U.P.B. Sci. Bull., Series A, Vol. 78, Iss. 3, 2016 ISSN 1223-7027

BIFURCATION OF LIMIT CYCLES IN A CLASS OF LIÉNARD SYSTEMS WITH A CUSP AND NILPOTENT SADDLE

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In this paper the asymptotic expansion of first-order Melnikov function of a heteroclinic loop connecting a cusp and a nilpotent saddle both of order one for a planar near-Hamiltonian system are given. Next, we consider the bifurcation of limit cycles of a class of hyper-elliptic Liénard system with this kind of heteroclinic loop. It is shown that this system can undergo Poincarè bifurcation from which at most three limit cycles for small positive ε can emerge in the plane. Also using this asymptotic expansion it was shown that there exist parameter values for which three limit cycles exist close to this loop.

Keywords: Melnikov function, Limit cycle, Heteroclinic loop, Chebyshev property.

MSC2000: 34C07; 34C08; 37G15; 34M50

1. Introduction and statements of the main results

Consider the planar differential system

$$\dot{x} = P_n(x,y), \ \ \dot{y} = Q_n(x,y) \tag{1}$$

in which P_n and Q_n are real polynomials of degree n in x,y. The second half of the famous Hilbert's 16th problem is related to maximum number of limit cycles and their relative locations in planar differential system (1) for all possible P_n and Q_n. A weaker version of this problem is proposed by Arnold to study the zeros of Abelian integrals obtained by integrating polynomial 1-forms along ovals of polynomial Hamiltonian, that is called the weak Hilbert's 16th problem [1]. More precisely, consider a perturbed Hamiltonian system

$$\dot{x} = H_y + \varepsilon p(x,y,\varepsilon,\delta), \quad \dot{y} = -H_x + \varepsilon q(x,y,\varepsilon,\delta) \tag{2}$$

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where p, q and H are C^{ω} (analytic) function, ε is a small positive parameter and δ is a vector parameter where $\delta \in D \subset \mathbb{R}^m$ and D is a compact set. Suppose the unperturbed system $(2)|_{\varepsilon=0}$ has a family of periodic orbits L_h continuously depending on $h \in (h_1, h_2)$ defined by the $H(x,y) = h$. Then, there exist a so-called first-order Melnikov function or Abelian integral of the form $I(h,\delta) = \oint_{L_h} qdx - pdy|_{\varepsilon=0}$, which is important in the study of bifurcation of limit cycles from system (2). Recall that an Abelian integral is the integral of a rational 1-form along an algebraic oval. In this paper first we study the asymptotic expansion of the Melnikov function of a Hamiltonian system near a heteroclinic loop through a cusp and a nilpotent saddle, both of order one. We recall that a heteroclinic orbit Γ is an orbit whose ω and α-limit set of its points consist of two different equilibruim points S_1 and S_2. Now a heteroclinic loop consist of equilibrium points S_1, S_2 and two heteroclinic orbits Γ_1 and Γ_2, heteroclinic to S_1 and S_2 and vice versa (see Figure 1). Then, we consider a Liénard system of type $(6,5)$ that is a small perturbation of Hamiltonian vector field with a hyper-elliptic Hamiltonian of degree seven. Our system has a non-degenerate center at $O(\gamma,0)$, a cusp at $S_1(\alpha,0)$ and a nilpotent saddle at $S_2(\beta,0)$ with a heteroclinic loop passing through S_1 and S_2. Without loss of generality we may assume that $\gamma = 0$ and $\alpha < 0 < \beta$. Such a system will have the following form

$$\frac{dX}{d\tau} = Y, \qquad \frac{dY}{d\tau} = X(X-\alpha)^2(X-\beta)^3 := f(X). \tag{3}$$

The Hamiltonian of (3) is $H(X,Y) = \frac{1}{2}Y^2 + F(X)$ where $F(X) = -\int_0^X f(t)dt$. Since this system has a loop passing through S_1 and S_2 then $F(\alpha) = F(\beta)$ which implies $\alpha = -\frac{3}{4}\beta$. Let $X = \beta x, Y = \beta^{-\frac{5}{2}}y$ and $\tau = \beta^{\frac{7}{2}}t$, then system (3) with $\alpha = -\frac{3}{4}\beta$ will be transformed into

$$\dot{x} = y, \qquad \dot{y} = x(x+\frac{3}{4})^2(x-1)^3. \tag{H_0}$$

The Hamiltonian function of (H_0) is

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{7}x^7 + \frac{1}{4}x^6 + \frac{3}{16}x^5 - \frac{29}{64}x^4 - \frac{1}{16}x^3 + \frac{9}{32}x^2, \tag{4}$$

which has a cusp point $S_1(-\frac{3}{4},0)$, a non-degenerate center $O(0,0)$, a nilpotent saddle $S_2(1,0)$ and a heteroclinic loop $L_{\frac{27}{448}}$ where $L_h:\ \{(x,y) \mid H(x,y) = h,\ h \in (0,\frac{27}{448})\}$ (see Fig. 1). Inside $L_{\frac{27}{448}}$, all orbits L_h are closed. We study a perturbation of (H_0) of the form:

$$\dot{x} = y, \qquad \dot{y} = x(x+\frac{3}{4})^2(x-1)^3 + \varepsilon(a+bx^2+x^4)xy, \tag{H_ε}$$

and especially, its Abelian integral given by $I(h) = aI_0(h) + bI_1(h) + I_2(h)$ where $I_k(h) = \oint_{L_h} x^{2k+1}ydx$, and L_h is oriented clockwise. Here $0 < \varepsilon \ll 1$ and (a,b) belongs to any bounded subset of $\mathbb{R}^2$.

