ON ZEROS OF ACCRETIVE OPERATORS WITH APPLICATION TO THE CONVEX FEASIBILITY PROBLEM

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*In this paper, we study some iterative algorithms for finding common zeros of finite family of accretive operators in Banach spaces. Our results improve the recent results of Kim and Tuyen [Appl. Math. Comput., **283**(2016), 265-281 & Bull. Korean Math. Soc., **54**(2017), 1347-1359] by removing some assumptions on the parameters. Note that these restrictive conditions on the parameters have been extensively used so far for different versions of iterative algorithms in this literature. Finally, under more relaxed conditions on the parameters, we solve the convex feasibility problem.*

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1. Introduction

Fixed point theory has been revealed as a very powerful and effective tool for studying a wide class of problems which arise from real world applications and can be translated into equivalent fixed point problems. This theory has been successfully applied in different topics, including split and convex feasibility, variational inequality and equilibrium problems as well as for finding zeros of accretive operators. In order to obtain approximate solution of the fixed point problems various iterative methods have been proposed (see, e.g., [10, 15, 19, 26, 28, 29, 30] and the references therein). The well-known convex feasibility problem, which captures applications in various disciplines such as sensor networking [5], radiation therapy treatment planning [9], computerized tomography [13], image restoration [11] is to find a point in the intersection of a family of closed convex sets in a Hilbert space.

The aim of this paper is to control the conditions on the parameters used in iterative algorithms for finding zeros of accretive operators in the setting of Hilbert and Banach spaces. One of the first and most popular method for finding zeros of a maximal monotone operator is the proximal point algorithm (PPA). Rockafellar [25] proved the weak convergence of the PPA. However, Güler's example shows that in an infinite dimensional Hilbert space, PPA has only weak convergence (see [12]). To obtain the strong convergence, several authors proposed modification of PPA; please, see: Kamimura and Takahashi [14], Kim and Xu [18], Qin and Su [22] and references therein. Recently, Kim and Tuyen [16] introduced a new iterative method for finding a common zero of two accretive operators in a Banach space. They considered the following sequence based on alternating resolvent method:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^B_{\gamma_n} J^A_{\beta_n} x_n, \quad n \geq 0, \tag{1}$$

where u and initial guess x_0 are arbitrarily taken from a closed convex set C and proved strong convergence theorem under the conditions:

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(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1;$

(iii) $\beta_n \geq \varepsilon$, $\gamma_n \geq \varepsilon$ for some $\varepsilon > 0$ and for all n, and

$$\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \quad \sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

In [17], Kim and Tuyen introduced another iterative method for finding a common zero of a finite family of monotone operators in a Hilbert space. They introduced iterative algorithm $\{x_n\}$, defined by

$$\begin{cases} y_n^0 = x_n, \quad n \geq 0, \\ y_n^i = (1-\beta_{i,n})\, y_n^{i-1} + \beta_{i,n}\, J_{i,n}\, y_n^{i-1}, \ i = 1, \ldots, m, \ n \geq 0, \ J_{i,n} = J_{r_{i,n}}^{A_i}, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n^m, \quad n \geq 0, \end{cases} \tag{2}$$

where u and initial guess x_0 are arbitrarily taken from a closed convex set C and proved its strong convergence under the assumptions:

(i) $\min_{i=1,2,\ldots,m} \{\inf\{r_{i,n}\}\} \geq \varepsilon > 0, \quad \sum_{n=0}^{\infty} |r_{i,n+1} - r_{i,n}| < \infty$, for all $i = 1, 2, \ldots, m$;

(ii) $\{\beta_{i,n}\} \subset (\alpha, \beta)$ with $\alpha, \beta \in (0,1)$ and $\sum_{n=0}^{\infty} |\beta_{i,n+1} - \beta_{i,n}| < \infty$, for all $i = 1, 2, \ldots, m$;

(iii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

These results bring us some natural questions:

(a) Does the result proved in [16, Theorem 3.2] remain true after removing the restrictive conditions $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ on $\{\alpha_n\}$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ on $\{\beta_n\}$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ on $\{\gamma_n\}$?

