ON THE CATEGORY OF FUZZY COVERING AND RELATED TOPICS

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This paper uses category theory to approach the topic of fuzzy coverings and coverages, and it establishes the connection between them. The notions of sets and partitions are generalized to the fuzzy world.

Keywords: fuzzy sets, category theory, covering, coverage.

1. Introduction

Fuzzy sets were first introduced by Lotfi Zadeh in 1965. Fuzzy sets are generalizations of sets. The statement that an element is in a fuzzy set can be neither false nor true, but something in between. A covering of a set is a collection of $[0,1]-$ fuzzy sets such that every element of the set has the degree 1 in at least one of the fuzzy sets. The notion of a covering is similar to a set of attributes such that every element of the set has at least one attribute.

Coverings have a high importance in fuzzy control. For example, let's say we are trying to optimize a process. The input data given by the system should be characterized by a state or an attribute therefore the input has the degree 1 in the given state. The rule base should associate the decisions to the corresponding states for optimizing the output and the corresponding decisions must be applied with degree 1. Or as a different example for the concept of covering we can think of a patient who is told to do two tests to be diagnosed. After the results the doctor should associate the test results with a disease or a set of diseases or the tests done are inconclusive. So the symptoms or test results have degree 1 for certain diseases.

A coverage is a family of $[0,1]-$ fuzzy sets, defined on a set X, which have sum 1 for all $x \in X$. They resemble probability distributions meaning that if we have the set X, then coverage $(X,(A_i)_{i=\overline{1,n}})$ has the all events $A_i : X \to [0,1]$ that can occur to the elements of X, so the probability that the event A_i happens to the element x is $A_i(x)$.

This paper is divided into four sections. The first two define the fuzzy set, the partition, the covering and the coverage. The next two sections describe $Covering$ $-$ the covering category and $Coverage$ $-$ the category of coverage spaces.

The main results are the limits and colimits of $Covering$ and the isomorphism between a subcategory of $Covering$ and $Coverage$.

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2. Preliminaries

Let X be a set.

Definition 2.1. A is a fuzzy set, or a fuzzy subset of X, if $A : X \to [0,1]$ is a function, where $A(x)$ is the membership degree to which x belongs to A.

Definition 2.2. $(X, (A_i)_{i\in I})$ is a crisp covering if:
a. $A_i \subseteq X$ sets $\forall i \in I$;
b. $\bigcup_{i\in I} A_i = X$.

Definition 2.3. $(X, (A_i)_{i\in I})$ is a partition if $(A_i)_{i\in I}$ is a collection of non-empty, disjoint sets whose union is X.

Remark 2.1. A crisp subset A of X can be identified with its characteristic function:

$$A : X \to \{0,1\}, A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

and the set X can be identified with the function $X : X \to \{0,1\}$, $X(x) = 1, \forall x \in X$.

Definition 2.4. $(X, (A_i)_{i\in I})$ is a fuzzy covering if $A_i : X \to [0,1]$ are fuzzy sets $\forall i \in I$ such that $\forall x \in X$, $\exists i \in I$ such that $A_i(x) = 1$. In this case we can also say that X is covered by the fuzzy sets A_i, $\forall i \in I$.

Example 2.1. (X, A_1, A_2) is a covering where $X = \{a, b\}$, $A_1(a) = 0.8$, $A_1(b) = 1$ and $A_2(a) = 1$, $A_2(b) = 0.3$.

Remark 2.2. We consider by convention that $(\emptyset, (\emptyset))$ is a covering since there is no need for any indexed fuzzy sets which take the value 1, where $\emptyset$ is the empty set and $(\emptyset) = (A_i)_{i\in\emptyset}$, $A_i : \emptyset \to [0,1]$.
Also we can consider $\left(\emptyset, (A_i)_{i\in I}\right)$ a covering because we still have $\forall x \in \emptyset$, $\exists i \in I$ such that $A_i(x) = 1$.

Definition 2.5. If $A : X \to [0,1]$ is a fuzzy set then we define:
a. $A^{\uparrow} : X \to [0,1]$, $A^{\uparrow}(x) = \begin{cases} 1, & A(x) > 0 \\ 0, & A(x) = 0 \end{cases}$
b. $A^{\downarrow} : X \to [0,1]$, $A^{\downarrow}(x) = \begin{cases} 1, & A(x) = 1 \\ 0, & A(x) < 1 \end{cases}$

Definition 2.6. The $\alpha -$ cut of the fuzzy set $A : X \to [0,1]$ is denoted A_α and it is defined by:

$$A_\alpha : X \to [0,1],\ A_\alpha(x) = \begin{cases} 1, & A(x) \geq \alpha \\ 0, & A(x) < \alpha \end{cases}$$

3. The Covering Category

Definition 3.1. Let *Covering* be the category which has:
1. $\mathrm{Ob}(Covering) = \left\{\left(X, (A_i)_{i\in I}\right) \mid \left(X, (A_i)_{i\in I}\right) \text{ is a fuzzy covering}\right\}$;
2. $\mathrm{Hom}\left(\left(X, (A_i)_{i\in I}\right), \left(Y, (B_j)_{j\in J}\right)\right) = \{(f, \rho) | f : X \to Y, \rho : I \to J,$

$A_i(x) \le B_{\rho(i)}(f(x)), \forall x \in X, \forall i \in I\}$;
3. $(g,\theta) \circ (f,\rho) = (g \circ f, \theta \circ \rho) \in \mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Z,(C_k)_{k\in K}\right)\right)$
$\forall (f,\rho) \in \mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Y,(B_j)_{j\in J}\right)\right)$,
$\forall (g,\theta) \in \mathrm{Hom}\left(\left(Y,(B_j)_{j\in J}\right),\left(Z,(C_k)_{k\in K}\right)\right)$;
4. $id_{\left(X,(A_i)_{i\in I}\right)} = (id_X, id_I)$, $\forall \left(X,(A_i)_{i\in I}\right) \in \mathrm{Ob}(Covering)$.

Remark 3.1. The condition for the morphism is equivalent with

$$f(A_i) \subseteq B_{\rho(i)}, \forall i \in I.$$

where $f(A_i) = \left\{(y,\alpha) | y \in Y, \alpha = \vee_{f(x)=y} A_i(x)\right\}$, $\forall i \in I$.