The paper is organized as follows. In Section 2, we obtain the asymptotic expansion of the Melnikov function for (2) near a heteroclinic loop connecting a cusp and a

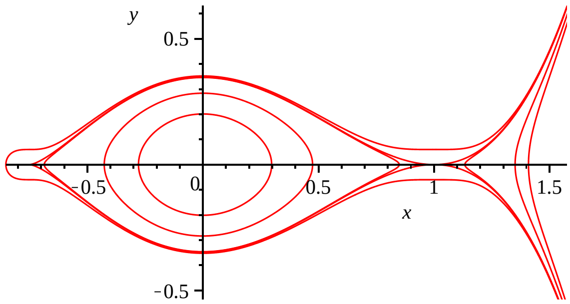

FIGURE 1. Level curves of equation (H_0) for $0 \le h \le \frac{27}{448}$

nilpotent saddle both of order one. In section 3 we show that, if $I(h)$ is not identically zero, system (H_ε) can undergo Poincaré bifurcation from which at most three limit cycles emerges in this period annulus.

2. Asymptotic expansions of Melnikov function $I(h,\delta)$

In this section we consider the Melnikov function $I(h,\delta)$ near a heteroclinic loop through a cusp and a nilpotent saddle both of order one for general planar near-Hamiltonian systems. Let us suppose $p(x,y,0,\delta) = \sum_{i+j\ge 0} a_{ij}x^iy^j$ and $q(x,y,0,\delta) = \sum_{i+j\ge 0} b_{ij}x^iy^j$. Suppose system $(2)|_{\varepsilon=0}$ has a cusp S_1 and a nilpotent saddle S_2. Moreover assume that this system

$(\mathbf{A}_1)$ has a heteroclinic loop denoted by $L_0 := \{(x,y) : H(x,y) = h_s\} = L_1 \cup L_2 \cup \{S_1,S_2\}$, where L_1 and L_2 are heteroclinic orbits connecting singular points S_1 and S_2 so that $\omega(L_1) = \alpha(L_2) = S_2$ and $\omega(L_2) = \alpha(L_1) = S_1$.

$(\mathbf{A}_2)$ In a neighborhood of L_0 there is a family of periodic orbit of $(2)|_{\varepsilon=0}$ denoted by $L_h = \{(x,y) : H(x,y) = h\}$ for $0 < -(h-h_s) \ll 1$.

Theorem 2.1. *Consider the C^ω system (2) and suppose $(2)|_{\varepsilon=0}$ satisfy assumptions (A_1) and (A_2). Then near $h = h_s$ corresponding to heteroclinic loop L_0, Melnikov function of system (2) has the following asymptotic expansion:*

$$\begin{aligned} I(h,\delta) &= \tilde{c}_1 + \tilde{c}_2|h-h_s|^{3/4} + \tilde{c}_3|h-h_s|^{5/6} + \tilde{c}_4|h-h_s|\ln|h-h_s| + \tilde{c}_5(h-h_s) \\ &+ \tilde{c}_6|h-h_s|^{7/6} + \tilde{c}_7|h-h_s|^{5/4} + \tilde{c}_8|h-h_s|^{7/4} + \tilde{c}_9(h-h_s)^2\ln|h-h_s| \\ &+ \tilde{c}_{10}|h-h_s|^{11/6} + O((h-h_s)^2) \end{aligned} \tag{5}$$

in which

$$\begin{aligned} \tilde{c}_1 &= M(h_s,\delta) = \oint_{L_0} qdx - pdy|_{\varepsilon=0} = \sum_{k=1}^{2} \oint_{L_k} (qdx-pdy)|_{\varepsilon=0}, \\ \tilde{c}_2 &= c_2(S_2,\delta), \quad \tilde{c}_3 = c_1(S_1,\delta), \quad \tilde{c}_4 = c_3(S_2,\delta), \quad \tilde{c}_6 = c_3(S_1,\delta), \\ \tilde{c}_7 &= c_5(S_2,\delta), \quad \tilde{c}_8 = c_6(S_2,\delta), \quad \tilde{c}_9 = c_7(S_2,\delta), \quad \tilde{c}_{10} = c_4(S_1,\delta), \end{aligned}$$

where $c_i(S_1,\delta)$, $i = 1,3,4$ are given in [4], *$c_i(S_2,\delta)$, $i = 2,3,5,6,7$ comes from* [11],

$$\tilde{c}_5 = \sum_{k=1}^{2} \int_{L_{0k}} (p_x + q_y - \sigma_k)|_{\varepsilon=0}dt + \int_{L_{03}} (p_x+q_y)|_{\varepsilon=0}dt + b_1\tilde{c}_2 + b_2\tilde{c}_3 + b_3\tilde{c}_4.$$

provided $b_{11}+2a_{20}|_{S_2}=0$ *where for* $k=1,2,\ \sigma_k=(p_x+q_y)|_{S_k}, L_{0k}=L_0\cap U_k, U_k$ *denotes a disk of diameter* $\varepsilon_k\geq 0$ *with centers at* S_k *and* $L_{03}=L_0-(L_{01}\cup L_{02})$ *In particular, if* $\tilde{c}_2=\tilde{c}_3=\tilde{c}_4=0$ *then*

$$\tilde{c}_5=\oint_{L_0}(p_x+q_y)|_{\varepsilon=0}dt=\sum_{k=1}^{2}\int_{L_k}(p_x+q_y)|_{\varepsilon=0}dt. \tag{6}$$

Proof. The idea of the proof is motivated by [8]. Also without loss of generality we may assume $h_s=0$ and we use Theorem 1.2 in [4] and Theorem 2 in [11]. First we use two linear transformations to move S_1 and S_2 to the origin and transform (2) into the forms that was considered in Theorem 1.2 in [4] and Theorem 2 in [11] respectively. For this let

$$\begin{pmatrix} x \\ y \end{pmatrix}=Q_k\begin{pmatrix} u \\ v \end{pmatrix}+S_k,\quad k=1,2, \tag{7}$$

where Q_k are 2×2 matrices satisfying $det(Q_k)=1$. Therefore system (2) become:

$$\dot{u}=\frac{\partial H_k}{\partial v}+\varepsilon p_k(u,v,\varepsilon,\delta),\qquad \dot{v}=-\frac{\partial H_k}{\partial u}+\varepsilon q_k(u,v,\varepsilon,\delta) \tag{8}$$

where

$$\begin{aligned}
H_1(u,v) &= \frac{1}{2}v^2+\sum_{i+j\geq 3}\bar{h}_{ij}u^iv^j=\frac{1}{2}v^2+\bar{h}_{30}u^3+O(u^4+|v||u,v|^2),\\
H_2(u,v) &= -\frac{1}{4}u^4+\sum_{j\geq 5}\tilde{h}_{j0}u^j+v^2\sum_{i+j\geq 0}\tilde{h}_{ij}u^iv^j,\\
p_1(u,v,0,\delta) &= \sum_{i+j\geq 0}\bar{a}_{i,j}u^iv^j,\quad q_1(u,v,0,\delta)=\sum_{i+j\geq 0}\bar{b}_{ij}u^iv^j,\\
p_2(u,v,0,\delta) &= \sum_{i+j\geq 0}\tilde{a}_{ij}u^iv^j,\quad q_2(u,v,0,\delta)=\sum_{i+j\geq 0}\tilde{b}_{ij}u^iv^j.
\end{aligned}$$