(b) Does the result proved in [17, Theorem 3.1] remain true after removing the restrictive conditions $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on $\{\alpha_n\}$, $\sum_{n=0}^{\infty} |r_{i,n+1} - r_{i,n}| < \infty$, for all $i = 1, 2, \ldots, m$, on $\{r_{i,n}\}$ and $\sum_{n=0}^{\infty} |\beta_{i,n+1} - \beta_{i,n}| < \infty$, for all $i = 1, 2, \ldots, m$, on $\{\beta_{i,n}\}$? Also $\{\beta_{i,n}\}$ are bounded between zero and one which excludes the important case $\beta_{i,n} \in [1,2]$, the over relaxed case. Can this condition be further relaxed as $0 < \liminf \beta_{i,n} \leq \limsup \beta_{i,n} < 2$?

In the present paper, we shall answer these questions affirmatively by removing the superfluous conditions on the parameters. Also, our approaches are simpler than those of Kim and Tuyen [16, 17]. Moreover, using our main results, we solve convex feasibility problem. Our results improve the results of Kim and Tuyen [16, 17], Qin and Su [22], Kim and Xu [18], Kamimura and Takahashi [14] and many others.

2. Preliminaries

Let X be a real Banach space and X^* be its dual space. Let J be the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If $X = H$ is a Hilbert space, then the duality mapping becomes the identity mapping.

The norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for $x, y \in U := \{x \in X : \|x\| = 1\}$. The norm is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit is attained uniformly for $x \in U$. The space X is said to have a Fréchet differentiable norm if for each $x \in U$, the limit in (3) is attained uniformly for $y \in U$. The space X is said to have a uniformly Fréchet differentiable norm (and X is said to be uniformly smooth) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$. It is known that X is smooth if and only if each duality mapping J is single valued.

The Banach space X is said to be strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$, for all $x, y \in X$, with $\|x\| = \|y\| = 1$ and $x \neq y$. Furthermore, the Banach space X is called uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

Let C be a nonempty closed convex subset of X. Let $T: C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Let D be a nonempty subset of C. Then a mapping Q is said to be retraction of C onto D if $Qx = x$, for all $x \in D$; Q is said to be sunny if $Q(Qx + \lambda(x - Qx)) = Qx$ holds whenever $x \in C$, $\lambda \geq 0$ and $Qx + \lambda(x - Qx) \in C$; D is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. In a Hilbert space, a sunny nonexpansive retraction Q coincides with the metric projection mapping.

A mapping $T: C \to X$ is said to be firmly nonexpansive [6] if

$$\|Tx - Ty\| \leq \|r(x - y) + (1 - r)(Tx - Ty)\|, \quad \forall\, x, y \in C,\ r \geq 0.$$

A sequence $\{T_n\}$ of mappings of C into X is said to be strongly nonexpansive sequence [1] if each T_n is nonexpansive and

$$\|x_n - y_n - (T_n x_n - T_n y_n)\| \to 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0.$$

Note that if we put $T_n = T$ for all $n \in \mathbb{N}$, then we have definition of strongly nonexpansive mapping defined in [7]. An operator A (possibly multivalued) with domain $D(A)$ and range $R(A)$ in X is called accretive, if for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0, \quad \forall u \in Ax,\ v \in Ay.$$

If $X = H$ is a Hilbert space, accretive operators are also called monotone. We denote by I the identity operator on X. An accretive operator A is said to be maximal if there is no proper accretive extension of A and m-accretive if $R(I + \lambda A) = X$, for all $\lambda > 0$. A is said to satisfy the range condition if $\overline{D(A)} \subseteq R(I + rA)$, for all $r > 0$. If A is accretive, one can define for each $r > 0$, a firmly nonexpansive single valued mapping

$$J_r^A: R(I + rA) \to D(A), \quad J_r^A = (I + rA)^{-1},$$

which is called resolvent of A and $F(J_r^A) = A^{-1}(0)$ (see [7]), where $F(J_r^A)$ is the fixed point set of J_r^A and $A^{-1}(0) = \{x \in D(A) : Ax = 0\}$.