Example 3.1. $(\{*\}, I)$ is a covering if and only if $I \subseteq [0,1]$ and $1 \in I$, where the values of the fuzzy sets identify with the elements of I. In *Covering* there are always morphisms between any two objects like that.

Lemma 3.1. Let $(X,(A_i)_{i\in I})$ be a covering. If $I = \emptyset$ then $X = \emptyset$ and the reverse isn't true.

Remark 3.2. Let U_1, U_2 : *Covering* $\to$ *Set* be the forgetful functors. U_1 on objects keeps the set that's been covered and on morphisms keeps the function between the covered sets. U_2 on objects keeps the set that indexes the fuzzy sets from the covering and on morphisms keeps the function between the sets of indexes. Then U_1 and U_2 are not full functors.

Example 3.2. Not every pair of functions $f : X \to Y$ and $\rho : I \to J$ can form a morphism in *Covering*. Let $|X| > 1$, $A_1 = X$, $Y = \{y_1, y_2\}$, $B_1 = \{y_1\}$, $B_2 = \{y_2\}$, $B_3 = \emptyset$ be six sets. If $(f,\rho) : (X, A_1) \to \left(Y,(B_i)_{i=\overline{1,3}}\right)$ is a morhipsm in *Covering* then $1 = A_1(x) \le B_{\rho(1)}(f(x))$, $\forall x \in X$, $\forall i \in I$ i.e. the image of the function f must be included in one of the sets B_1, B_2, B_3. This means that:

- $\rho(1) \ne 3$;
- f must be constant.

This implies that for given coverings $\left(X,(A_i)_{i\in I}\right)$ and $\left(Y,(B_j)_{j\in J}\right)$ we can't find a morphism (f,ρ) such that $U_1(f,\rho) = f$, $\forall f : X \to Y$ nor $U_2(f,\rho) = \rho$, $\forall \rho : I \to J$.

Remark 3.3. The functors U_1 and U_2 have the faithful right adjoint functors V_1 and respectively V_2, where V_1, V_2 : *Set* $\to$ *Covering*, $V_1(X) = (X,X)$, $V_1(f) = \left(f, id_{\{1\}}\right)$ and $V_2(I) = \left(\{*\},(\{*\})_{i\in I}\right)$, $V_2(\rho) = \left(id_{\{*\}}, \rho\right)$.

Definition 3.2. Let *Crisp* be the full subcategory of *Covering* for which the objects are crisp coverings.

Remark 3.4. Let $F^{\downarrow}, F^{\uparrow}, F_{\alpha}$: *Covering* $\to$ *Crisp* be the functors which apply the $\downarrow$, $\uparrow$ and $\alpha -$ cut on the fuzzy sets of the coverings, where $\alpha \in (0,1]$, and don't modify the morphisms. Then:
a. $F^{\downarrow}, F^{\uparrow}, F_{\alpha}$ are surjective on objects;
b. $F^{\downarrow}, F^{\uparrow}, F_{\alpha}$ are faithful and full;
c. $F^{\downarrow} \circ F^{\downarrow} = F^{\downarrow}$, $F^{\uparrow} \circ F^{\uparrow} = F^{\uparrow}$, $F_{\alpha} \circ F_{\alpha} = F_{\alpha}$.

Remark 3.5. The initial object and the terminal object in *Covering* are $(\emptyset, (\emptyset))$ and $(\{*\}, \{*\})$.

To clarify the index of the fuzzy set from the terminal object we will consider $\{*\} = (A_i)_{i\in\{1\}}$, where $A_1(*) = 1$.

Remark 3.6. In *Covering* a point of an object $\big(X,(A_i)_{i\in I}\big)$ is denoted by (x,i) where $x:\{*\}\to X$, $x(*)=x$, $x\in X$ and $i:\{1\}\to I$, $i(1)=i$, $i\in I$ such that $A_i(x)=1$. The functions are notated with their image since there is a unique bijection.

Theorem 3.1. Let $(f,\rho):\big(X,(A_i)_{i\in I}\big)\to(Y,(B_j)_{j\in J})$ be a *Covering* morphism.
a. (f,ρ) is a monomorphism if and only if $f|_{A_i^{\downarrow}}$ and ρ are injective, $\forall i\in I$.
b. (f,ρ) is an epimorphism if and only if f and ρ are surjective.
c. (f,ρ) is an isomorphism if and only if $f|_{A_i^{\downarrow}}:A_i^{\downarrow}\to B_{\rho(i)}^{\downarrow}$ and ρ are bijective, $\forall i\in I$.

Proof:

a. " $\Rightarrow$ " Let (f,ρ) be a monomorphism. We will divide the proof in two parts:

i. Assume ρ isn't injective i.e. $\exists i_1,i_2\in I$, $i_1\neq i_2$ so that $\rho(i_1)=\rho(i_2)$.
$(\emptyset,C_1)$ is an object in *Covering*, where $C_1=\emptyset$.
Let $(u,\tau),(v,\theta):(\emptyset,C_1)\to\big(X,(A_i)_{i\in I}\big)$ be two functions.
Regardless how we define $\tau,\theta:\{1\}\to I$ we have:

$$C_1\subseteq A_{\tau(1)}^{\downarrow} \text{ and } C_1\subseteq A_{\theta(1)}^{\downarrow}$$

Therefore (u,τ), (v,θ) are morphisms in *Covering*.
We can choose $\tau(1)=i_1$ and $\theta(1)=i_2$.
It is easy to see that $f\circ u=f\circ v$. Then:

$$(f,\rho)\circ(u,\tau)=(f,\rho)\circ(v,\theta)$$

But since (f,ρ) is a monomorphism we get $(u,\tau)=(v,\theta)$. Contradiction with the assumption. ρ is injective.

