MILD SOLUTIONS FOR NEUTRAL CONFORMABLE FRACTIONAL ORDER FUNCTIONAL EVOLUTION EQUATIONS USING MEIR-KEELER TYPE FIXED POINT THEOREM

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Our mission is to demonstrate the existence, uniqueness, attractiveness, and controllability of mild solutions to neutral conformable fractional-order functional evolution equations, specifically of order between 1 and 2. These intriguing equations encompass finite delay, all while adhering to local conditions within a separable Banach space. By invoking Meir-Keeler's fixed-point Theorem and enhancing it with measures of noncompactness, we establish the existence of these solutions. To highlight the potency of our approach, we present a captivating example.

Keywords: Neutral functional differential equation, mild solution, finite delay, infinite delay, fixed point, condensing operator, measure of noncompactness, conformable fractional.
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1. Introduction

This paper establishes criteria for the existence, uniqueness, attractivity, and controllability of mild solutions to conformable fractional-order neutral functional evolution equations with finite delay. The analysis is conducted within the framework of a separable Banach space, where the completeness of the space and local conditions are leveraged to prove the existence of a unique mild solution, forming the cornerstone of our study.

The conformable derivative, introduced by Khalil et al. [29], has advanced fractional calculus beyond classical definitions [33, 30] and enabled diverse applications (see eg. [9, 11]). Recent work has extended this framework to complex settings: Liang et al. [32] and Bouaouid et al. [21, 20] studied impulsive differential equations using semigroup theory, while Bouaouid et al. [12, 8] and Atraoui et al. [13] applied fixed-point theorems to prove existence and controllability. Further contributions by Baghli et al. [14] and Agarwal et al. [7] extended this approach to controllability, also, see [1, 16, 17, 18, 19, 4, 5, 6, 26, 31, 34]. Researchers have also utilized measures of noncompactness to address solution existence challenges across various contexts (see eg. [10, 15, 27]).

Firstly, we study in Section 3 the conformable fractional order functional evolution equations with local conditions of the form:

$$D^c[D^c(\psi(s) - \mathfrak{Y}(s, \psi_s))] = \mathfrak{P}(\psi(s) - \mathfrak{Y}(s, \psi_s)) + \Psi(s, \psi_s), \quad \text{a.e. } s \in I := [0, +\infty); \qquad (1)$$

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$$\psi(s)=\eta(s), \quad s\in\mathcal{H}:=[-b,0],\ where\ 0<b<+\infty, \quad D^c\psi(0)=\vartheta\in\mathcal{D}; \tag{2}$$

Let Ψ and $\mathcal{Y}$: $I\times C([-b,0],\mathcal{D})\to\mathcal{D}$ denote given functions, η: $[-b,0]\to\mathcal{D}$ represent a continuous function, $\mathfrak{P}$: $D(\mathfrak{P})\subset\mathcal{D}\to\mathcal{D}$ serves as the infinitesimal generator of a strongly continuous cosine function, represented by a family of bounded linear operators $(\mathfrak{C}(s))_{s\in\mathbb{R}}$. This elegant framework ensures that $\mathfrak{S}(s)=\int_0^s\mathfrak{C}(x)dx$, weaving a seamless integration of the operators over the real line, and $\mathcal{D}$ denote a real separable Banach space equipped with norm $|\cdot|$, while D^c represents a fractional conformable derivative of order $0<c\leq 1$.

Let ψ_s denote, for all $s\geq 0$, the function in $C([-b,0],\mathcal{D})$ defined as $\psi_s(\theta)=\psi(s+\theta)$. Here, $\psi_s(\cdot)$ captures the state history from $s-b$ up to the current time s. Additionally, we will explore the attractiveness of mild solutions to conformable fractional-order functional evolution equations subject to local conditions $(1)-(2)$. Moreover, we will delineate adequate conditions to guarantee the controllability of mild solutions across the semi-infinite interval $I=[0,+\infty)$ for conformable fractional-order functional evolution equations characterized by the following conditions

$$D^c[D^c(\psi(s)-\mathcal{Y}(s,\psi_s))]=\mathfrak{P}(\psi(s)-\mathcal{Y}(s,\psi_s))+\Psi(s,\psi_s)+\mathcal{B}\mathcal{U}(s); \tag{3}$$

$$\psi(s)=\eta(s), \quad s\in\mathcal{H}, \quad D^c\psi(0)=\vartheta\in\mathcal{D}; \tag{4}$$

where $\mathfrak{P}$, Ψ, $\mathcal{Y}$, and η are defined as in problem $(1)-(2)$, the control function $\mathcal{U}(\cdot)$ is provided in $L^2(I,\mathcal{D})$, representing the Banach space of admissible control functions, and $\mathcal{B}$ stands as a bounded linear operator mapping from $\mathcal{D}$ to $\mathcal{D}$.

Ultimately, we furnish an illustrative example demonstrating the abstract theory expounded in the preceding results.

2. Introductory concepts

In this part, we present symbols, explanations, and fundamental principles drawn from multivalued analysis.

The notation $BC(I,\mathcal{D})$ represents the Banach space comprising all functions that are both bounded and continuous from I to $\mathcal{D}$, where the norm is defined as: $\|\psi\|_{BC}=\sup\{\ |\psi(s)|\ :\ s\in I\}$.