For ε_k sufficiently small we can write $I(h,\delta)=\sum_{k=1}^{3}I_k(h,\delta)$, for $0<-h\ll 1$ where

$$I_k(h,\delta)=\oint_{L_{hk}}(qdx-pdy)|_{\varepsilon=0},\ k=1,2,3, L_{hk}=L_h\cap U_k,\ k=1,2,\ L_{h3}=\overline{(L_h\setminus\cup_{k=1}^{2}L_{hk})}.$$

By Theorem 1.2 in [4] we can apply the formula for the local coefficients $c_i(S_1,\delta)$, $i=1,3,4$ to the system (8) with $k=1$ and obtain the following expansion of I_1:

$$I_1(h)=c_1(S_1,\delta)|h|^{5/6}+c_3(S_1,\delta)|h|^{7/6}+c_4(S_1,\delta)|h|^{11/6}+O(h^2)+\varphi_1(h,\delta) \tag{9}$$

By Theorem 2 in [11] we can apply the formula for the local coefficients $c_i(S_2,\delta)$, $i=2,3,5,6,7$ to the system (8) with $k=2$ and obtain the following expansion of I_2:

$$\begin{aligned}
I_2(h) &= c_2(S_2,\delta)|h|^{3/4}+c_3(S_2,\delta)|h|\ln|h|+c_5(S_2,\delta)|h|^{5/4}+c_6(S_2,\delta)|h|^{7/4}\\
&\quad +c_7(S_2,\delta)h^2\ln|h|+O(h^2)+\varphi_2(h,\delta),
\end{aligned} \tag{10}$$

for $0<-h\ll 1$ and $\varphi_k\in C^{\omega}$ at $h=0$, with $\varphi_k(0,\delta)=O(\varepsilon_k)$. We set $\tilde{c}_2=c_2(S_2,\delta),\tilde{c}_3=c_1(S_1,\delta),\tilde{c}_4=c_3(S_2,\delta),\tilde{c}_6=c_3(S_1,\delta),\tilde{c}_7=c_5(S_2,\delta),\tilde{c}_8=c_6(S_2,\delta),$

$\tilde{c}_9 = c_7(S_2,\delta)$ and $\tilde{c}_{10} = c_4(S_1,\delta)$. According to (9)-(10) for $0 < -h \ll 1$ we have

$$\begin{aligned} I(h,\delta) &= \tilde{c}_2|h|^{3/4} + \tilde{c}_3|h|^{5/6} + \tilde{c}_4|h|\ln|h| + \tilde{c}_6|h|^{7/6} + \tilde{c}_7|h|^{5/4} + \tilde{c}_8|h|^{7/4} \\ &\quad + \tilde{c}_9 h^2\ln|h| + \tilde{c}_{10}|h|^{11/6} + O(h^2) + N(h,\delta) \end{aligned} \tag{11}$$

where $N(h,\delta) = \varphi_1(h,\delta)+\varphi_2(h,\delta)+I_3(h,\delta)$. Let $N(h,\delta) = \tilde{c}_1(\delta)+\tilde{c}_5(\delta)h+O(h^2)$. It is easy to see that

$$\tilde{c}_1(\delta) = \varphi_1(0,\delta)+\varphi_2(0,\delta)+I_3(0,\delta) = \lim_{\varepsilon_{1,2}\to 0}[\varphi_1(0,\delta)+\varphi_2(0,\delta)+I_3(0,\delta)]$$

$$= \lim_{\varepsilon_{1,2}\to 0} I_3(0,\delta) = \oint_{L_0}(q dx - p dy)|_{\varepsilon=0} = \sum_{i=1}^{2}\oint_{L_i}(q dx - p dy)|_{\varepsilon=0} = I(0,\delta) \tag{12}$$

since $\varphi_k(0,\delta) = O(\varepsilon_k), k = 1,2$. By (11) and $N(h,\delta) = \tilde{c}_1(\delta)+\tilde{c}_5(\delta)h+O(h^2)$, we have

$$\tilde{c}_5(\delta)+O(h) = N_h(h,\delta) = I_h(h,\delta)+\frac{3}{4}\tilde{c}_2|h|^{-\frac{1}{4}}+\frac{5}{6}\tilde{c}_3|h|^{-\frac{1}{6}}+\tilde{c}_4(1+\ln|h|)+O(|h|^{\frac{1}{6}}).$$

According to [2] we know that $I_h(h,\delta) = \oint_{L_h}(p_x+q_y)|_{\varepsilon=0}dt$, then

$$\begin{aligned} \tilde{c}_5(\delta) = N_h(0,\delta) &= \lim_{h\to 0}\left[\oint_{L_h}(p_x+q_y)|_{\varepsilon=0}dt + \frac{3}{4}\tilde{c}_2|h|^{-\frac{1}{4}}+\frac{5}{6}\tilde{c}_3|h|^{-\frac{1}{6}}+\tilde{c}_4(1+\ln|h|)\right] \\ &= \lim_{h\to 0}\left[\sum_{k=1}^{2}\int_{L_{hk}}(p_x+q_y-\sigma_k)|_{\varepsilon=0}dt + \int_{L_{h3}}(p_x+q_y)|_{\varepsilon=0}dt\right. \\ &\quad \left. + \sum_{k=1}^{2}\int_{L_{hk}}\sigma_k dt + \frac{3}{4}\tilde{c}_2|h|^{-\frac{1}{4}}+\frac{5}{6}\tilde{c}_3|h|^{-\frac{1}{6}}+\tilde{c}_4(1+\ln|h|)\right], \end{aligned}$$