ii. If $X=\emptyset$ then there is nothing to prove, but if $X\neq\emptyset$ assume there $\exists i_0\in I$ where $f|_{A_{i_0}^{\downarrow}}$ isn't injective i.e. $\exists x_1,x_2\in X, x_1\neq x_2$ such that $f(x_1)=f(x_2)$ and $A_{i_0}(x_1)=A_{i_0}(x_2)=1$. Let $(u,\tau),(v,\theta):(\{*\},D_1)\to\big(X,(A_i)_{i\in I}\big)$ be two functions, where $D_1=\{*\}$. If we define $u(*)=x_1$, $\tau(1)=i_0$ and $v(*)=x_2$, $\theta(1)=i_0$ we notice that:

- (u,τ) and (v,θ) are morphisms in *Covering* because:
$$D_1(*)\le A_{\tau(1)}(u(*)) \text{ and } D_1(*)\le A_{\theta(1)}(v(*));$$
- $(f,\rho)\circ(u,\tau)=(f,\rho)\circ(v,\theta)$ which in particular means that $f|_{A_{i_0}^{\downarrow}}\circ u(*)=f|_{A_{i_0}^{\downarrow}}\circ v(*)$, but having (f,ρ) monomorphism leads us to $x_1=x_2$. Contradiction. $f|_{A_i^{\downarrow}}$ is injective $\forall i\in I$.

" $\Leftarrow$ " Let $f|_{A_i^{\downarrow}}$ and ρ be injective, $\forall i\in I$.
Let (u,τ), $(v,\theta):\big(Z,(C_k)_{k\in K}\big)\to\big(X,(A_i)_{i\in I}\big)$ such that

$$(f,\rho)\circ(u,\tau)=(f,\rho)\circ(v,\theta)$$

then:
i. $\rho(\tau(k))=\rho(\theta(k))$, $\forall k\in K$ and ρ injective gives us $\tau=\theta$;
ii. $f\circ u=f\circ v$.
We can safely assume $K\neq\emptyset$, since there is nothing to prove otherwise. Then we particularly have $f|_{A_i^{\downarrow}}\circ u|_{C_{\tau^{-1}(i)}}=f|_{A_i^{\downarrow}}\circ v|_{C_{\tau^{-1}(i)}}$, $\forall i\in Im(\tau)$, but $f|_{A_i^{\downarrow}}$ being injective $\forall i\in I$ gives us

$u|_{C_k} = v|_{C_k}, \forall k \in K$.
Then $(u, \tau) = (v, \tau)$ and (f, ρ) is indeed a monomorphism.

b. It is a consequence of the forgetful functors U_1 and U_2 being left adjoints.

c. It is a consequence of a. and b.

Lemma 3.2. If (f, ρ) is a monomorphism, an epimorphism or an isomorphism in *Covering* then $F^{\downarrow}(f, \rho)$ is a monomorphism, an epimorphism or an isomorphism respectively in *Crisp*.

Definition 3.3. Let $(X, (A_i)_{i\in I})$ and $(X, (B_j)_{j\in J})$ be two coverings. $(X, (A_i)_{i\in I})$ is finer than $(X, (B_j)_{j\in J})$ if there exists a function $\rho : I \to J$ such that $(1_X, \rho)$ is a morphism in *Covering*. We denote that by:

$$(X, (A_i)_{i\in I}) \subseteq (X, (B_j)_{j\in J})$$

and $(X, (A_i)_{i\in I})$ is strictly finer than $(X, (B_j)_{j\in J})$ if $(X, (A_i)_{i\in I})$ is finer than $(X, (B_j)_{j\in J})$ and there $\exists x \in X$ such that $A_i(x) < B_{\rho(i)}(x)$. We denote that by:

$$(X, (A_i)_{i\in I}) \subset (X, (B_j)_{j\in J})$$

Remark 3.7. For all $(X, (A_i)_{i\in I})$ coverings we have:
$(X, \emptyset, \{x_1\}, \{x_2\}, ...) \subseteq (X, (A_i)_{i\in I}) \subseteq (X, X)$.

Remark 3.8. Let $(X, (A_i)_{i\in I})$ be a covering and $L = [0, 1]^X$. Then a covering of X is a family of $L-$ fuzzy sets which have at least a 1 for all the coordinates.

Example 3.3. The covering from example 2.6. can be written as the fuzzy set $A : X \to [0, 1]^X$ where $A(a) = (0.8, 1)$ and $A(b) = (1, 0.3)$.

Theorem 3.2. In *Covering* we have the following:
a. The product of the two coverings $(X, (A_i)_{i\in I})$ and $(Y, (B_j)_{j\in J})$ is the covering

$$\left(X \times Y, (A_i \wedge B_j)_{(i,j)\in I\times J}\right)$$

b. The coproduct of the two coverings $(X, (A_i)_{i\in I})$ and $(Y, (B_j)_{j\in J})$ is the covering $\left(X \coprod Y, (C_k)_{k\in I \coprod J}\right)$ where $\forall k \in I \coprod J$ we have:

$$C_k = \begin{cases} A|_k^{X \coprod Y}, & k \in I \\ B|_k^{X \coprod Y}, & k \in J \end{cases}$$

c. The equalizer of $(f, \rho), (g, \theta) : (X, (A_i)_{i\in I}) \to (Y, (B_j)_{j\in J})$ is the object $\left(X_0, (A_i|_{X_0})_{i\in I_0}\right)$ where:

- $I_0 = \{i \in I | \rho(i) = \theta(i)\}$;
- $X_0 = \{x \in X | f(x) = g(x), \exists i \in I_0, A_i(x) = 1\}$;
- $A_i|_{X_0}$ is the restriction of A_i to X_0 i.e. $A_i|_{X_0} : X_0 \to [0, 1]$, $A_i|_{X_0}(x) = A_i(x), \forall x \in X_0, \forall i \in I_0$.

d. The coequalizer of $(f, \rho), (g, \theta) : (X, (A_i)_{i\in I}) \to (Y, (B_j)_{j\in J})$ is the object $\left(Y/R, \left(B_{\widetilde{j}}\right)_{\widetilde{j}\in J/S}\right)$ where:

- S is the equivalence relation on J generated by $\rho(i)S\theta(i)$;
- R is the equivalence relation on X generated by $f(x)Rg(x)$;
- $B_{\widetilde{j}}(\widehat{y}) = \vee_{y\in\widehat{y}}\left(\vee_{j\in\widetilde{j}} B_j(y)\right), \forall \widehat{y} \in Y/R, \forall \widetilde{j} \in J/S$.

