists $x_0 \in X_F$. As the graph $G$ is weakly connected, i.e., $\tilde{G}$ is connected. Thus, we have $[x_0]_{\tilde{G}} = X$, then by (i), $F$ has a fixed point in X.
(iii) Given that $Y = \bigcup\{[x]_{\tilde{G}} : x \in X_F\}$. By (i) for any $x \in X_F$, $F|_{[x]_{\tilde{G}}}$ has a fixed point. Since $[x]_{\tilde{G}} \subset Y$. Thus $F|_Y$ has a fixed point in $Y$.
(iii) Let $F \subset E$. Then for all $x \in X$ and $y \in Fx$, we have $(x, y) \in E(G)$. Hence $X_F = X$ and $Y = X$. By using (iii), $F$ has a fixed point.
(iv) If $FixF \neq \emptyset$, then there exists $x \in Fx$. Since $\Delta \subset E$, we have that $(x, x) \in E$ and thus $x \in X_F$. Now, if $X_f \neq \emptyset$, from (i), we have that $FixF \neq \emptyset$.

**Remark 3.3** *Nadler's fixed point theorem [15] can be obtained from Theorem 3.2 by taking $E = X \times X$ and $\varphi(x) = x$ for each $x \geq 0$.*

**Remark 3.4** *Let us remark that than in the case of a partial metric space $X$, Nadler's type fixed point theorem (Theorem 3.2 of [1] and Theorem 3.3 of [7]) can be obtained from Theorem 3.2 by taking $E = X \times X$ and $\varphi(x) = x$ for each $x \geq 0$.*

**Example 3.5** *Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$ and graph $G = (V, E)$ is defined as $V = X$ and $E = \{(x, y) : x, y \geq 0\} \cup \{(x, x) : x \in \mathbb{R}\}$. Define $T : X \to CB(X)$ by*

$$Fx = \begin{cases} [0, \frac{2x}{3}] & \text{if } x \geq 0 \\ & \text{otherwise.} \end{cases}$$

Take $\varphi(x) = \sqrt{x}$ for each $x \geq 0$ and $\alpha = \sqrt{\frac{2}{3}}$. Then for each $(x, y) \in E$, we have

$$\varphi(H(Fx, Fy)) = \alpha\varphi(d(x, y)).$$

Moreover, for $(x, y) \in E$, if $u \in Fx$ and $v \in Fy$ are such that

$$\varphi(d(u, v)) \leq \sqrt{\alpha}\varphi(d(x, y)),$$

then $(u, v) \in E$. Therefore, Theorem 3.2 ensures the existence of fixed point of $F$.

## 4. Application

Let $X = C([a, b], \mathbb{R})$ be the space of all continuous realvalued functions, endowed with the metric $d(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|$ and graph

$G=(V(G),E(G))$ such that $V=X$ and $E(G)=\{(x,y):x(t)\le y(t),\forall t\in[a,b]\}$. Clearly $(X,d)$ is a complete metric space.

As an application, we give existence theorems for Fredholm integral equation of following type.

$$x(t)=g(t)+\int_a^b K(t,s,x(s))ds, \tag{8}$$

where $g:[a,b]\to \mathbb{R}$ is a continuous function, and $K:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing function.

**Theorem 4.1** *Let* $X=C([a,b],\mathbb{R})$ *and let the operator* $F:X\to X$ *is defined by*

$$Fx(t)=g(t)+\int_a^b K(t,s,x(s))ds$$

*where* $g:[a,b]\to\mathbb{R}$ *is a continuous function, and* $K:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ *is continuous and nondecreasing function. Assume that the following conditions hold:*

*(i) for each* $t,s\in[a,b]$ *and* $x,y\in X$ *with* $(x,y)\in E(G)$, *there exists a continuous mapping* $p:[a,b]\to[0,\infty)$ *such that*

$$|K(t,s,x(s))-K(t,s,y(s))|\le p(s)|x(s)-y(s)|;$$

*(ii)* $\left|\int_a^b p(s)ds\right|=q<1$*;*

*(iii) there exists* $x_0\in X$ *such that* $(x_0,Fx_0)\in E(G)$.

*Then the integral equation (8) has atleast one solution.*

*Proof.* First we show that for each $(x,y)\in E(G)$, the the inequalities (2) and (3), hold by assuming $\varphi(t)=t$. As for each $(x,y)\in E(G)$

$$\begin{aligned}|Fx(t)-Fy(t)| &\le \int_a^b |K(t,s,x(s))-K(t,s,y(s))|ds\\ &\le \int_a^b p(s)|x(s)-y(s)|ds\\ &= qd(x,y).\end{aligned}$$

Thus, we get $d(Fx,Fy)\le qd(x,y)$ for each $(x,y)\in X$. Since $K$ is nondecreasing, for each $(x,y)\in E(G)$, we have $(Fx,Fy)\in E(G)$. Moreover, by hypothesis (iii), we have $X_F\ne\varnothing$. Therefore by Theorem 3.2-(v), there exists atleast one fixed point of the operator $F$, that is, integral equation (8) has atleast one solution.

The proof of following theorem runs on the same lines as the proof of above theorem.

**Theorem 4.2** *Let* $X=C([a,b],\mathbb{R})$ *and let the operator* $F:X\to X$ *is defined by*

$$Fx(t) = g(t) + \int_a^b K(t,s,x(s))ds$$

*where* $g:[a,b]\to\mathbb{R}$ *is a continuous function, and* $K:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ *is continuous and nondecreasing function. Assume that the following conditions hold:*

*(i) for each* $t,s\in[a,b]$ *and* $x,y\in X$ *with* $(x,y)\in E(G)$*, we have*

$$|K(t,s,x(s))-K(t,s,y(s))|\le \xi\,|x(s)-y(s)|;$$

*(ii)* $\xi = \dfrac{1}{(b-a)+\tau}$*, for some* $\tau>0$*;*

*(iii) there exists* $x_0\in X$ *such that* $(x_0,Fx_0)\in E(G)$.

*Then the integral equation (8) has at least one solution.*

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