Lemma 2.1 ([21]). *Let X be a Banach space. Then for $x, y \in X$, there exists $j(x+y) \in J(x+y)$ such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 2.2 ([27]). *Let $\{a_n\}$ be a sequences of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\alpha_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([3] The Resolvent Identity). *For each $\lambda, \mu > 0$,*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x\right).$$

Lemma 2.4 ([23]). *Let X be a uniformly smooth Banach space and let $T\colon C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $x \mapsto tu + (1-t)Tx$ converges strongly to a fixed point of T as $t \to 0$. Define $Q\colon C \to F(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retraction from C onto $F(T)$; that is, Q satisfies the property*

$$\langle u - Q(u),\ J(p - Q(u))\rangle \leq 0, \quad u \in C,\ p \in F(T).$$

Lemma 2.5 ([8]). *If all the conditions of Lemma 2.4 are satisfied and $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then*

$$\limsup_{n\to\infty} \langle u - Q(u), J(x_n - Q(u))\rangle \leq 0.$$

Lemma 2.6 ([7]). *If X is a uniformly convex Banach space, then every firmly nonexpansive mapping is strongly nonexpansive.*

Lemma 2.7 ([7]). *If $\{T_i : 1 \leq i \leq k\}$ is a family of strongly nonexpansive mappings and $\bigcap_{i=1}^{k} \{F(T_i) : 1 \leq i \leq k\} \neq \emptyset$, then $\bigcap_{i=1}^{k} \{F(T_i) : 1 \leq i \leq k\} = F(T_k T_{k-1} \cdots T_2 T_1)$.*

Lemma 2.8 ([2]). *Let C be a nonempty subset of a uniformly convex Banach space X. Let $\{S_n\}$ be a sequence of firmly nonexpansive mappings of C into X. Then $\{S_n\}$ is a strongly nonexpansive sequence.*

Lemma 2.9 ([2]). *Let C be a nonempty subset of a uniformly convex Banach space X. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into X and $\{\lambda_n\}$ a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \lambda_n > 0$. Then, a sequence $\{S_n\}$ of mappings of C into X defined by $S_n = \lambda_n I + (1-\lambda_n)T_n$, for $n \in \mathbb{N}$, is a strongly nonexpansive sequence, where I is the identity mapping on C.*

Lemma 2.10 ([2]). *Let C and D be two nonempty subsets of a Banach space X. Let $\{S_n\}$ be a sequence of mappings of C into X and $\{T_n\}$ a sequence of mappings of D into X. Suppose that both $\{S_n\}$ and $\{T_n\}$ are strongly nonexpansive sequences such that $T_n(D) \subset C$ for each $n \in \mathbb{N}$. Then $\{S_n T_n\}$ is a strongly nonexpansive sequence.*

Lemma 2.11 ([20]). *Let $\{c_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $c_{n_i} < c_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_q\} \subset \mathbb{N}$ such that $m_q \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $q \in \mathbb{N}$:*

$$c_{m_q} \leq c_{m_q+1}, \quad c_q \leq c_{m_q+1}.$$

In fact, $m_q = \max\{j \leq q : c_j < c_{j+1}\}$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X. Let $A_i\colon D(A_i) \subseteq C \to 2^X$ be accretive operators such that $S := \bigcap_{i=1}^{m} A_i^{-1}0 \neq \emptyset$ and $\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$, for all $i = 1, 2, \ldots, m$. Let $\{\alpha_n\} \subset (0, 1)$ and $\{r_{i,n}\}$ be sequences of positive numbers satisfying the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$

(*ii*) $\{r_{i,n}\} \geq \varepsilon$ *for some* $\varepsilon > 0$, *for all* $n \geq 0$ *and for all* $i = 1, 2, \ldots, m$.