e. The pullback of $(f,\rho):\left(X,(A_i)_{i\in I}\right)\to(Z,(C_k)_{k\in K})$ and $(g,\theta):\left(Y,(B_j)_{j\in J}\right)\to\left(Z,(C_k)_{k\in K}\right)$ is the covering:

$$\left(X\times_Z Y,(A_i(x)\wedge B_j(y))_{(i,j)\in I\times_K J}\right)$$

where $X\times_Z Y=\{(x,y)|f(x)=g(y),\exists(i,j)\in I\times_K J, A_i(x)\wedge B_j(y)=1\}$ and $I\times_K J=\{(i,j)|\rho(i)=\theta(j)\}$.

f. The pushout of the morphisms $(f,\rho):\left(X,(A_i)_{i\in I}\right)\to\left(Y,(B_j)_{j\in J}\right)$ and $(g,\theta):\left(X,(A_i)_{i\in I}\right)\to\left(Z,(C_k)_{k\in K}\right)$ is the covering:

$$\left(Y\coprod_X Z,\left(D_{\widetilde{(j,k)}}\right)_{\widetilde{(j,k)}\in J\coprod_I K}\right)$$

where:

- on $J\coprod_I K$ we have the equivalence relation S generated by $\rho(i)S\theta(i)$;
- on $Y\coprod_X Z$ we have the equivalence relation R generated by $f(x)Rg(x)$;
- $D_{\widetilde{(j,k)}}:Y\coprod_X Z\to[0,1]$, is given by

$$D_{\widetilde{(j,k)}}\left(\widehat{(y,z)}\right)=\vee_{(j,k)\in\widetilde{(j,k)}}\left(\vee_{(y,z)\in\widehat{(y,z)}}(B_j(y)\vee C_k(z))\right),$$

$\forall\widehat{(y,z)}\in Y\coprod_X Z$, $\forall\widetilde{(j,k)}\in J\coprod_I K$.

g. The exponential of the objects $\left(X,(A_i)_{i\in I}\right)$ and $\left(Y,(B_j)_{j\in J}\right)$ is the covering:

$$\left(Y,(B_j)_{j\in J}\right)^{\left(X,(A_i)_{i\in I}\right)}=\left(\overline{Y^X},\left(\wedge_{(x,i)\in X\times I}\left(A_i(x)\to B_{\rho(i)}(f(x))\right)\right)_{\rho\in J^I}\right),$$

where:

- $\overline{Y^X}=U_1\left(\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Y,(B_j)_{j\in J}\right)\right)\right)$;
- J^I is the set of all functions $I\to J$;
- $a\to b=\sup\{c\in[0,1]|a\wedge c\le b\}$, $\forall a,b\in[0,1]$.

Proof:

a. First of all we will verify that $\left(X\times Y,(A_i\wedge B_j)_{(i,j)\in I\times J}\right)$ is a covering. We know that $\forall x\in X$, $\exists i\in I$ such that $A_i(x)=1$ and $\forall y\in Y$, $\exists j\in J$ such that $B_j(y)=1$. Then $\forall(x,y)\in X\times Y$, $\exists(i,j)\in I\times J$ such that

$$(A_i\wedge B_j)(x,y)=A_i(x)\wedge B_j(y)=1$$

The projection morphisms associated to the product are:

$$(p_1,\pi_1):\left(X\times Y,(A_i\wedge B_j)_{(i,j)\in I\times J}\right)\to\left(X,(A_i)_{i\in I}\right),\ p_1(x,y)=x,\ \pi_1(i,j)=i$$

and

$$(p_2,\pi_2):\left(X\times Y,(A_i\wedge B_j)_{(i,j)\in I\times J}\right)\to\left(Y,(B_j)_{j\in J}\right),\ p_2(x,y)=y,\ \pi_2(i,j)=j.$$

Let $\left(Z,(C_k)_{k\in K}\right)$ be a covering and $(f,\rho):\left(Z,(C_k)_{k\in K}\right)\to\left(X,(A_i)_{i\in I}\right)$ and $(g,\theta):\left(Z,(C_k)_{k\in K}\right)\to\left(Y,(B_j)_{j\in J}\right)$ be two morphisms in *Covering*. We have to prove that for all the morphisms (f,ρ), (g,θ) there exists a unique morphism $(h,\tau):\left(Z,(C_k)_{k\in K}\right)\to\left(X\times Y,(A_i\wedge B_j)_{(i,j)\in I\times J}\right)$ such that

$$(p_1,\pi_1)\circ(h,\tau)=(f,\rho)$$

and

$$(p_2,\pi_2)\circ(h,\tau)=(g,\theta)$$

Then $h(z) = (f(z), g(z))$, $\forall z \in Z$ and $\tau(k) = (\rho(k), \theta(k))$, $\forall k \in K$, because $C_k(z) \leq A_{\rho(k)}(f(z))$ and $C_k(z) \leq B_{\theta(k)}(g(z))$ imply
$C_k(z) \leq A_{\rho(k)}(f(z)) \wedge B_{\theta(k)}(g(z))$.

b. We define $(i_X, \iota_I) : \left(X, (A_i)_{i\in I}\right) \to \left(X \coprod Y, (C_k)_{k\in I \coprod J}\right)$ and $(i_Y, \iota_J) : \left(Y, (B_j)_{j\in J}\right) \to \left(X \coprod Y, (C_k)_{k\in I \coprod J}\right)$ to be the inclusion morphisms.