Consider the space BC_∞ defined as $\{\psi:[-b,+\infty)\to\mathcal{D},\ \psi|_{[0,s]}$ is bounded and continuous for $s>0\}$, with the norm: $\|\psi\|_{BC_\infty}=\sup\{\ |\psi(s)|\ :\ s\in[0,T]\}$, here $T=\sup\{s>0:\ \psi|_{[0,s]}$ is bounded and continuous$\}$.

Definition 2.1. *(Khalil et al.* [29]*) The conformable fractional derivative* of order $0<c\leq 1$ for a function $\psi(\cdot)$ is defined as

$$D^c\psi(s)=\lim_{t\to 0}\frac{\psi(s+ts^{1-c})-\psi(s)}{t}, \quad s>0;$$
$$D^c\psi(0)=\lim_{t\to 0}D^c\psi(t),$$

Definition 2.2. *(see eg.* [10, 15, 27]*)* Let $\mathcal{F}_{\mathcal{D}}$ the bounded subsets of $\mathcal{D}$ so the map $\mathfrak{A}:\mathcal{F}_{\mathcal{D}}\to[0,+\infty)$ denotes *the Kuratowski measure of noncompactness which is given by*

$$\mathfrak{A}(\mathcal{F})=\inf\{\alpha>0:\mathcal{F}\subseteq\bigcup_{j+1}^{k}\mathcal{F}_j\ and\ diam(\mathcal{F}_j)\leq\alpha\},\ here\ \mathcal{F}\in\mathcal{F}_{\mathcal{D}}.$$

Definition 2.3. Let's say we have a nonempty subset $\mathcal{F}$ within the Banach space $\mathcal{D}$, and consider any arbitrary measure of noncompactness $\mathfrak{A}$ defined on $\mathcal{D}$. We define $\mathfrak{M}:\mathcal{F}\to\mathcal{D}$ as a *Meir-Keeler condensing operator* if it meets the following criteria: $\mathfrak{M}$ is both continuous and bounded, and for any given $\beta>0$, there exists $\mu>0$ such that if $\beta<\mathfrak{A}(\mathcal{R})<\beta+\mu$, then $\mathfrak{A}(\mathfrak{M}(\mathcal{R}))\leq\beta$ holds true for every bounded subset $\mathcal{R}$ of $\mathcal{F}$.

Lemma 2.1. *(see [25]) Consider $\mathcal{D}$ as a Banach space, and let $\mathcal{F}$ be a subset of $C(I,\mathcal{D})$ that is both bounded and equicontinuous. Then, the function $\mathfrak{A}(\mathcal{F}(s))$ remains continuous over the interval I, and $\mathfrak{A}_I(\mathcal{F})$ equals the maximum value of $\mathfrak{A}(\mathcal{F}(s))$ for s in I.*

Theorem 2.1. *(Meir-Keeler's Theorem [8]) Let $\mathcal{F}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathcal{D}$. If $\mathfrak{M}:\mathcal{F}\to\mathcal{F}$ is a continuous Meir-Keeler condensing operator, then $\mathfrak{M}$ guarantees at least one fixed point, and the collection of all such fixed points within $\mathcal{F}$ forms a compact set.*

Definition 2.4. *(see [23]) We characterize solutions of Equations (1) − (2) as locally attractive if there exists a closed ball $\overline{B}(\psi^*,\sigma)$ within the space BC, where $\psi^*\in BC$, such that for any solutions ψ and $\widetilde{\psi}$ of Equations (1) − (2) within $\overline{B}(\psi^*,\sigma)$, the following convergence occurs:* $\lim_{s\to+\infty}(\psi(s)-\widetilde{\psi}(s))=0$.

3. Existence results

In this section, we reveal our main findings regarding the existence of solutions to problems (1) − (2). Before presenting and verifying this result, we introduce the notion of its mild solution.

Definition 3.1. *We define the mild solution $\psi\in C([-b,+\infty),\mathcal{D})$ of the problem (1) − (2) as follows*

$$\psi(s)=\begin{cases}\eta(s), & \text{if } s\in\mathcal{H};\\[2mm] \mathfrak{C}\left(\frac{s^c}{c}\right)\,[\eta(0)-\mathcal{Y}(0,\eta(0))]+\mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta+\mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\left(\frac{s^c-t^c}{c}\right)\,\Psi(t,\psi_t)\,dt, & \text{if } s\in I;\end{cases}$$

We must introduce the following hypotheses, which will be utilized subsequently:

(i) The function $\Psi: I\times C(\mathcal{H},\mathcal{D})\to\mathcal{D}$ is carathéodory function and there exist a continuous function $\mathcal{O}: I\to I$ such that

$$|\Psi(s,u)|\le\mathcal{O}(s)\|u\|,$$

$$\mathfrak{A}(\Psi(s,\mathcal{F}))\le\mathcal{O}(s)\mathfrak{A}(\mathcal{F}),$$

and $\mathcal{O}^*:=\sup_{s\in I}\int_0^s t^{c-1}\mathcal{O}(t)dt<\infty$, for all $s\in I,\ u\in C(\mathcal{H},\mathcal{D})$, *bounded set* $\mathcal{F}\subset C([-b,+\infty),\mathcal{D})$ *and* $0<c\le 1$;

(ii) The cosine operator $\mathfrak{C}(s)_{s\in\mathbb{R}}$ is uniformly continuous and there exist constants $\mathcal{M}_c^{\mathfrak{C}}$, $\mathcal{M}_c^{\mathfrak{S}}$ both greater than zero, such that

$$\sup_{s\in I}\|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\|\le\mathcal{M}_c^{\mathfrak{C}}\ \text{and}\ \sup_{s\in I}\|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\|\le\mathcal{M}_c^{\mathfrak{S}}.$$