By corollary 1.2 in [8], corollary 4.4 in [6], and considering $\tilde{c}_2 = c_2(S_2,\delta) = -\frac{4\sqrt{2}\sigma_2\Delta_{02}}{3}$, $\tilde{c}_4 = c_3(S_2,\delta) = \frac{\sqrt{2}}{2}((2\tilde{h}_{50}-\tilde{h}_{1,2})\sigma_2+(\tilde{b}_{11}+2\tilde{a}_{20}))$ and $\tilde{c}_3 = c_1(S_1,\delta) = 2\sqrt{2}\sigma_1\bar{h}_3^{-\frac{1}{3}}$ and under condition $\tilde{b}_{11}+2\tilde{a}_{20} = 0$ we have

$$\tilde{c}_5 = \sum_{k=1}^{2}\int_{L_{0k}}(p_x+q_y-\sigma_k)|_{\varepsilon=0}dt + \int_{L_{03}}(p_x+q_y)|_{\varepsilon=0}dt + b_1\tilde{c}_2+b_2\tilde{c}_3+b_3\tilde{c}_4.$$

Obviously $\tilde{c}_2{=}\tilde{c}_3{=}\tilde{c}_4{=}0$ then $\tilde{c}_5(\delta){=}\oint_{L_0}(p_x+q_y)|_{\varepsilon=0}dt{=}\sum_{k=1}^{2}\int_{L_k}(p_x+q_y)|_{\varepsilon=0}dt$. □

Now assume $(2)|_{\varepsilon=0}$ has an elementary center $C(0,0)$ with $H(0,0) = 0$ and our assumption for p and q in the beginning of this section holds. For (x,y) near $C(0,0)$, we may assume $H(x,y) = \frac{1}{2}(x^2+y^2)+\sum_{i+j\geq 3}h_{ij}x^i y^j$. Then $I(h,\delta)$ near the elementary center $C(0,0)$ has the following expansion (see [3])

$$I(h,\delta) = \sum_{j\geq 1} b_j(\delta)h^j,\ 0 < h \ll 1. \tag{13}$$

To obtain more limit cycles we consider the limit cycles bifurcated from the annulus not only near the center $C(0,0)$ but also near the heteroclinic loop L_0, by the following theorem.

Theorem 2.2. *Consider system (2) and assume (5) and (13)* ***hold****. Also suppose there exists* $\delta_0 \in \mathbb{R}^N$ *such that*

$$\begin{aligned} \tilde{c}_1(\delta_0) = \tilde{c}_2(\delta_0) =, \cdots, = \tilde{c}_m(\delta_0) = 0,\ \tilde{c}_{m+1}(\delta_0) \neq 0, \\ b_1(\delta_0) = b_2(\delta_0) = \cdots = b_k(\delta_0) = 0, b_{k+1}(\delta_0) \neq 0 \end{aligned} \tag{14}$$

and

$$\operatorname{rank} \frac{\partial(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_m, b_1, b_2, \cdots, b_k)}{\partial \delta} = m + k. \tag{15}$$

Then system (2) *can have* $m + k + \frac{1 - sgn(I(h_1,\delta_0)I(h_2,\delta_0))}{2}$ *limit cycles for some* (ε, δ) *near* $(0, \delta_0)$ *in which m limit cycles are near the heteroclinic loop* L_0*, k limit cycles are near the center* $C(0,0)$ *and* $\frac{1 - sgn(I(h_1,\delta_0)I(h_2,\delta_0))}{2}$ *limit cycle are surrounding the center* $C(0,0)$*, where* $h_1 = h_s - \varepsilon_1$*,* $h_2 = 0 + \varepsilon_2$ *with* ε_1 *and* ε_2 *are positive and small.*

Theorem 2.2 can be proved similarly as theorem 2.1 proved in [10] by using implicit function theorem, then here we omit its proof for the sake of brevity.

3. Limit Cycles of System (H_ε)

In this section we provide a complete description of the number and the possible configurations of limit cycles for system (H_ε).

3.1. Bifurcation of limit cycles from the period annulus

In this subsection we study the least upper bound of the number of limit cycles which can bifurcate from the period annulus of system (H_0). We use an algebraic criterion developed in [5] and [7] to study the related Abelian integral $I(h)$ of system (H_ε). But first we give the following definition:

Definition 3.1. *The base functions* $\{I_i(h),\ i = 1, \ldots, n\}$ *in the Melnikov function* $I(h)$ *is said to be a Chebyshev system with accuracy k, if number of zeros of any nontrivial linear combination* $\alpha_0 I_0(h) + \alpha_1 I_1(h) + \cdots + \alpha_n I_n(h)$ *counted with multiplicity is at most* $n + k - 1$*. If* $k = 0$*, it simply said to be a Chebyshev system.*

We will show that the base functions $\{I_0(h),\ I_1(h),\ I_2(h)\}$ in $I(h)$ form a Chebyshev system with accuracy one. Hence, the number (multiplicity taken into account) of isolated zeros of $I(h)$ in the open interval $(0, \frac{27}{448})$ is at most three.
Let us consider Abelian integral $I(h)$ with Hamiltonian function (4), which is a linear combination of $n := 3$ Abelian integral $\{I_0(h), I_1(h), I_2(h)\}$, where $I_i(h) = \oint_{\gamma_h} x^{2i+1} y^{2s-1} dx,\ i = 0,1,2$ with $s = 1$ and γ_h is a closed curve defined by