Then the sequence $\{x_n\}$ defined by x_0, $u \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{m,n} J_{m-1,n} \cdots J_{1,n} x_n, \quad n \geq 0, \ J_{i,n} = J^{A_i}_{r_{i,n}}, \tag{4}$$

converges strongly to $Q_s u$, where $Q_s\colon C \to S$ is a sunny nonexpansive retraction from C onto S.

Proof. We start the proof with the boundedness of the sequence $\{x_n\}$.

Let $z = Q_s u (\in S)$ and put $T_n = J_{m,n} J_{m-1,n} \cdots J_{1,n}$, for each $n \geq 0$. Clearly, T_n is nonexpansive for each $n \geq 0$. By using Lemma 2.6, for each $n \geq 0$, T_n is composition of strongly nonexpansive mappings. Therefore, from Lemma 2.7, we get

$$\emptyset \neq S := \bigcap_{i=1}^{m} A_i^{-1}0 = \bigcap_{i=1}^{m} F(J_{i,n}) = F(T_n),$$

for each $n \geq 0$. From (4), we have

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|T_n x_n - z\| \\ &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_n - z\|\}. \end{aligned}$$

By induction on n, we get

$$\|x_{n+1} - z\| \leq \max\{\|u - z\|, \|x_0 - z\|\}.$$

This implies that $\{x_n\}$ is bounded.

Again from (4), we see that

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1 - \alpha_n) \|T_n x_n - z\| \\ &\leq \alpha_n \|u - z\| + \|T_n x_n - z\|. \end{aligned} \tag{5}$$

Using the nonexpansiveness of T_n and (5), we observe that

$$\begin{aligned} 0 &\leq \|x_n - z\| - \|T_n x_n - z\| \\ &\leq \|x_n - z\| - \|x_{n+1} - z\| + \alpha_n \|u - z\|. \end{aligned} \tag{6}$$

Again from (4), we get

$$\begin{aligned}\|x_{n+1}-z\| &\le \alpha_n\|u-z\| + (1-\alpha_n)\|J_{m,n}J_{m-1,n}\cdots J_{1,n}x_n - z\|\\ &\le \alpha_n\|u-z\| + \|J_{m-1,n}\cdots J_{1,n}x_n - z\|. \qquad (7)\end{aligned}$$

Using the nonexpansiveness of $J_{m-1,n}J_{m-2,n}\cdots J_{1,n}$ and (7), we observe that

$$\begin{aligned}0 &\le \|x_n - z\| - \|J_{m-1,n}J_{m-2,n}\cdots J_{1,n}x_n - z\|\\ &\le \|x_n - z\| - \|x_{n+1}-z\| + \alpha_n\|u-z\|. \qquad (8)\end{aligned}$$

Continuing like (6) and (8), for each $i = 1, 2, \ldots, m$, we can obtain

$$\begin{aligned}0 &\le \|x_n - z\| - \|J_{i,n}J_{i-1,n}\cdots J_{1,n}x_n - z\|\\ &\le \|x_n - z\| - \|x_{n+1}-z\| + \alpha_n\|u-z\|. \qquad (9)\end{aligned}$$

Now, in order to prove that $x_n \to z$ as $n \to \infty$, we consider two possible cases.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the real sequence $\{\|x_n - z\|\}$ is nonincreasing for all $n \ge n_0$. Since $\{\|x_n - z\|\}$ is bounded, $\{\|x_n - z\|\}$ is convergent. Therefore, using the given condition $\alpha_n \to 0$ in (9), for each $i = 1, 2, \ldots, m$, we obtain

$$\|x_n - z\| - \|J_{i,n}J_{i-1,n}\cdots J_{1,n}x_n - z\| \to 0 \quad \text{as } n \to \infty.$$