(iii) The function $\mathcal{Y}: I\times C([-b,0],\mathcal{D})\to\mathcal{D}$ is carathéodory function and there exist $\mathcal{Y}^*>0$ such that

$$|\mathcal{Y}(s,u)|\le\mathcal{Y}^*\|u\|,$$

$$\mathfrak{A}(\mathcal{Y}(s,\mathcal{F}))\le\mathcal{Y}^*\,\mathfrak{A}(\mathcal{F}),$$

$$\{s\mapsto\mathcal{Y}(s,u),\ u\in\mathcal{F}\}\text{ is equicontinuous on each compact interval of } I,$$

for all $s\in I,\ u\in C(\mathcal{H},\mathcal{D})$, *bounded set* $\mathcal{F}\subset C([-b,+\infty),\mathcal{D})$.

Theorem 3.1. *Given assumptions $(i)-(iii)$, if $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*<1$, then problem (1) − (2) admits at least one mild solution over BC.*

Proof. We initiate the transformation of problem $(1)-(2)$ into a fixed-point problem. Let's examine the operator $\mathfrak{M} : BC([-b,+\infty),\mathcal{D}) \to BC([-b,+\infty),\mathcal{D})$, which is delineated as follows:

$$\mathfrak{M}(\psi)(s) = \begin{cases} \eta(s), & \text{if } s \in [-b,0]; \\ \mathfrak{C}\Big(\frac{s^c}{c}\Big)\ [\eta(0) - \mathcal{Y}(0,\eta(0))] + \mathfrak{S}\Big(\frac{s^c}{c}\Big)\vartheta + \mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\ \Psi(t,\psi_t)\ dt, & \text{if } s \in I; \end{cases}$$

The operator $\mathfrak{M}$ maps BC into BC. Specifically, for $\psi \in BC$ and for any $s \in I$ we have:

$$\begin{aligned} |\mathfrak{M}(\psi)(s)| &\le \|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\||\eta(0) + \mathcal{Y}(0,\eta(0))| + \|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\|\|\vartheta\| + |\mathcal{Y}(s,\psi_s)| \\ &+ \int_0^s t^{c-1}\|\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\|\ |\Psi(t,\psi_t)|dt \\ &\le \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*) + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)\|\psi\|_{BC}. \end{aligned}$$

which imply $\mathfrak{M} \in BC$.

Furthermore, suppose $l \ge \frac{\mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*)+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|}{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)}$, and let B_l denote the closed ball in BC centered at the origin with radius l. consider $\psi \in B_l$ and $s \in I$, we get

$$|\mathfrak{M}(\psi)(s)| \le \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*) + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)l$$

Thus, $\|\mathfrak{M}(\psi)\|_{BC} \le l$.

Now we prove that $\mathfrak{M} : B_l \to B_l$ satisfies the assumptions of Meir-Keeler's fixed point Theorem.

Firstly, we establish that $\mathfrak{M}$ exhibits continuity within B_l. Let $\{\psi_n\}$ be a sequence such that $\psi_n \to \psi$ in B_l. We have

$$|\mathfrak{M}(\psi_n)(s) - \mathfrak{M}(\psi)(s)| \le |\mathcal{Y}(s,(\psi_s)_n) - \mathcal{Y}(s,\psi_s)| + \mathcal{M}_c\int_0^s t^{c-1}|\Psi(t,(\psi_t)_n) - \Psi(t,\psi_t)|dt$$

and by (i) and (iii) we get $\Psi(t,(\psi_t)_n) \to \Psi(t,\psi_t)$ and $\mathcal{Y}(t,(\psi_t)_n) \to \mathcal{Y}(t,\psi_t)$ as $n \to +\infty$ for ae. $t \in I$ and by the Lebesgue dominated convergence Theorem we conclude that

$$\|\mathfrak{M}(\psi_n) - \mathfrak{M}(\psi)\|_{BC} \to 0,\ as\ n \to \infty.$$

Thus, $\mathfrak{M}$ is continuous.

Secondly, we observe that $\mathfrak{M}(B_l) \subset B_l$, which is evident.

Moving on, we note that $\mathfrak{M}(B_l)$ demonstrates equicontinuity on every compact interval X' of I, let $x_1,\ x_2 \in X'$ with $x_2 > x_1$ we have

$$\begin{aligned} |\mathfrak{M}(\psi)(x_1) - \mathfrak{M}(\psi)(x_2)| &\le \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ (1+\mathcal{Y}^*)\|\eta\| \\ &+ \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ \|\vartheta\| + |\mathcal{Y}(x_1,\psi_{x_1}) - \mathcal{Y}(x_2,\psi_{x_2})| \\ &+ \int_0^{x_1} t^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c-t^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c-t^c}{c}\Big)\|_{B(\mathcal{D})}\ |\Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\int_{x_1}^{x_2} t^{c-1}\ |\Psi(t,\psi_t)|dt \end{aligned}$$

As $x_1 \to x_2$, the uniformly continuity property of the operators $\mathfrak{C}(s)$ and $\mathfrak{S}(s)$ indicate that the right part of the previous enequality converges to zero. This confirms the equicontinuity of $\mathfrak{M}$.