$$\gamma_h = \left\{(x,y) :\ A(x) + B(x) y^{2m} = h,\ 0 < h < 27/448\right\},$$

with $A(x) = -\frac{1}{7}x^7 + \frac{1}{4}x^6 + \frac{3}{16}x^5 - \frac{29}{64}x^4 - \frac{1}{16}x^3 + \frac{9}{32}x^2$, $m = 1$ and $B(x) = \frac{1}{2}$. First we check if the hypothesis of Theorem A in [7] are satisfied. We notice that the projection of period annulus on the x-axis is $(-\frac{3}{4},1)$ and $xA'(x)>0$ for all $x\in(-\frac{3}{4},1)\setminus\{0\}$. Therefore, there exists an invertible function $z(x)$ with $-\frac{3}{4}<z(x)<0$ such that $A(x)=A(z(x))$ for $0<x<1$. But in this case $m=1$, $n=4$ and $s=1$, so one of the hypothesis of Theorem A in [7], i.e. $s>m(n+k-2)$ is not fulfilled. However it is possible to overcome this problem using Lemma 4.1 in [5], and obtain new Abelian integrals for which the corresponding s is large enough to verify the inequality. Here we have to promote the power s to three such that the condition $s>n+k-2$ holds. On the oval γ_h for $i=0,1,2$ we have

$$I_i(h)=\frac{1}{h}\oint_{\gamma_h}\left(A(x)+\frac{y^2}{2}\right)x^{2i+1}ydx=\frac{1}{2h}\left(\oint_{\gamma_h}2x^{2i+1}A(x)ydx+\oint_{\gamma_h}x^{2i+1}y^3dx\right). \tag{16}$$

Using Lemma 4.1 in [5] with $k=3$ and $F(x)=2x^{2i+1}A(x)$ to get $\oint_{\gamma_h}2x^{2i+1}A(x)ydx=\oint_{\gamma_h}G_i(x)y^3dx$, where $G_i(x)=\frac{d}{3dx}(\frac{2x^{2i+1}A(x)}{A'(x)})=\frac{g_i}{42(4x+3)^3(x-1)^4}$, and

$$\begin{aligned} g_i &= x^{2i+1}\Big[512(1+i)x^7-(1024i+960)x^6-(832i+560)x^5+(2464i+1708)x^4\\ &\quad +(322i+21)x^3-(2282i+1036)x^2+(84i+126)x^1+756(i+1)\Big] \end{aligned}$$

By (16) we obtain

$$\begin{aligned} I_i(h) &= \frac{1}{2h}\oint_{\gamma_h}\left(x^{2i+1}+G_i(x)\right)y^3dx=\frac{1}{4h^2}\oint_{\gamma_h}(2A(x)+y^2)(x^{2i+1}+G_i(x))y^3dx\\ &= \frac{1}{4h^2}\left(\oint_{\gamma_h}2(x^{2i+1}+G_i(x))A(x)y^3dx+\oint_{\gamma_h}(x^{2i+1}+G_i(x))y^5dx\right). \end{aligned} \tag{17}$$

Again using Lemma 4.1 in [5] with $k=5$ and $F(x)=2(x^{2i+1}+G_i(x))A(x)$ to get

$$\oint_{\gamma_h}2(x^{2i+1}+G_i(x))A(x)y^3dx=\oint_{\gamma_h}H_i(x)y^5dx,$$

where $H_i(x)=\frac{d}{5dx}(\frac{2(x^{2i+1}+G_i(x))A(x)}{A'(x)})=\frac{h_i}{2940(4x+3)^6(x-1)^8}$, and

$$\begin{aligned} h_i=\big[&(262144i^2+1900544i+1638400)x^{14}-(1048576i^2+7159808i+5955584)x^{13}\\ &+(196608i^2+1388544i+1490944)x^{12}+(4227072i^2+24812544i+17810944)x^{11}\\ &-(4024320i^2+22312704i+16308992)x^{10}-(7096320i^2+34743744i+20726048)x^9\\ &+(10295040i^2+47087376i+28808416)x^8+(5986176i^2+24246516i+12042730)x^7\\ &-(12830076i^2+48586986i+26187560)x^6-(2313640i^2+7908278i+3090675)x^5\\ &+(8987188i^2+29109332i+14592984)x^4+(103488i^2+360444i-338394)x^3\\ &-(3443328i^2+10638684i+4836888)x^2+(127008i^2+444528i+381024)x\\ &+(571536i^2+2000376i+1428840)\big]\,x^{2i+1} \end{aligned}$$

From (17) we obtain $4h^2I_i(h)=\oint_{\gamma_h}f_i(x)y^5dx\equiv\tilde{I}_i(h)$, where $f_i(x)=x^{2i+1}+G_i(x)+H_i(x)$. It is clear that $\{\tilde{I}_0,\tilde{I}_1,\tilde{I}_2\}$ is a Chebyshev system with accuracy 1 on $(0,\frac{27}{448})$ if and only if $\{I_0,I_1,I_2\}$ is as well. Now since $s=3$ and the condition $s>m(n+k-2)$

satisfies, we can apply Theorem A in [7]. We set $l_i(x) = \left(\frac{f_i}{A'}\right)(x) - \left(\frac{f_i}{A'}\right)(z(x))$, where $z(x)$ is an analytic involution defined by $A(x) = A(z(x))$ with $A(1) = A(-\frac{3}{4})$, then $z = z(x)$ for $x \in (0,1)$ satisfy $A(x) - A(z) = -\frac{1}{448}(x-z)q(x,z) = 0$, where

$$\begin{aligned}(x-z)q(x,z) &= 64(x^7-z^7) - 112(x^6-z^6) - 84(x^5-z^5)\\ &\quad +203(x^4-z^4) + 28(x^3-z^3) - 126(x^2-z^2).\end{aligned}$$

So $\frac{d}{dx}l_i(x) = \frac{d}{dx}(\frac{f_i}{A'})(x) - [\frac{d}{dz}(\frac{f_i}{A'})(z(x))].\frac{dz}{dx}$, where $\frac{dz}{dx} = -\frac{\partial q(x,z)}{\partial x}/\frac{\partial q(x,z)}{\partial z}$. We will show that $\{l_0, l_1, l_2\}$ is a Chebyshev system with accuracy 1 on $x \in (0,1)$:

Lemma 3.1.