Using Lemma 2.8 and Lemma 2.10, for each $i = 1, 2, \ldots, m$, $\{J_{i,n}J_{i-1,n}\cdots J_{1,n}\}$ is strongly nonexpansive sequence. Therefore, we have

$$\|x_n - J_{i,n}J_{i-1,n}\cdots J_{1,n}x_n\| \to 0 \quad \text{as } n \to \infty. \qquad (10)$$

Next, we show that

$$\|x_n - J_{i,n}x_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{for each } i = 1, 2, \ldots, m. \qquad (11)$$

Clearly, from (10) for $i = 1$, (11) is true. Now for $i = 2, 3, \ldots, m$, we see that

$$\begin{aligned}\|x_n - J_{i,n}x_n\| &\le \|x_n - J_{i,n}J_{i-1,n}\cdots J_{1,n}x_n\| + \|J_{i,n}J_{i-1,n}\cdots J_{1,n}x_n - J_{i,n}x_n\|\\ &\le \|x_n - J_{i,n}J_{i-1,n}\cdots J_{1,n}x_n\| + \|J_{i-1,n}J_{i-2,n}\cdots J_{1,n}x_n - x_n\|.\end{aligned}$$

Thus, we have $\|x_n - J^{A_i}_{r_{i,n}}x_n\| \to 0$ as $n \to \infty$, for each $i = 1, 2, \ldots, m$.

Take a fixed number s such that $\varepsilon > s > 0$ and using Lemma 2.3, for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\|x_n - J^{A_i}_s x_n\| &\le \|x_n - J^{A_i}_{r_{i,n}}x_n\| + \|J^{A_i}_{r_{i,n}}x_n - J^{A_i}_s x_n\|\\ &\le \|x_n - J^{A_i}_{r_{i,n}}x_n\| + \left\|J^{A_i}_s\Big(\frac{s}{r_{i,n}}x_n + \Big(1-\frac{s}{r_{i,n}}\Big)J^{A_i}_{r_{i,n}}x_n\Big) - J^{A_i}_s x_n\right\|\Big)\\ &\le \|x_n - J^{A_i}_{r_{i,n}}x_n\| + \left\|\frac{s}{r_{i,n}}x_n + \Big(1-\frac{s}{r_{i,n}}\Big)J^{A_i}_{r_{i,n}}x_n - x_n\right\|\\ &= \|x_n - J^{A_i}_{r_{i,n}}x_n\| + \Big(1-\frac{s}{r_{i,n}}\Big)\|J^{A_i}_{r_{i,n}}x_n - x_n\|\\ &\le 2\|x_n - J^{A_i}_{r_{i,n}}x_n\|.\end{aligned}$$

Hence, we obtain

$$\|x_n - J^{A_i}_s x_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{for each } i = 1, 2, \ldots, m. \qquad (12)$$

Next, we show that $\|x_n - Wx_n\| \to 0$ as $n \to \infty$, where $W = \dfrac{1}{m}\Big(\sum_{i=1}^{m} J^{A_i}_s\Big)$ and $0 < s < \varepsilon$. Using (12), we obtain

$$\|x_n - Wx_n\| \le \frac{1}{m}\sum_{i=1}^{m}\|x_n - J^{A_i}_s x_n\| \to 0 \quad \text{as } n \to \infty.$$

Next, we show that $\limsup_{n\to\infty}\langle u - z, J(x_n - z)\rangle \le 0$, where $z = Q_S u$.

For $t \in (0,1)$, let $x_t \in C$ be unique fixed point of the contraction mapping $R_t x = tu + (1-t)Wx$, $x \in C$. Then, by Lemma 2.4, $x_t = tu + (1-t)Wx_t$ converges strongly to $Q_s u \in F(W) = S$.