Additionally, we establish the equiconvergence of $\mathfrak{M}(B_l)$. For $s \in I$ and $\psi \in B_l$, we find

$$|\mathfrak{M}(\psi)(s)| \leq \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*) + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)dx + \mathcal{Y}^*)l$$

Consequently, $|\mathfrak{M}(\psi)(s)| \to l',\ as\ s \to +\infty$. Where $l' \leq \mathcal{M}_c^{\mathfrak{C}}\|\eta\|(1+\mathcal{Y}^*)+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)l$. Here $\mathcal{O}^* := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{O}(x)dx$. Therefore,

$$|\mathfrak{M}(\psi)(s) - \mathfrak{M}(\psi)(+\infty)| \to 0,\ \ s \to +\infty.$$

Finally, we confirm that the Meir-Keeler type condition is satisfied.

For any given $\beta > 0$, there exists $\mu > 0$ such that if $\beta < \mathfrak{A}_I(\mathcal{R}) < \beta + \mu$, then $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq \beta$ for any $\mathcal{R} \subset B_l$ where $\mathfrak{A}_I(\mathcal{R}) = \max_{x\in I}\mathfrak{A}(\mathcal{R}(x))$.
We have

$$\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)) \leq (\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}).$$

Since $\mathfrak{M}(\mathcal{R})$ is bounded and equicontinuous of all $\mathcal{R} \subset B_l$. Then

$$\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) = \max_{s\in I}\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)).$$

Therefore $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq (\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}) \leq \beta \Rightarrow \mathfrak{A}_I(\mathcal{R}) \leq \frac{\beta}{\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}$.
Then for any given $\beta > 0$ and taking $\mu = \left(\frac{1-\mathcal{Y}^*-\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}{\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*}\right)\beta - \epsilon$ such that $\epsilon > 0$, we obtain

$$\beta < \mathfrak{A}_I(\mathcal{R}) < \beta + \mu \Rightarrow \mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq \beta,\ \ for\ any\ \mathcal{R} \subset B_l$$

Hence $\mathfrak{M}$ is a Meir-Keeler condensing operator.
Through these steps, we ensure that the conditions required for Meir-Keeler' s fixed-point Theorem [8] are satisfied by $\mathfrak{M} : B_l \to B_l$. Therefore, we may conclude that $\mathfrak{M}$ has a fixed point ψ that provides a mild solution to the problem $(1)-(2)$.

3.1. **Uniqueness results.** Subsequently, we present our main finding concerning the existence and uniqueness of solutions to problem $(1)-(2)$. Before proceeding with the demonstration of this outcome, we establish the following conditions.

$(i)'$ The function $\Psi : I \times C(\mathcal{H}, \mathcal{D}) \to \mathcal{D}$ is carathéodory function and there exist a continuous function $\mathcal{O} : I \to I$ such that

$$|\Psi(s,u) - \Psi(s,v)| \leq \mathcal{O}(s)\|u - v\|,$$

$$\Psi^* = \sup_{s\in I}\int_0^s t^{c-1}\Psi(t,0)dt < \infty$$

$$\mathfrak{A}(\Psi(s,\mathcal{F})) \leq \mathcal{O}(s)\mathfrak{A}(\mathcal{F}),$$

and $\mathcal{O}^* := \sup_{s\in I}\int_0^s t^{c-1}\mathcal{O}(t)dt < \infty$, for all $s \in I,\ u,v \in C(\mathcal{H},\mathcal{D}),$ *bounded set* $\mathcal{F} \subset C([-b,+\infty),\mathcal{D})$ *and* $0 < c \leq 1$;

$(iii)'$ The function $\mathcal{Y} : I \times C([-b,0],\mathcal{D}) \to \mathcal{D}$ is carathéodory function, continuous according to its first variable and there exist $\mathcal{Y}^* > 0$ such that

$$|\mathcal{Y}(s,u) - \mathcal{Y}(s,v)| \leq \mathcal{Y}^*\|u - v\|,$$

$$\mathfrak{A}(\mathcal{Y}(s,\mathcal{F})) \leq \mathcal{Y}^*\mathfrak{A}(\mathcal{F}),$$

$$\{s \mapsto \mathcal{Y}(s,u),\ u \in \mathcal{F}\} \text{ is equicontinuous on each compact interval of } I,$$

$$\mathcal{Y}' = \sup_{s\in I}|\mathcal{Y}(s,0)| < +\infty,$$

for all $s \in I,\ u \in C(\mathcal{H},\mathcal{D}),$ *bounded set* $\mathcal{F} \subset C([-b,+\infty),\mathcal{D})$.

Theorem 3.2. *Given assumptions* $(i)' - (ii)$ *and* $(iii)'$, *if* $\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* < 1$, *then problem* $(1)-(2)$ *possesses a unique mild solution over BC.*

Proof. By following analogous procedures as those in the proof of Theorem 3.2, we confirm the presence of a sole mild solution. Particularly noteworthy is the adjustment of the radius estimation to $l \geq \frac{\mathcal{M}_c^{\mathfrak{C}}[\|\eta\|(1+\mathcal{Y}^*)+\mathcal{Y}']+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\Psi^*+\mathcal{Y}'}{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)}$.