Let $\left(Z, (D_n)_{n\in N}\right)$ be a covering and let $(f, \rho) : \left(X, (A_i)_{i\in I}\right) \to \left(Z, (D_n)_{n\in N}\right)$ and $(g, \theta) : \left(Y, (B_j)_{j\in J}\right) \to \left(Z, (D_n)_{n\in N}\right)$ be two morphisms in *Covering*. Then the unique morphism $(h, \tau) : \left(X \coprod Y, (C_k)_{k\in I \coprod J}\right) \to \left(Z, (D_n)_{n\in N}\right)$ with the properties:

$$(f, \rho) = (h, \tau) \circ (i_X, \iota_I) \text{ and } (g, \theta) = (h, \tau) \circ (i_Y, \iota_J)$$

is given by:

$$h(x) = \begin{cases} f(x), & x \in X \\ g(x), & x \in Y \end{cases} \text{ and } \tau(i) = \begin{cases} \rho(i), & i \in I \\ \theta(i), & i \in J \end{cases}$$

c. From the way $\left(X_0, (A_i|_{X_0})_{i\in I_0}\right)$ is defined it is clear that it is a covering even if X_0 and I_0 are the empty. We can even find a set I_0 such that $A_i(x) < 1$, $\forall x \in X$, $\forall i \in I_0$ but that would lead to $X_0 = \emptyset$. So we can't have $X_0 \neq \emptyset$ and $I_0 = \emptyset$. Then lemma 3.4. is respected and the covering is well defined.

Let $(i_{X_0}, \iota_{I_0}) : \left(X_0, (A_i|_{X_0})_{i\in I_0}\right) \to \left(X, (A_i)_{i\in I}\right)$, be a morphism in *Covering* where $i_{X_0} : X_0 \to X$, $i_{X_0}(x) = x$ and $\iota_{I_0} : I_0 \to I$, $\iota_{I_0}(i) = i$. We have:

$$(f, \rho) \circ (i_{X_0}, \iota_{I_0}) = (g, \theta) \circ (i_{X_0}, \iota_{I_0})$$

If $(h, \tau) : \left(Z, (C_k)_{k\in K}\right) \to \left(X, (A_i)_{i\in I}\right)$ is a morphism such that:

$$(f, \rho) \circ (h, \tau) = (g, \theta) \circ (h, \tau)$$

then:

- $f(h(z)) = g(h(z))$, $\forall z \in Z \Rightarrow h(z) \in \{x \in X | f(x) = g(x)\}$, $\forall z \in Z$;
- $\rho(\tau)(k) = \theta(\tau(k))$, $\forall k \in K \Rightarrow \tau(k) \in I_0$, $\forall k \in K$.

But in the covering $\left(Z, (C_k)_{k\in K}\right)$ we know that $\forall z \in Z$, $\exists k \in K$ such that $C_k(z) = 1$. From the construction of the morphisms we also know that
$C_k(z) \leq A_{\tau(k)}(h(z))$. This means that $A_{\tau(k)}(h(z)) = 1$.

So $\left\{h(z) | f(h(z)) = g(h(z)), \exists k \in K, \tau(k) \in I_0, A_{\tau(k)}(h(z)) = 1\right\} \subseteq X_0$. Then the unique morphism $(u, \sigma) : \left(Z, (C_k)_{k\in K}\right) \to \left(X_0, (A_i|_{X_0})_{i\in I_0}\right)$ with the property:

$$(i_{X_0}, \iota_{I_0}) \circ (u, \sigma) = (h, \tau)$$

is given by $u(z) = h(z)$, $\forall z \in Z$ and $\sigma(k) = \tau(k)$, $\forall k \in K$.

d. Let $(p, \pi) : \left(Y, (B_j)_{j\in J}\right) \to \left(Y/R, \left(B_{\widehat{j}}\right)_{\widehat{j}\in J/S}\right)$ where $p(y) = \widehat{y}$ (modulo Y/R) and $\pi(j) = \widetilde{j}$ (modulo J/S). We notice that:

- $\left(Y/R, \left(B_{\widehat{j}}\right)_{\widehat{j}\in J/S}\right)$ is a covering;
- $(p, \pi) \circ (f, \rho) = (p, \pi) \circ (g, \theta)$.

Let $(h, \tau) : \left(Y, (B_j)_{j\in J}\right) \to \left(Z, (C_k)_{k\in K}\right)$ be a morphism in covering with the property: $(h, \tau) \circ (f, \rho) = (h, \tau) \circ (g, \theta)$. If $\left(Z, (C_k)_{k\in K}\right) = (\{*\}, \{*\})$ then the previous property is satisfied. We know about h that:

- $h(f(x)) = h(g(x))$, $\forall x \in X \Rightarrow f(x)Rg(x)$ (composition property);

- $h(f(x_1)) = h(g(x_2)), \forall g(x_1) = f(x_2) \Rightarrow f(x_1)Rg(x_2)$ (composition property).

Then there exists a unique function $u : Y/R \to Z$ such that $u \circ p = h$.

Similarly for $\tau : J \to K$ noticing that $B_j(y) \leq C_{\tau(j)}(h(y)), \forall y \in \widehat{y}$. We have $B_j(y) \leq \vee_{y \in \widehat{y}} \left(\vee_{j \in \widetilde{j}} B_j(y) \right), \forall \widehat{y} \in Y/R$, but since the factorization gives us the most diverse classes there exists a unique function $\sigma : J/S \to K$ such that $\sigma \circ \pi = \tau$ and $B_{\widetilde{j}}(\widehat{y}) \leq C_{\sigma(\widetilde{j})}(v(\widehat{y})))$.

e. From the way $\left(X \times_Z Y, (A_i(x) \wedge B_j(y))_{(i,j) \in I \times_K J} \right)$ is defined it is clear that it is a covering. Let:
$(p_X, \pi_I) : \left(X \times_Z Y, (A_i(x) \wedge B_j(y))_{(i,j) \in I \times_K J} \right) \to \left(X, (A_i)_{i \in I} \right)$ and
$(p_Y, \pi_J) : \left(X \times_Z Y, (A_i(x) \wedge B_j(y))_{(i,j) \in I \times_K J} \right) \to \left(Y, (B_j)_{j \in J} \right)$ where
$(p_X, \pi_I)((x, y), (i, j)) = (x, i)$ and $(p_Y, \pi_J)((x, y), (i, j)) = (y, z)$,
$\forall (x, y) \in X \times_Z Y, \forall (i, j) \in I \times_K J$ be two morphisms. They have the property:

$$(f, \rho) \circ (p_X, \pi_I) = (g, \theta) \circ (p_Y, \pi_J).$$

Let $(h_1, \tau_1) : \left(V, (D_n)_{n \in N} \right) \to \left(X, (A_i)_{i \in I} \right)$ and
$(h_2, \tau_2) : \left(V, (D_n)_{n \in N} \right) \to \left(Y, (B_j)_{j \in J} \right)$ be two morphisms in *Covering* so that:

$$(f, \rho) \circ (h_1, \tau_1) = (g, \theta) \circ (h_2, \tau_2).$$

Firstly $(\rho \circ \tau_1)(n) = (\theta \circ \tau_2)(n)$ implies means that $(\tau_1(n), \tau_2(n)) \in I \times_K J$. Then we define the unique function $\sigma : N \to I \times_K J$, where $\sigma(n) = (\tau_1(n), \tau_2(n))$.
Secondly $(f \circ h_1)(v) = (g \circ h_2)(v)$ implies that $(h_1(v), h_2(v)) \in X \times_Z Y$. Then we define the unique function $u : V \to X \times_Z Y$, where $u(v) = (h_1(v), h_2(v))$.
We notice that:

- $(p_X, \pi_I) \circ (u, \sigma) = (h_1, \tau_1)$ and $(p_Y, \pi_J) \circ (u, \sigma) = (h_2, \tau_2)$;
- We have $D_n(v) \leq A_{\tau_1(n)}(h_1(v))$ and $D_n(v) \leq B_{\tau_2(n)}(h_2(v))$ since (h_1, τ_1) and (h_2, τ_2) are morphisms in *Covering*, which imply that:
$$D_n(v) \leq A_{\tau_1(n)}(h_1(v)) \wedge B_{\tau_2(n)}(h_2(v)), \forall v \in V, \forall n \in N.$$

Therefore (u, σ) is a unique morphism in *Covering* which completes the proof.

f. $\left(Y \coprod_X Z, (D_n)_{n \in J \coprod_I K} \right)$ is a covering. Let's define the functions:

- $p_1 : Y \to Y \coprod_X Z,\ p_1(y) = \begin{cases} (\widehat{f(x), g(x)}), & y \in Im(f) \\ \widehat{0}, & y \notin Im(f) \end{cases}$
- $p_2 : Z \to Y \coprod_X Z,\ p_2(z) = \begin{cases} (\widehat{f(x), g(x)}), & z \in Im(g) \\ \widehat{0}, & z \notin Im(g) \end{cases}$
- $\pi_1 : J \to J \coprod_I K,\ \pi_1(j) = \begin{cases} (\widetilde{\rho(i), \theta(i)}), & j \in Im(\rho) \\ \widetilde{0}, & j \notin Im(\rho) \end{cases}$
- $\pi_2 : K \to J \coprod_I K,\ \pi_2(k) = \begin{cases} (\widetilde{\rho(i), \theta(i)}), & k \in Im(\theta) \\ \widetilde{0}, & k \notin Im(\theta) \end{cases}$

Then (p_1, π_1) and (p_2, π_2) are morphisms in *Covering* because of the way we defined the fuzzy sets $D_{\widetilde{(j,k)}}, \forall \widetilde{(j, k)} \in J \coprod_I K$ and it is easy to see that:

$$(p_1, \pi_1) \circ (f, \rho) = (p_2, \pi_2) \circ (g, \theta).$$

Now we have to prove that $\left(Y\coprod_X Z, \left(D_{\widetilde{(j,k)}}\right)_{\widetilde{(j,k)}\in J\coprod_I K}\right)$ has the universal property. Let $\left(U,(E_m)_{m\in M}\right)$ be a covering and let
$(u_1,\tau_1):\left(Y,(B_j)_{j\in J}\right)\to\left(U,(E_m)_{m\in M}\right)$ and
$(u_2,\tau_2):\left(Z,(C_k)_{k\in K}\right)\to\left(U,(E_m)_{m\in M}\right)$ be two morphisms with the property:

$$(u_1,\tau_1)\circ(f,\rho)=(u_2,\tau_2)\circ(g,\theta).$$

Then in *Covering* there exists the unique morphism
$(h,\sigma):\left(Y\coprod_X Z,(D_n)_{n\in J\coprod_I K}\right)\to\left(U,(E_m)_{m\in M}\right)$, where:

- $h\left(\widehat{(y,z)}\right)=(u_1,u_2)\left(\widehat{(y,z)}\right)$, $\forall\widehat{(y,z)}\in Y\coprod_X Z$;
- $\sigma\left(\widetilde{(j,k)}\right)=(\pi_1,\pi_2)\left(\widetilde{(j,k)}\right)$, $\forall\widetilde{(j,k)}\in J\coprod_I K$.

g. First we will prove that $\left(\overline{Y^X},\left(\wedge_{(x,i)\in X\times I}\left(A_i(x)\to B_{\rho(i)}(f(x))\right)\right)_{\rho\in J^I}\right)$ is an object in *Covering*. Let $f\in\overline{Y^X}$. Then there exists $\rho\in J^I$ such that $(f,\rho):\left(X,(A_i)_{i\in I}\right)\to\left(Y,(B_j)_{j\in J}\right)$ is a morphism in *Covering*, hence $A_i(x)\le B_{\rho(i)}(f(x))$, $\forall x\in X$, $\forall i\in I$ and

$$A_i(x)\to B_{\rho(i)}(f(x))=1,\ \forall x\in X,\forall i\in I,$$

because $a\to b=1$, $\forall a\le b$. And for any $f\in Y^X\backslash\overline{Y^X}$ the result would be less than 1.

Let $D_\rho:\overline{Y^X}\to[0,1]$, $D_\rho(f)=\wedge_{(x,i)\in I\times X}\left(A_i(x)\to B_{\rho(i)}(f(x))\right)$, $\forall x\in X$ be fuzzy sets $\forall\rho\in J^I$. Then we have:

$$(ev,\epsilon):\left(\overline{Y^X},(D_\rho)_{\rho\in J^I}\right)\times\left(X,(A_i)_{i\in I}\right)\to\left(Y,(B_j)_{j\in J}\right)$$

where $(ev,\epsilon)\left((f,x),(\rho,i)\right)=(f(x),\rho(i))$, $\forall(f,x)\in\overline{Y^X}\times X$, $\forall(\rho,i):J^I\times I$.