Note that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$ as $n \to \infty$. Thus, using Lemma 2.5, we obtain $\limsup_{n\to\infty}\langle u - z, J(x_n - z)\rangle \leq 0$. Finally, we claim that $x_n \to z$ as $n \to \infty$. Using Lemma 2.1, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)T_n x_n - z\|^2 \\ &\leq (1-\alpha_n)^2\|T_n x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle. \end{aligned} \tag{13}$$

By Lemma 2.2, we get $x_n \to z (= Q_s u)$ as $n \to \infty$.

Case 2. Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\|x_{n_j} - z\| < \|x_{n_j+1} - z\|, \quad \forall\, j \geq 0.$$

Then, by Lemma 2.11, there exists a nondecreasing sequence of integers $\{m_q\} \subset \mathbb{N}$ such that $m_q \to \infty$ as $q \to \infty$ and

$$\|x_{m_q} - z\| \leq \|x_{m_q+1} - z\| \ and \ \|x_q - z\| \leq \|x_{m_q+1} - z\|, \tag{14}$$

for all $q \in \mathbb{N}$. Now, using (14) in (9), we have

$$\begin{aligned} 0 &\leq \|x_{m_q} - z\| - \|J_{i,m_q}J_{i-1,m_q}\cdots J_{1,m_q}x_{m_q} - z\| \\ &\leq \|x_{m_q} - z\| - \|x_{m_q+1} - z\| + \alpha_{m_q}\|u - z\| \\ &\leq \alpha_{m_q}\|u - z\|. \end{aligned}$$

Since $\alpha_{m_q} \to 0$, we obtain

$$\|x_{m_q} - z\| - \|J_{i,m_q}J_{i-1,m_q}\cdots J_{1,m_q}x_{m_q} - z\| \to 0 \ as \ q \to \infty$$

As $\{J_{i,m_q}J_{i-1,m_q}\cdots J_{1,m_q}\}$ is strongly nonexpansive sequence for each $i = 1, 2, ..\ ., m$, we have

$$\|x_{m_q} - J_{i,m_q}J_{i-1,m_q}\cdots J_{1,m_q}x_{m_q}\| \to 0 \quad \text{as } q \to \infty.$$

Following similar arguments as in case 1, we can obtain

$$\limsup_{q\to\infty}\langle u - z, J(x_{m_q} - z)\rangle \leq 0, \tag{15}$$

where $z = Q_s u$. Now, from (13), we have

$$a_{m_q+1} \leq (1-\alpha_{m_q})a_{m_q} + \alpha_{m_q}\delta_{m_q}, \tag{16}$$

where $a_{m_q} = \|x_{m_q} - z\|^2$ and $\delta_{m_q} = 2\langle u - z, J(x_{m_q+1} - z)\rangle$.
Thus, (14) and (16) implies that

$$\begin{aligned} \alpha_{m_q}a_{m_q} &\leq a_{m_q} - a_{m_q+1} + \alpha_{m_q}\delta_{m_q} \\ \alpha_{m_q}a_{m_q} &\leq \alpha_{m_q}\delta_{m_q}. \end{aligned}$$

Using the fact that $\alpha_{m_q} > 0$, we obtain

$$a_{m_q} \leq \delta_{m_q},$$

that is,

$$\|x_{m_q} - z\|^2 \leq 2\langle u - z, J(x_{m_q+1} - z)\rangle,$$

it follows from (15) that

$$\|x_{m_q} - z\| \to 0 \ as \ q \to \infty.$$

This together with (16) implies that $\|x_{m_q+1} - z\| \to 0 \ as \ q \to \infty$. But $\|x_q - z\| \leq \|x_{m_q+1} - z\|$ for all $q \in \mathbb{N}$, which gives that $x_q \to z$ as $q \to \infty$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A_i\colon D(A_i) \subseteq C \to 2^H$ be monotone operators such that $S := \bigcap_{i=1}^{m} A_i^{-1}0 \neq \emptyset$ and $\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$, for all $i = 1, 2, \ldots, m$. If the sequences $\{\alpha_n\}$, $\{\beta_{i,n}\}$ and $\{r_{i,n}\}$ satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$, $\quad \lim_{n\to\infty} \alpha_n = 0$, $\quad \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{r_{i,n}\} \geq \varepsilon$, for some $\varepsilon > 0$, for all $n \geq 0$ and for all $i = 1, 2, \ldots, m$;