Now, we proceed to demonstrate uniqueness. Suppose ψ and ψ^* are both mild solutions of the problem $(1)-(2)$, then,

$$\begin{aligned}|\psi(s)-\psi^*(s)| &= |\mathfrak{M}\psi(s)-\mathfrak{M}\psi^*(s)|\\ &\leq |\mathcal{Y}(s,\psi_s)-\mathcal{Y}(s,\psi_s^*)| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}|\Psi(t,\psi_t)-\Psi(t,\psi_t^*)|dt\\ &\leq (\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\|\psi-\psi^*\|_{BC}\end{aligned}$$

then $(1-(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*))\|\psi-\psi^*\|_{BC} \leq 0$ therefore $\psi=\psi^*$. Hence the uniqueness of the mild solution.

3.2. Attractivity of Mild Solutions. In this section, we explore the local attractiveness of solutions to problem $(1)-(2)$.

Theorem 3.3. *Given assumptions $(i)'-(ii)$ and $(iii)'$, if $\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*<1$, and let ψ^* be a solution of $(1)-(2)$, and $\overline{B}(\psi^*,\sigma)$ represent the closed ball in BC such that: $\sigma \geq \frac{\mathcal{M}_c^{\mathfrak{C}}[\|\eta\|(1+\mathcal{Y}^*)+\mathcal{Y}']+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|+\mathcal{M}_c^{\mathfrak{S}}\Psi^*+\mathcal{Y}'}{1-(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)}$ then the problem $(1)-(2)$ exhibits attractivity.*

Proof. For $\psi \in \overline{B}(\psi^*,\sigma)$, by $(i)'-(ii)$ *and* $(iii)'$, we get

$$\begin{aligned}|\mathfrak{M}(\psi)(s)-\psi^*(s)| &= |\mathfrak{M}(\psi)(s)-\mathfrak{M}(\psi^*)(s)|\\ &\leq |\mathcal{Y}(s,\psi_s^*)-\mathcal{Y}(s,\psi_s)| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}|\Psi(t,\psi_t^*)-\Psi(t,\psi_t)|dt\\ &\leq \mathcal{Y}^*\|\psi_s^*-\psi_s\| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}\mathcal{O}(t)\|\psi_t^*-\psi_t\|dt\\ &\leq (\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\sigma \leq \sigma\end{aligned}$$

consequently, $\mathfrak{M}(\overline{B}(\psi^*,\sigma)) \subset \overline{B}(\psi^*,\sigma)$ then for each solutions $\psi,\ \widetilde{\psi} \in \overline{B}(\psi^*,\sigma)$ of $(1)-(2)$ and $s \in I$, we have

$$|\psi(s)-\widetilde{\psi}(s)| \leq (\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\ \|\widetilde{\psi}-\psi\|_{BC}$$

hence

$$\|\widetilde{\psi}-\psi\|_{BC}=0$$

As a result, the problem solutions $(1)-(2)$ are locally attractive.

3.3. Controllability results. This section delineates the controllability outcomes for the system $(3)-(4)$. Before delving into this, we introduce a specific type of solutions for problem $(3)-(4)$.

Definition 3.2. *We define the mild solution $\psi \in C([-b,+\infty),\mathcal{D})$ of the problem $(3)-(4)$ as follows*

$$\psi(s)=\begin{cases}\eta(s), & \text{if } s\in\mathcal{H};\\ \mathfrak{C}\left(\frac{s^c}{c}\right)[\eta(0)-\mathcal{Y}(0,\eta(0))]+\mathfrak{S}\left(\frac{s^c}{c}\right)\vartheta+\mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\left(\frac{s^c-t^c}{c}\right)\Psi(t,\psi_t)\,dt,+\int_0^s t^{c-1}\mathfrak{S}\left(\frac{s^c-t^c}{c}\right)\mathcal{B}\mathcal{U}(t)dt, & \text{if } s\in I;\end{cases}$$

Definition 3.3. *The system $(3)-(4)$ is considered controllable if, for every initial function $\eta \in C([-b,0],\mathcal{D})$ and $\widehat{\psi} \in \mathcal{D}$, there exists some natural number $n \in \mathbb{N}$ and a control function $\mathcal{U} \in L^2([0,n],\mathcal{D})$ such that the resulting mild solution $\psi(\cdot)$ satisfies the terminal condition $\psi(n)=\widehat{\psi}$.*

We will adopt the assumptions $(i)-(iii)$ from Section 3, along with the introduction of the following additional assumption, which will be consistently assumed hereafter:

(iv) For all n integer, the linear operator $\mathfrak{V} : L^2([0,n],\mathcal{D}) \to \mathcal{D}$ defined by

$$\mathfrak{V}\mathcal{U} = \int_0^n x^{c-1}\mathfrak{S}\Big(\frac{n^c - x^c}{c}\Big)\ \mathcal{B}\mathcal{U}(x)dx,$$

possesses a pseudo-invertible operator $\widetilde{\mathfrak{V}}^{-1}$, which maps functions from $L^2([0,n],\mathcal{D})$ to the space $L^2([0,n],\mathcal{D})$ excluding the kernel of $\mathfrak{V}$, and is bounded. Additionally, $\mathcal{B}$ is bounded, satisfying:

$$\|\mathcal{B}\| \leq \widetilde{\mathcal{N}}\ and\ \|\widetilde{\mathfrak{V}}^{-1}\| \leq \widetilde{\mathcal{N}}_1.$$

(v) There exists a continuous function function $\mathcal{K}_{\mathfrak{V}} : [0,n] \to \mathbb{R}_+$ such that: for any bounded subset $\mathcal{F} \subset \mathcal{D}$, we have : $\mathfrak{A}(\widetilde{\mathfrak{V}}^{-1}(\mathcal{F})(s)) \leq \mathcal{K}_{\mathfrak{V}}(s)\mathfrak{A}(\mathcal{F})$, $s \in I$ and $\mathcal{K}' := \sup_{s\in I}\int_0^s t^{c-1}\mathcal{K}_{\mathfrak{V}}(t)dt < \infty$ for all $0 < c \leq 1$.