Second we will prove that (ev,ϵ) is a morphism in *Covering*. With the product already found, we have to show that:

$$D_\rho(f)\wedge A_i(x)\le B_{\epsilon(\rho,i)}\left(ev(f,x)\right),\ \forall(f,x)\in\overline{Y^X}\times X,\ \forall(\rho,i):J^I\times I,$$

meaning

$$D_\rho(f)\wedge A_i(x)\le B_{\rho(i)}\left(f(x)\right),\ \forall(f,x)\in\overline{Y^X}\times X,\ \forall(\rho,i):J^I\times I.$$

But we know that:

$$D_\rho(f)\wedge A_i(x)=\wedge_{(x',i')\in I\times X}\left(A_{i'}(x')\to B_{\rho(i')}(f(x'))\right)\wedge A_i(x)\le$$
$$\left(A_i(x)\to B_{\rho(i)}(f(x))\right)\wedge A_i(x)\le B_{\rho(i)}\left(f(x)\right),\ \forall(f,x)\in\overline{Y^X}\times X,\ \forall(\rho,i):J^I\times I.$$

Now we will prove the universal property of the exponential.
Let $(g,\theta):\left(Z,(C_k)_{k\in K}\right)\times\left(X,(A_i)_{i\in I}\right)\to\left(Y,(B_j)_{j\in J}\right)$ be a morphism in *Covering*. We have:

$$C_k(z)\wedge A_i(x)\le B_{\theta(k,i)}\left(g(z,x)\right),\ \forall(z,x)\in Z\times X,\ \forall(k,i)\in K\times I.$$

From here we get:

$$C_k(z)\le A_i(x)\to B_{\theta(k,i)}\left(g(z,x)\right),\ \forall(z,x)\in Z\times X,\ \forall(k,i)\in K\times I,$$

because $a\wedge b\le c\Longleftrightarrow a\le b\to c$, $\forall a,b,c\in[0,1]$.
We define $(\lambda g,\lambda\theta):\left(Z,(C_k)_{k\in K}\right)\to\left(\overline{Y^X},(D_\rho)_{\rho\in J^I}\right)$, where
$(\lambda g,\lambda\theta)(z,k)=(g(z,-),\theta(k,-))$. First we have to show that $\lambda g:Z\to Y^X$ takes values in $\overline{Y^X}$. Let $z\in Z$, then there exists $k_z\in K$ with $C_{k_z}(z)=1$. Then $(\lambda g(z),\lambda\theta(k_z))$:

$\left(X,(A_i)_{i\in I}\right)\to\left(Y,(B_j)_{j\in J}\right)$ is a morphism in *Covering*, therefore $\lambda g(z)\in\overline{Y^X}$, $\forall z\in Z$. From the inequality above we get that:

$$C_k(z)\le \wedge_{(x,i)\in X\times I}\left(A_i(x)\to B_{\theta(k,i)}\left(g(z,x)\right)\right),\ \forall(z,x)\in Z\times X,\ \forall(k,i)\in K\times I,$$

which gives us

$$C_k(z)\le \wedge_{(x,i)\in X\times I}\left(A_i(x)\to B_{\lambda\theta(k)}\left((\lambda g(z))(x)\right)\right),\ \forall z\in Z,\ \forall k\in K.$$

This proves that $(\lambda g,\lambda\theta)$ is a morphism in *Covering*. $(\lambda g,\lambda\theta)$ is the unique morphism with the property:

$$(ev,\epsilon)\circ((\lambda g,\lambda\theta)\times(id_X,id_I))=(g,\theta).$$

Remark 3.9. *Covering* is a Cartesian closed category, meaning:

$$\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right)\times\left(Y,(B_j)_{j\in J}\right),\left(Z,(C_k)_{k\in K}\right)\right)\simeq$$
$$\mathrm{Hom}\left(\left(X,(A_i)_{i\in I}\right),\left(Z,(C_k)_{k\in K}\right)^{\left(Y,(B_j)_{j\in J}\right)}\right).$$

4. The Category of Coverage Spaces

Consider I and J non-empty, finite sets.

Definition 4.1. Let X be a non-empty set. The pair $\left(X,\left(\overline{A}_i\right)_{i\in I}\right)$ is a fuzzy coverage, or simply a coverage if $\overline{A}_i:X\to[0,1],\forall i\in I$ and $\sum_{i\in I}\overline{A}_i(x)=1$, $\forall x\in X$.

Remark 4.1. For all $x\in X$ and all $\left(X,\left(\overline{A}_i\right)_{i=\overline{1,n}}\right)$ fuzzy coverages we get the probability distribution:

$$\begin{pmatrix}1 & 2 & ... & n\\ \overline{A}_1(x) & \overline{A}_2(x) & ... & \overline{A}_n(x)\end{pmatrix}$$

Definition 4.2. Let *Coverage* be the category of fuzzy coverage spaces:
1. $\mathrm{Ob}(Covering)=\left\{\left(X,\left(\overline{A}_i\right)_{i\in I}\right)|\overline{A}:X\to[0,1],\sum_{i\in I}\overline{A}_i(x)=1,\forall x\in X\right\}$;
2. $\mathrm{Hom}\left(\left(X,\left(\overline{A}_i\right)_{i\in I}\right),\left(Y,\left(\overline{B}_j\right)_{j\in J}\right)\right)=\{\left(\overline{f},\overline{\rho}\right)|\overline{f}:X\to Y,\ \overline{\rho}:I\to J$, such that $\forall i_0\in I$, $\frac{\overline{A}_{i_0}(x)}{\vee_i\overline{A}_i(x)}\le\frac{\overline{B}_{\overline{\rho}(i_0)}(f(x))}{\vee_j\overline{B}_j(f(x),)}\}$.

Remark 4.2. *Coverage* is well defined.

Definition 4.3. We define the category $f-Covering$ as the full subcategory of *Covering* which has a finite number of fuzzy sets in the covering and the covered sets are non-empty.