(i) $W[l_0](x) \neq 0$ *for all* $x \in (0,1)$;

(ii) $W[l_0,l_1](x) \neq 0$ *for all* $x \in (0,1)$;

(iii) $W[l_0,l_1,l_2](x)$ *has a unique simple root* $x^* \in (0,1)$.

Proof. Using Maple we find that

$$\begin{aligned}W[l_0](x) &= \frac{8(x-z)w_0(x,z)}{21xz(4x+3)^5(x-1)^7(4z+3)^5(z-1)^7},\\ W[l_0,l_1](x) &= -\frac{64(x-z)^3 w_1(x,z)}{441(4x+3)^{10}(x-1)^{14}(4z+3)^{10}(z-1)^{14}W_{01}(x,z)},\\ W[l_0,l_1,l_2](x) &= -\frac{1024(x-z)^6 w_2(x,z)}{3087(4x+3)^{14}(x-1)^{21}(4z+3)^{14}(z-1)^{21}W_{01}^3(x,z)},\end{aligned}$$

where $w_i(x,z)$, $i = 0,1,2$ are polynomials with long expression in (x,z) and

$$\begin{aligned}W_{01}(x,z) &= 320xz^4 - 448xz^3 + 128zx^4 + 256x^2z^3 + 192z^2x^3 - 168zx^2 - 224zx^3\\ &\quad -336z^2x^2 - 252z^2x + 406zx - 126 + 64x^5 - 112x^4 - 84x^3 + 203x^2\\ &\quad +28x - 336z^3 - 560z^4 + 384z^5 + 56z + 609z^2.\end{aligned}$$

The resultant with respect to z between $W_{01}(x,z)$ and $q(x,z)$ is

$$p_{01}(x) = 4619872982007808(64x^5 - 112x^4 - 84x^3 + 203x^2 + 28x - 126)(4x+3)^{11}(x-1)^{14}.$$

It is easy to see that $p_{01}(x)$ does not have a zero in $(0,1)$. This implies that $W[l_0,l_1]$ and $W[l_0,l_1,l_2](x)$ are well defined in $-\frac{3}{4} < z < 0 < x < 1$.

In order to determine if these three Wronskians have zeros on $(0,1)$, we shall rely on the symbolic computations by Maple to compute the resultant between $w_i(x,z)$, $i = 0,1,2$ and $q(x,z)$ with respect to z, and then we apply Sturm's Theorem.

Case (i). The resultant with respect to z between $q(x,z)$ and $w_0(x,z)$ is $R(q,w_0,z) = (4x+3)^{28}(x-1)^{38}p_0(x)$, where $p_0(x)$ is a polynomial of degree 42 in x. By Sturm's Theorem we get that $p_0(x) \neq 0$ for all $x \in (0,1)$. Thus, $w_0(x,z) = 0$ and $q(x,z) = 0$ have no common roots. This fact implies that $W[l_0](x) \neq 0$ for all $x \in (0,1)$.

Case (ii). The resultant with respect to z between $q(x,z)$ and $w_1(x,z)$ is $R(q,w_1,z) =$

$(4x+3)^{62}(x-1)^{86}p_1(x)$, where $p_1(x)$ is a polynomial of degree 98 in x. By applying Sturm's Theorem, there is a unique root $x^0 \approx 0.9662426637 \in (0,1)$, applying Sturm's Theorem to another resultant $R(q,w_2,z)$, we get that it has a root on $(-\frac{3}{4},0)$. In order to make sure if there exist a common root of $w_1(x,z)$ and $q(x,z)$ satisfying $-\frac{3}{4} < z < 0 < x < 1$, we use a program to compute the intervals in which all common roots exist (see [9]). From the result of the program, there are 6 pairs of common roots of $w_1(x,z)$ and $q(x,z)$ in 6 pairs of intervals as follow:

```
[[x=[1.424285889, 1.424293518], z=[-.2157721701, -.2157721701]],
 [x=[.9662399292, .9662475586], z=[-.7610659790, -.7610583496]],
 [x=[1.019927979, 1.019935608], z=[-.7454480087, -.7454480087]],
 [x=[-.2157745361, -.2157669067], z=[1.424286746, 1.424286746]],
 [x=[-.7454528809, -.7454452515], z=[1.019933805, 1.019933805]],
 [x=[-.7610659790, -.7610583496], z=[.9662399292, .9662475586]]].
```

But none pair satisfy $-\frac{3}{4} < z < 0 < x < 1$. Therefore $W[l_0,l_1] \neq 0$ for all $x \in (0,1)$.
Case (iii). The resultant with respect to z between $q(x,z)$ and $w_2(x,z)$ is $R(w_2,q,z) = (4x+3)^{97}(x-1)^{141}p_2(x)$, where $p_2(x)$ is a polynomial of degree 164 in x. By Sturm's Theorem we get that $p_2(x)$ has a unique root in the interval $(0,1)$ at $x_1^* \approx .7990077271$. Substituting $x = x_1^*$ into $q(x,z)$, we find that $q(x_1^*,z)$ has also a unique root in the interval $(-\frac{3}{4},0)$ at $z_1^* \approx -.6569892317$. To make sure if (x_1^*,z_1^*) is the common root of $q(x,z)$ and $w_2(x,z)$, by program of the previous stage we obtain 4 pairs of common roots of $w_2(x,z)$ and $q(x,z)$ in 4 pairs of intervals as follow:

```
[[x=[0.981296921, 0.981302643], z=[-.7186737061, -.7186660767]],
 [x=[.7990055084, .7990112305], z=[1.171028761, 1.171028761]],
 [x=[-.7186737061, -.7186660767], z=[0.981296921, 0.981302643]],
 [x=[1.171028137, 1.171035767], z=[.7990077271, .7990077271]]].
```

Only the first pair denoted by (x^*,z^*) satisfies $-\frac{1}{2} < z < 0 < x < 1$. Therefore, there is a unique $x^* \in (0,1)$ such that $W[l_0,l_1,l_2](x^*) = 0$. Now we want to show that x^* is a simple root. We denote $W[l_0,l_1,l_2](x)$ by $W_3(x,z(x))$. Its derivative is

$$\frac{dW_3}{dx} = -\frac{1024(x-z)^5 w_3(x,z)}{3087(4x+3)^{15}(x-1)^{22}(4z+3)^{15}(z-1)^{22}W_{01}^5(x,z)}$$

where $w_3(x,z)$ is polynomials with long expression in (x,z). Taking $w_3(x,z)$ in place of $w_2(x,z)$ in our program, we obtain seven common roots of $w_3(x,z)$ and $q(x,z)$, but non of them is equal (x^*,z^*) and because $q(x^*,z^*) = q(x^*,z(x^*)) = 0$, then $w_3(x^*,z(x^*)) \neq 0$, hence x^* is a simple root of $w_2(x,z)$. □

Based on the above arguments we obtain the following theorem.