Theorem 3.4. *Assume that* $(i)-(v)$ *hold. If* $\max\{\mathcal{M}_c\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}+(\mathcal{Y}^*+\mathcal{M}_c\mathcal{O}^*)[1+\mathcal{M}_c\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}],\ (\mathcal{Y}^*+\mathcal{M}_c\mathcal{O}^*)(1+\mathcal{M}_c\widetilde{\mathcal{N}}\ \mathcal{K}')\} < 1$, *then the problem* $(3)-(4)$ *is controllable on* $[-b,+\infty)$.

Proof. Convert problem $(3)-(4)$ into a fixed-point problem. We examine the operator $\mathfrak{M} : BC_\infty([-b,+\infty),\mathcal{D}) \to BC_\infty([-b,+\infty),\mathcal{D})$, defined as:

$$\mathfrak{M}(\psi)(s) = \begin{cases} \eta(s), & \text{if } s \in [-b,0];\\ \mathfrak{C}\Big(\frac{s^c}{c}\Big)\ [\eta(0) - \mathcal{Y}(0,\eta(0))] + \mathfrak{S}\Big(\frac{s^c}{c}\Big)\vartheta + \mathcal{Y}(s,\psi_s) & \\ +\int_0^s t^{c-1}\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\ \Psi(t,\psi_t)\ dt, + \int_0^s t^{c-1}\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\ \mathcal{B}\mathcal{U}(t)dt, & \text{if } s \in I; \end{cases}$$

Using assumption (iv), for arbitrary function $\psi(\cdot)$, we define the control

$$\mathcal{U}_\psi(s) = \widetilde{\mathfrak{V}}^{-1}\Big[\widehat{\psi} - \mathfrak{C}\Big(\frac{n^c}{c}\Big)\ [\eta(0) - \mathcal{Y}(0,\eta(0))] - \mathfrak{S}\Big(\frac{n^c}{c}\Big)\vartheta - \mathcal{Y}(n,\psi_n)$$
$$-\int_0^n t^{c-1}\mathfrak{S}\Big(\frac{n^c-t^c}{c}\Big)\ \Psi(t,\psi_t)\ dt\Big](s)$$

Noting that, we have

$$|\mathcal{U}_\psi(s)| \leq \widetilde{\mathcal{N}}_1\Big[|\widehat{\psi}| + \mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\| + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)\ \|\psi\|_{BC_\infty}\Big]$$

The operator $\mathfrak{M}$ maps BC_∞ into BC_∞. Specifically, the mapping $\mathfrak{N}(\psi)$ is continuous on $[-b,n]$ for any $\psi \in BC_\infty$ we have:

$$|\mathfrak{M}(\psi)(s)| \leq \|\mathfrak{C}\Big(\frac{s^c}{c}\Big)\|[|\eta(0)| + |\mathcal{Y}(0,\eta(0))|] + \|\mathfrak{S}\Big(\frac{s^c}{c}\Big)\|\|\vartheta\| + |\mathcal{Y}(s,\psi_s)|$$
$$+ \int_0^s t^{c-1}\|\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\||\Psi(t,\psi_t)|dt + \int_0^s t^{c-1}\|\mathfrak{S}\Big(\frac{s^c-t^c}{c}\Big)\|\|\mathcal{B}\|\ |\mathcal{U}_\psi(t)|dt$$
$$\leq (\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + \|\psi\|_{BC_\infty}\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}$$
$$+ (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big].$$

Which imply $\mathfrak{M} \in BC_\infty$.

Furthermore, suppose $l \geq \frac{(\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\|+\mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})}{1-\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}+(\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*+\mathcal{Y}^*)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big]}$, and let B_l denote the closed ball in BC_∞ centered at the origin with radius l. Let $\psi \in B_l$ and $s \in I$, we get

$$|\mathfrak{M}(\psi)(s)| \leq l\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c} + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^* + \mathcal{Y}^*)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big]$$
$$+ (\mathcal{M}_c^{\mathfrak{C}}(1 + \mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}).$$

Thus, $\|\mathfrak{M}(\psi)\|_{BC_\infty} \leq l$.

We now aim to demonstrate that $\mathfrak{M} : B_l \to B_l$ fulfills the prerequisites of Meir-Keeler's fixed-point Theorem.

Firstly, we establish that $\mathfrak{M}$ exhibits continuity within B_l. Let $\{\psi_k\}$ be a sequence such that $\psi_k \to \psi$ in B_l. We have

$$\begin{aligned}|\mathfrak{M}(\psi_k)(s) - \mathfrak{M}(\psi)(s)| &\leq |\mathcal{Y}(s,(\psi_s)_k) - \mathcal{Y}(s,\psi_s)| + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}|\Psi(t,(\psi_t)_k) - \Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_0^s t^{c-1}|\mathcal{U}_{\psi_k}(t) - \mathcal{U}_{\psi}(t)|dt \\ &\leq |\mathcal{Y}(s,(\psi_s)_k) - \mathcal{Y}(s,\psi_s)| \\ &+ \mathcal{M}_c^{\mathfrak{S}}\Big(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}\Big)\int_0^n t^{c-1}|\Psi(t,(\psi_t)_k) - \Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}\Big[|\widehat{\psi}_k - \widehat{\psi}| + |\mathcal{Y}(n,(\psi_n)_k) - \mathcal{Y}(n,\psi_n)|\Big]\end{aligned}$$

Using (i) and (iii), we have $\Psi(s,(\psi_s)_k) \to \Psi(s,\psi_s)$ and $\mathcal{Y}(s,(\psi_s)_k) \to \mathcal{Y}(s,\psi_s)$ as $k \to +\infty$ for almost every $s \in [0,n]$. Then, by the Lebesgue dominated convergence Theorem: $\|\mathfrak{M}(\psi_n) - \mathfrak{M}(\psi)\|_{BC_\infty} \to 0,\ as\ n \to \infty$. Thus, $\mathfrak{M}$ is continuous.