(iii) $\{\beta_{i,n}\} \subset (0,2)$, $\quad 0 < \liminf_{n\to\infty} \beta_{i,n} \leq \limsup_{n\to\infty} \beta_{i,n} < 2$, for all $i = 1, 2, \ldots, m$.

Then the sequence $\{x_n\}$ defined by (2) converges strongly to $P_S u$, where $P_S\colon C \to S$ is a metric projection from C onto S.

Proof. Firstly, we rewrite (2) as

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) S_{m,n} S_{m-1,n} \cdots S_{1,n} x_n \\ S_{i,n} = (1-\beta_{i,n}) I + \beta_{i,n} J_{i,n}, \quad J_{i,n} = J_{r_{i,n}}^{A_i}, \ i = 1, 2, \ldots, m, \ n \geq 0. \end{cases} \tag{17}$$

Also, note that for each $i = 1, 2, \ldots, m$, $S_{i,n}$ can further be written as

$$\begin{aligned} S_{i,n} &= \Big(1 - \frac{\beta_{i,n}}{2}\Big) I + \frac{\beta_{i,n}}{2}(2J_{i,n} - I) \\ &= (1-\gamma_{i,n}) I + \gamma_{i,n} R_{i,n}, \end{aligned} \tag{18}$$

where $\gamma_{i,n} = \frac{\beta_{i,n}}{2}$ and $R_{i,n} = 2J_{i,n} - I$. Clearly, $0 < \liminf_{n\to\infty} \gamma_{i,n} \leq \limsup_{n\to\infty} \gamma_{i,n} < 1$ and $R_{i,n}$ is nonexpansive mapping for each $i = 1, 2, \ldots, m$.

Let $T_n = S_{m,n} S_{m-1,n} \cdots S_{1,n}$ and $z = P_S u (\in S)$. Using Lemma 2.9, $S_{i,n}$ is strongly nonexpansive for each $i = 1, 2, \ldots, m$ and $n \geq 0$. Also, from Lemma 2.7, we obtain $F(T_n) = \bigcap_{i=1}^{m} F(S_{i,n})$, for all $n \geq 0$. Note that

$$\emptyset \neq S = \bigcap_{i=1}^{m} A_i^{-1}(0) = \bigcap_{i=1}^{m} F(J_{i,n}) = \bigcap_{i=1}^{m} F(R_{i,n}) = \bigcap_{i=1}^{m} F(S_{i,n}) = F(T_n).$$

Further, being composition of nonexpansive mappings, T_n is nonexpansive for each $n \geq 0$. It follows from (17) that

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|T_n x_n - z\| \\ &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_n - z\|\}. \end{aligned}$$

By induction on n, we get

$$\|x_{n+1} - z\| \leq \max\{\|u - z\|, \|x_0 - z\|\}.$$

Thus, $\{x_n\}$ is bounded.

Again from (17), we see that

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|T_n x_n - z\| \\ &\leq \alpha_n \|u - z\| + \|T_n x_n - z\|. \end{aligned} \tag{19}$$

Using the nonexpansiveness of T_n and (19), we obtain

$$\begin{aligned} 0 &\leq \|x_n - z\| - \|T_n x_n - z\| \\ &\leq \|x_n - z\| - \|x_{n+1} - z\| + \alpha_n \|u - z\|. \end{aligned} \tag{20}$$

Since $S_{m,n}$ is nonexpansive, we obtain from (19)

$$\|x_{n+1} - z\| \leq \alpha_n \|u - z\| + \|S_{m-1,n} \cdots S_{1,n} x_n - z\|. \tag{21}$$