Theorem 3.1. *The collection $\{I_0(h),I_1(h),I_2(h)\}$ is a Chebyshev system with accuracy one on the interval $(0,\frac{27}{448})$. Hence, if the Abelian integral $I(h)$ is not identically zero then in any compact subinterval of $(0,\frac{27}{448})$ and for all values of parameters (a,b) it has at most three zeros, counting the multiplicities, And the number of limit cycles bifurcating from the period annulus is at most three.*

3.2. Asymptotic expansion of $I(h)$ near the endpoints of $(0, \frac{27}{448})$

In this subsection we study the asymptotic expansion of Abelian integral $I(h)$ at the end points $h=0$ and $h=\frac{27}{448}$, respectively. To obtain the asymptotic expansion of $I(h)$ when $h \to 0^+$, we compute $I(h)$ near the elementary equilibrium $(0,0)$. Let $x = r\cos\theta$, $y = r\sin\theta$, then the oval $\gamma_h := \{H(x,y)=h\}$ will be transformed into

$$\begin{aligned} &r\Big(224 - 64r^5\cos^7\theta + 112r^4\cos^6\theta + 84r^3\cos^5\theta \\ &-203r^2\cos^4\theta - 28r\cos^3\theta - 98\cos^2\theta\Big)^{\frac{1}{2}} - \sqrt{448h} = 0, \end{aligned}$$

with $0 < h, r \ll 1$. Let $\rho = \sqrt{448h}$ and define $F(r,\rho)$ to be left hand expression of the above equality. Then by applying the Implicit Function Theorem to $F(r,\rho)=0$ at $(r,\rho)=(0,0)$, we obtain that there exists a unique smooth function $r=\varphi(\rho)$ and a small positive number $0<\delta \ll 1$ such that $F(\varphi(\rho),\rho) \equiv 0$ for $0<\rho<\delta$. It can be checked that $\varphi(\rho)$ has the following expansion

$$\begin{aligned} \varphi(\rho) =& \frac{\rho}{\sqrt{-98\cos^2\theta+224}} + \frac{\rho^2\cos^3\theta}{14(49\cos^4\theta - 224\cos^2\theta + 256)} \\ &+ \frac{\cos^4\theta(193\cos^2\theta - 464)\rho^3}{56\sqrt{-98\cos^2\theta+224}(343\cos^6\theta - 2352\cos^4\theta + 5376\cos^2\theta - 4096)} + O(\rho^4). \end{aligned} \tag{18}$$

Let us compute $I(h)$ in the coordinate system (r,θ). From (18) we have

$$\begin{aligned} I(h) &= \oint_{\gamma_h} (a+bx^2+x^4)xy\,dx = \iint_{\mathrm{int}\gamma_h} (a+bx^2+x^4)x\,dx\,dy \\ &= \int_0^{2\pi} d\theta \int_0^{\varphi(\rho)} \left(a + br^2\cos^2\theta + r^4\cos^4\theta\right) r^2\cos\theta\, dr. \end{aligned} \tag{19}$$

Note that $h = \frac{\rho^2}{448}$. Thus we obtain the asymptotic expansion of $I(h)$ as $h \to 0^+$,

$$\begin{aligned} I(h) =& \frac{64}{81}\pi h^2 \left[a + \left(\frac{5980}{729}a + \frac{80}{27}b\right)h + \left(\frac{74894971}{2834352}a + \frac{269360}{6561}b + \frac{2240}{243}\right)h^2 \right. \\ &\left. + \left(\frac{30174980537}{1033121304}a + \frac{3602369650}{14348907}b + \frac{10721536}{59049}\right)h^3 + O(h^4)\right]. \end{aligned} \tag{20}$$

We set

$$b_1 = \frac{64}{81}\pi a,\ b_2 = \frac{64}{81}\pi\left(\frac{5980}{729}a + \frac{80}{27}b\right),\ b_3 = \frac{64}{81}\pi\left(\frac{74894971}{2834352}a + \frac{269360}{6561}b + \frac{2240}{243}\right)$$

Now let us apply Theorem 2.1 to system (H_ε) and obtain the asymptotic expansion of Abelian integral $I(h)$ as $h \to (\frac{27}{448})^-$. It is clear that on the loop $L_{\frac{27}{448}}$ we have $H(x,y) = \frac{27}{448}$, which implies that $y^{\pm} = \pm\frac{1}{56}\sqrt{(56x+42)}(x-1)^2(4x+3)$, thus

$$\tilde{c}_1(\delta) = I(0,\delta) = 2\int_{-\frac{3}{4}}^{1} (a+bx^2+x^4)xy^+\,dx = \frac{343\sqrt{2}}{266048640}(8398a + 3876b + 2049)$$

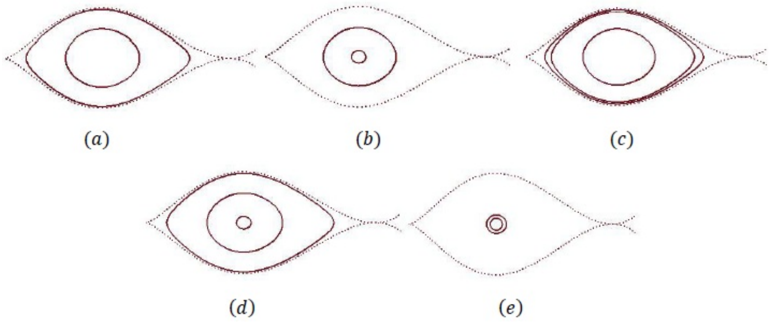

FIGURE 2. Distribution of limit cycles bifurcated from the period annulus of system (H_ε).