Secondly, we observe that $\mathfrak{M}(B_l) \subset B_l$, which is evident.

Moving on, we note that $\mathfrak{M}(B_l)$ demonstrates equicontinuity on every compact interval $X' = [0,n]$, let $x_1,\ x_2 \in X'$ with $x_2 > x_1$ we have

$$\begin{aligned}|\mathfrak{M}(\psi)(x_1) - \mathfrak{M}(\psi)(x_2)| &\leq \|\mathfrak{C}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{C}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ (1 + \mathcal{Y}^*)\|\eta\| \\ &+ \|\mathfrak{S}\Big(\frac{x_2^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c}{c}\Big)\|_{B(\mathcal{D})}\ \|\vartheta\| + |\mathcal{Y}(x_1,\psi_{x_1}) - \mathcal{Y}(x_2,\psi_{x_2})| \\ &+ \int_0^{x_1} t^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c - t^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - t^c}{c}\Big)\|_{B(\mathcal{D})}\ |\Psi(t,\psi_t)|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\int_{x_1}^{x_2} t^{c-1}\ |\Psi(t,\psi_t)|dt \\ &+ \int_0^{x_1} t^{c-1}\|\mathfrak{S}\Big(\frac{x_2^c - t^c}{c}\Big) - \mathfrak{S}\Big(\frac{x_1^c - t^c}{c}\Big)\|_{B(\mathcal{D})}\ \|\mathcal{B}\|\|\mathcal{U}_\psi(t)\|dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_{x_1}^{x_2} t^{c-1}\ \|\mathcal{U}_\psi(t)\|dt\end{aligned}$$

As $x_1 \to x_2$, the uniformly continuity property of $\mathfrak{C}(s)$ and $\mathfrak{S}(s)$ indicate that the right part of the previous inequality converges to zero. This confirms the equicontinuity of $\mathfrak{M}$.

Additionally, we establish the equiconvergence of $\mathfrak{M}(B_l)$. For $s \in X'$ and $\psi \in B_l$, we find

$$|\mathfrak{M}(\psi)(s)| \leq (\mathcal{M}_c^{\mathfrak{C}}(1 + \mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + l\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}$$
$$+ (\mathcal{M}_c^{\mathfrak{S}}\int_0^s x^{c-1}\mathcal{O}(x)dx + \mathcal{Y}^*)(1 + \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big].$$

Consequently, $|\mathfrak{M}(\psi)(s)| \to l',\ as\ s \to +\infty$. Where $l' \leq (\mathcal{M}_c^{\mathfrak{C}}(1+\mathcal{Y}^*)\|\eta\| + \mathcal{M}_c^{\mathfrak{S}}\|\vartheta\|)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c}) + l\Big[\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c} + (\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*dx + \mathcal{Y}^*)(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}_1\frac{n^c}{c})\Big]$. Here $\mathcal{O}^* := \sup_{s\in I}\int_0^s x^{c-1}\mathcal{O}(x)dx$. Therefore,

$$|\mathfrak{M}(\psi)(s) - \mathfrak{M}(\psi)(+\infty)| \to 0,\ s \to +\infty.$$

Finally, we confirm that the Meir-Keeler type condition is satisfied.
For any given $\beta > 0$, there exists $\mu > 0$ such that if $\beta < \mathfrak{A}_I(\mathcal{R}) < \beta + \mu$, then $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq \beta$ for any $\mathcal{R} \subset B_l$ where $\mathfrak{A}_I(\mathcal{R}) = \max_{s\in I}\mathfrak{A}(\mathcal{R}(s))$. We have

$$\mathfrak{A}(\mathcal{U}_{\mathcal{R}}(s)) \leq \mathcal{K}_{\mathfrak{V}}(s)(\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}),$$

which imply

$$\begin{aligned}\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)) &\leq \mathcal{Y}^*\mathfrak{A}_I(\mathcal{R}) + \mathcal{M}_c^{\mathfrak{S}}\int_0^s t^{c-1}\mathcal{O}(t)\mathfrak{A}_I(\mathcal{R})dt \\ &+ \mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\int_0^s t^{c-1}\mathcal{K}_{\mathfrak{V}}(t)(\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R})dt \\ &\leq (1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\ \mathcal{K}')(\mathcal{Y}^* + \mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R})\end{aligned}$$

Since $\mathfrak{M}(\mathcal{R})$ is bounded and equicontinuous of all $\mathcal{R} \subset B_l$. Then

$$\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) = \max_{s\in I}\mathfrak{A}(\mathfrak{M}(\mathcal{R})(s)).$$