Definition 4.4. We define the functor $F:f-Covering\to Coverage$ such that:
a. $\forall\left(X,(A_i)_{i\in I}\right)\in\mathrm{Ob}\left(f-Covering\right)$, $F\left(X,(A_i)_{i\in I}\right)=\left(X,\left(\overline{A}_i\right)_{i\in I}\right)$ where $\forall x\in X$ we have:

$$\overline{A}_i(x)=\frac{A_i(x)}{\sum_{j\in J}A_j(x)}$$

b. For every morphism $\left(\overline{f},\overline{\rho}\right):\left(X,(A_i)_{i\in I}\right)\to\left(Y,(B_j)_{j\in J}\right)$

$$F(f,\rho):\left(X,\left(\overline{A}_i\right)_{i\in I}\right)\to\left(Y,\left(\overline{B}_j\right)_{j\in J}\right),\ F(f,\rho)=(f,\rho).$$

Remark 4.3. The functor F is well defined.

Definition 4.5. We define another functor $G : Coverage \to f - Covering$ such that:
a. For every coverage $\left(X, \left(\overline{A}_i\right)_{i\in I}\right) \in \mathrm{Ob}\left(Coverage\right)$,
$G\left(X, \left(\overline{A}_i\right)_{i\in I}\right) = \left(X, \left(A_i\right)_{i\in I}\right)$ where $\forall x \in X$:

$$A_i(x) = \frac{\overline{A}_i(x)}{\vee_{j\in I}\overline{A}_j(x)};$$

b. On morphisms $\forall \left(\overline{f}, \overline{\rho}\right) : \left(X, \left(\overline{A}_i\right)_{i\in I}\right) \to \left(Y, \left(\overline{B}_j\right)_{j\in j}\right)$
$G\left(\overline{f}, \overline{\rho}\right) : \left(X, \left(A_i\right)_{i\in I}\right) \to \left(Y, \left(B_j\right)_{j\in J}\right)$ where $G\left(\overline{f}, \overline{\rho}\right) = \left(\overline{f}, \overline{\rho}\right)$

Theorem 4.1. The categories $f - Covering$ and $Coverage$ are isomorphic.

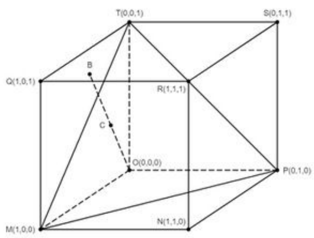
Proof: The functors F and G are fully faithful and we only have to prove that:
a. $G \circ F = 1_{\mathrm{Ob}(f-Covering)}$.
For all $\left(X, \left(A_i\right)_{i\in I}\right) \in \mathrm{Ob}\left(f - Covering\right)$ we have:
$$\left(G \circ F\right)\left(A_i(x)\right) = G\left(\left(F \circ A_i\right)(x)\right) = G\left(\frac{A_i(x)}{\sum_{j\in I} A_j(x)}\right) = \frac{\frac{A_i(x)}{\sum_{j\in I} A_j(x)}}{\vee_{j\in I}\frac{A_j(x)}{\sum_{k\in I} A_k(x)}} = \frac{A_i(x)}{\vee_{j\in I} A_k(x)} = A_i(x).$$

b. $F \circ G = 1_{\mathrm{Ob}(Coverage)}$.
For all $\left(X, \left(\overline{A}_j\right)_{j\in J}\right) \in \mathrm{Ob}\left(Coverage\right)$ we have:

$$\left(F \circ G\right)\left(\overline{A}_j\right)(x) = F\left(\left(G \circ \overline{A}_j\right)(x)\right) = F\left(\frac{\overline{A}_j(x)}{\vee_{i\in J}\overline{A}_i(x)}\right) = \frac{\frac{\overline{A}_j(x)}{\vee_{i\in j}\overline{A}_i(x)}}{\sum_{k\in J}\frac{\overline{A}_k(x)}{\vee_{i\in J}\overline{A}_i(x)}}$$
$$= \frac{\frac{\overline{A}_j(x)}{\vee_{i\in J}\overline{A}_i(x)}}{\frac{\sum_{k\in J}\overline{A}_k(x)}{\vee_{i\in J}\overline{A}_i(x)}} = \frac{\overline{A}_j(x)}{\sum_{i\in J}\overline{A}_i(x)} = \overline{A}_j(x).$$

Remark 4.4. The consequence of defining the coverages on non-empty sets is the loss of the initial object, the equalizer and the pullback in $Coverage$.

Remark 4.5. The geometric interpretation of the previous theorem is that there exists an isomorphism between the $n-$ dimensional simplex
$\Delta_n = \{(x_1, x_2, ..., x_n) \,|\, \sum_{i=1}^{n} x_i = 1, x_i \in [0, 1]\}$ and the union of the faces of the $n-$ dimensional unit cube $\cup_{i=1}^{n} \{(x_1, x_2, ..., x_n) \,|\exists i \text{ such that } x_i = 1\}$.



Example 4.1. If $X = \{*\}$ is a set and $A_i : X \to [0, 1]$, $\forall i = \overline{1, 3}$ are fuzzy sets then the covering $\left(X, \left(A_i\right)_{i=\overline{1,3}}\right)$ can be represented as the point $B\left(A_1(*), A_2(*), A_3(*)\right)$, which is inside or on the sides of the squarse $QRST$, $NPSR$ or $MNRQ$ and $F\left(X, \left(A_i\right)_{i=\overline{1,3}}\right)$ is the intersection between the OB and the plane MTP.

And vice versa if $\left(X, \left(\overline{A}_i\right)_{i=\overline{1,3}}\right)$ is a coverage then it can be represented as the point $C\left(\overline{A}_1(*), \overline{A}_2(*), \overline{A}_3(*)\right)$, which is inside or on the sides of the triangle ΔMPT and $G\left(X, \left(\overline{A}_i\right)_{i=\overline{1,3}}\right)$ is the only intersection of OC with one of the three squares $QRST$, $NPSR$ and $MNRQ$.

We plan to discuss the connection between coverings and fuzzy tolerances in a subsequent paper, with emphasis on the categorical aspects. While the classical crisp case is well-studied, much less is known in the fuzzy case. We consider that category theory can be useful here in explaining why and how fuzzy coverings are related to fuzzy tolerances and providing the right intuition of the corresponding concepts.

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