For $S_1 = (-\frac{3}{4}, 0)$ let $X = x + \frac{3}{4}$, $Y = y$, and still denote X, Y by x, y , respectively. Then system (H_ε) becomes $\dot{x} = y,\ \dot{y} = -(x - \frac{3}{4})x^2(x - \frac{7}{4})^3 + \varepsilon y q_1(x)$, where

$$q_1(x) = x^5 - \frac{15}{4}x^4 + (b + \frac{45}{8})x^3 - (\frac{9}{4}b + \frac{135}{32})x^2 + (a + \frac{27}{16}b + \frac{405}{256})x - \frac{3}{4}a - \frac{27}{64}b - \frac{243}{1024}.$$

For $\varepsilon = 0$ the Hamiltonian function is $H(x,y) = -\frac{343}{256}x^3 + \frac{49}{16}x^4 - \frac{21}{8}x^5 + x^6 - \frac{1}{7}x^7$.

Thus from Theorem 2.1 we see that $\tilde{c}_3 = c_1(S_1, \delta) = \frac{2^{\frac{1}{6}}(-343)^{\frac{2}{3}}}{21952}(768a + 432b + 243)$. For the nilpotent saddle $S_2 = (1, 0)$, we make the transformations $X = \frac{\sqrt{7}}{2}(1 - x)$, $Y = y$ and $T = -\frac{\sqrt{7}}{2}t$ and still denote X, Y and T by x, y and t , respectively. Then system (H_ε) becomes $\dot{x} = y,\ \dot{y} = x^3 - \frac{30}{49}\sqrt{7}x^4 + \frac{288}{343}x^5 - \frac{128}{2401}\sqrt{7}x^6 + \varepsilon y q_2(x)$, where

$$q_2(x) = \frac{64x^5 - 160\sqrt{7}x^4}{343} + \frac{16(b+10)x^3 - 8\sqrt{7}(3b+10)x^2}{49} + \frac{4(a+3b+5)x}{7} - \frac{(1+a+b)}{\sqrt{7}}.$$

For $\varepsilon = 0$ the Hamiltonian function is $H(x,y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{6}{49}\sqrt{7}x^5 - \frac{48}{343}x^6 + \frac{128}{16807}\sqrt{7}x^7$. Therefore from Theorem 2.1 we have

$$\tilde{c}_2 = c_2(S_2, \delta) = \frac{8\sqrt{14}}{21}(1 + a + b)\Delta_{0,2}, \qquad \tilde{c}_4 = c_3(S_2, \delta) = \frac{4\sqrt{2}}{98}(29 + a + 15b)$$

3.3. Distribution of limit cycles of system (H_ε)

In this section, we will use the coefficients given in previous section and apply Theorem 2.2, to discuss distributions of limit cycles of system (H_ε).

(1) $\tilde{c}_1(\delta) = 0$ yields $b = -\frac{13}{6} - \frac{683}{1292}a$. If $\delta_0 = (a, -\frac{13}{6} - \frac{683}{1292}a, 1)$, then

$$b_1(\delta_0) = \frac{64}{81}\pi a, \quad \tilde{c}_2(\delta_0) = \frac{8\sqrt{14}}{21}(\frac{609}{1292} - \frac{7}{6}a)\Delta_{0,2},$$

if we fix $a \in (-\infty, 0) \cup (\frac{261}{646}, \infty)$, then $b_1(\delta_0)\tilde{c}_2(\delta_0) < 0$, and $\frac{1 - sgn(I(h_1,\delta_0)I(h_2,\delta_0))}{2} = 1$ for $h_1 = \varepsilon_1$ $h_2 = \frac{27}{448} - \varepsilon_2$ with ε_1 and ε_2 positive and small. Note that $rank\left(\frac{\partial(\tilde{c}_1)}{\partial(a,b,1)}\right) = 1$ and we can apply theorem (2.2) to deduce that there exists some $(a, b, 1)$ near

$(a, -\frac{13}{6} - \frac{683}{1292}a, 1)$ for $a \in (-\infty, 0) \cup (\frac{261}{646}, \infty)$ and ε positive and small, such that system (H_ε) has 2 limit cycles, 1 limit cycle is near the heteroclinic loop $L_{\frac{27}{448}}$ and 1 limit cycle is surrounding the center L_0, see Fig. 2(a).
Using the similar method as in (1) we obtain other cases as bellow:
(2) For ε positive and sufficiently small, there exists some $(a, b, 1)$ near $(0, b, 1)$ for $b \in (-\frac{683}{1292}, 0)$, such that system (H_ε) has 2 limit cycles, 1 limit cycle is near the center L_0, 1 limit cycle is surrounding the center L_0(See Fig. 2(b)).
(3) For ε positive and sufficiently small, there exists some $(a, b, 1)$ near $(\frac{261}{646}, -\frac{907}{646}, 1)$, such that system (H_ε) has 3 limit cycles, two limit cycles are near the heteroclinic loop $L_{\frac{27}{448}}$ and one limit cycle is surrounding the center L_0(See Fig. 2(c)).
(4) For ε positive and small, there exists some $(a, b, 1)$ near $(0, -\frac{683}{1292}, 1)$, such that system (H_ε) has 3 limit cycles, for which one is near the heteroclinic loop $L_{\frac{27}{448}}$, one is near the center L_0 and one is surrounding the center L_0(See Fig.2(d)).
(5) For ε positive and small, there exists some $(a, b, 1)$ near $(0, 0, 1)$, such that system (H_ε) has 2 limit cycles which are near center L_0(See Fig.2(e)).

4. **Conclusions**

Based on the expansions of Melnikov function and the results of subsections 3.1 and 3.2 we proved the following theorem.

Theorem 4.1. *There exist some parameter values such that the Abelian integral $I(h)$ has three isolated zeros in $(0, \frac{27}{448})$.*

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