Therefore $\mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq (1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\ \mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)\mathfrak{A}_I(\mathcal{R}) \leq \beta \Rightarrow \mathfrak{A}_I(\mathcal{R}) \leq \frac{\beta}{(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\ \mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)}$.
Then for any given $\beta > 0$ and taking $\mu = \Big(\frac{1-(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\ \mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)}{(1+\mathcal{M}_c^{\mathfrak{S}}\widetilde{\mathcal{N}}\ \mathcal{K}')(\mathcal{Y}^*+\mathcal{M}_c^{\mathfrak{S}}\mathcal{O}^*)}\Big)\beta - \epsilon$ such that $\epsilon > 0$, we obtain

$$\beta < \mathfrak{A}_I(\mathcal{R}) < \beta + \mu \Rightarrow \mathfrak{A}_I(\mathfrak{M}(\mathcal{R})) \leq \beta,\ for\ any\ \mathcal{R} \subset B_l$$

Hence $\mathfrak{M}$ is a Meir-Keeler condensing operator.
Through these steps, we ensure that the conditions required for Meir-Keeler' s fixed-point Theorem [8] are satisfied by $\mathfrak{M} : B_l \to B_l$. Therefore, we may conclude that $\mathfrak{M}$ has a fixed point ψ that provides the controllability of the problem $(3) - (4)$.

4. Examples

Example 4.1. *To showcase the practical application of our results, let $\mathcal{E}$ denote a nonempty bounded open set in $\mathbb{R}^2$. We explore the following conformable fractional differential equation:*

$$D_s^{\frac{2}{3}}[D_s^{\frac{2}{3}}\psi(s,x) - \mathcal{Y}(s,\psi(s-b,x))] = D_x^2[\psi(s,x) - \mathcal{Y}(s,\psi(s-b,x))] + \Psi(s,\psi(s-b,x)),$$
$$x \in \mathcal{E},\ s \in [0,+\infty); \tag{5}$$
$$\psi(s,x) = 0,\ s \in [0,+\infty),\ x \in \partial\mathcal{E}; \tag{6}$$
$$\psi(s,x) = \eta(s,x);\quad D_s^{\frac{2}{3}}[\psi(0,x)] = \vartheta,\ s \in [-b,0],\ x \in \mathcal{E}. \tag{7}$$

Here, $b > 0$ and we have

$$\Psi(s,\psi(s-b,x)) = \frac{\exp -s}{7}\sin\psi(s-b,x),$$
$$\mathcal{Y}(s,\psi(s-b,x)) = \frac{\exp -s}{2}\tanh\psi(s-b,x),$$

taking $\mathcal{D} = L^2(\mathcal{E})$ and defining $\mathfrak{P}$ as follows: $\mathfrak{P}\psi = D_x^2\psi,\ \psi \in D(\mathfrak{P})$ and

$$D(\mathfrak{P}) = \{\psi \in \mathcal{H}(\mathcal{D}),\ \psi(x)|_{x\in\partial\mathcal{E}} = 0\}$$

It is well know the operator $\mathfrak{P}$ generates a cosine family $((\mathfrak{C}(s))_{s\in\mathbb{R}}, (\mathfrak{S}(s))_{s\in\mathbb{R}})$. Additionally, it follows that

$$\|\mathfrak{C}(s)\| \leq 1\ and\ \|\mathfrak{S}(s)\| \leq 1,\ for\ all\ s \in [0,+\infty).$$

Thus, to apply our Theorems on existence and attractivity, we require $\mathcal{Y}^* + \mathcal{O}^* < 1$.
The function $\Psi(s, \psi(s-b,x)) = \frac{\exp -s}{7} \sin \psi(s-b,x)$ *is carathéodory and*

$$|\Psi(s, \psi_1(s-b,x)) - \Psi(s, \psi_2(s-b,x))| \leq \frac{\exp -s}{7} |\psi_1(s-b,x) - \psi_2(s-b,x)|$$

thus $\mathcal{O}(s) = \frac{\exp -s}{7}$. *Moreover, we have*

$$\mathcal{O}^* = \sup\{\int_0^s x^{-\frac{1}{3}} \frac{\exp -x}{7} dx,\ s \in [0, +\infty)\} = \frac{\Gamma(\frac{2}{3})}{7} \backsimeq 0.19302,\ \Psi_0 = 0.$$

Also, $\mathcal{Y}(s, \psi(s-b,x)) = \frac{\exp -s}{2} \tanh \psi(s-b,x)$ *is carathéodory and*

$$|\mathcal{Y}(s, \psi_1(s-b,x)) - \mathcal{Y}(s, \psi_2(s-b,x))| \leq \frac{1}{2} |\psi_1(s-b,x) - \psi_2(s-b,x)|$$

thus $\mathcal{Y}^* = \frac{1}{2}$. *Moreover, we have*

$$\mathcal{Y}(s, 0) = \frac{\exp -s}{2} \tanh(0) = 0 = \mathcal{Y}'.$$

Thus $\mathcal{Y}^* + \mathcal{O}^* \mathcal{M}_c^{\mathfrak{S}} \leq \frac{1}{2} + \frac{\Gamma(\frac{2}{3})}{7} \backsimeq 0.693 < 1$.
Then, by [15, 24]*, the problem* (1)-(3) *is an abstract formulation of the problem* (5)-(7)*, and conditions* $(i)-(iii)$ *are satisfied. Theorem 3.3 implies that the problem* (5)-(7) *has a unique mild solution on* BC*, which is attractive by Theorem 3.4.*

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