Also, from the nonexpansiveness of $S_{m-1,n} S_{m-2,n} \cdots S_{1,n}$ and (21), it follows that

$$\begin{aligned} 0 &\leq \|x_n - z\| - \|S_{m-1,n} S_{m-2,n} \cdots S_{1,n} x_n - z\| \\ &\leq \|x_n - z\| - \|x_{n+1} - z\| + \alpha_n \|u - z\|. \end{aligned} \tag{22}$$

Continuing like (20) and (22), for each $i = 1, 2, \ldots, m$, we can obtain

$$\begin{aligned} 0 &\leq \|x_n - z\| - \|S_{i,n} S_{i-1,n} \cdots S_{1,n} x_n - z\| \\ &\leq \|x_n - z\| - \|x_{n+1} - z\| + \alpha_n \|u - z\|. \end{aligned} \tag{23}$$

Now, in order to prove that $x_n \to z$ as $n \to \infty$, we consider two possible cases.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that the real sequence $\{\|x_n - z\|\}$ is nonincreasing for all $n \geq n_0$. Since $\{\|x_n - z\|\}$ is bounded, $\{\|x_n - z\|\}$ is convergent. Therefore, using the given condition $\alpha_n \to 0$ in (23), for each $i = 1, 2, \ldots, m$, we obtain

$$\|x_n - z\| - \|S_{i,n} S_{i-1,n} \cdots S_{1,n} x_n - z\| \to 0 \quad \text{as } n \to \infty.$$

Using Lemma 2.9 and Lemma 2.10, for each $i = 1, 2, \ldots, m$, $\{S_{i,n} S_{i-1,n} \cdots S_{1,n}\}$ is strongly nonexpansive sequence. Therefore, we have

$$\|x_n - S_{i,n} S_{i-1,n} \cdots S_{1,n} x_n\| \to 0 \quad \text{as } n \to \infty. \tag{24}$$

Next, we show that

$$\|x_n - S_{i,n} x_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{for each } i = 1, 2, \ldots, m. \tag{25}$$

Clearly, from (24) for $i = 1$, (25) is true. Now, for $i = 2, 3, \ldots, m$, we see that

$$\begin{aligned} \|x_n - S_{i,n} x_n\| &\leq \|x_n - S_{i,n} S_{i-1,n} \cdots S_{1,n} x_n\| + \|S_{i,n} S_{i-1,n} \cdots S_{1,n} x_n - S_{i,n} x_n\| \\ &\leq \|x_n - S_{i,n} S_{i-1,n} \cdots S_{1,n} x_n\| + \|S_{i-1,n} S_{i-2,n} \cdots S_{1,n} x_n - x_n\|. \end{aligned}$$

Thus, using (24), we obtain (25).
Now, for each $i = 1, 2, \ldots, m$, it follows from (18) that

$$\|x_n - S_{i,n} x_n\| = \|\gamma_{i,n} x_n - \gamma_{i,n} R_{i,n} x_n\| = 2\gamma_{i,n} \|x_n - J_{i,n} x_n\|. \tag{26}$$

Using (25) and $\liminf_{n \to \infty} \gamma_{i,n} > 0$ in (26), we obtain

$$\|x_n - J_{i,n} x_n\| \to 0 \quad \text{as } n \to \infty, \quad \text{for each } i = 1, 2, \ldots, m.$$

Now, following the proof lines of Theorem 3.1, we can obtain

$$\limsup_{n \to \infty} \langle u - z, x_n - z \rangle \leq 0,$$

where $z = P_S u$. Finally, we claim that $x_n \to z$ as $n \to \infty$. Using Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n u + (1 - \alpha_n) T_n x_n - z\|^2 \\ &\leq (1 - \alpha_n)^2 \|T_n x_n - z\|^2 + 2\alpha_n < u - z, x_{n+1} - z > \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n < u - z, x_{n+1} - z > . \end{aligned}$$

By Lemma 2.2, we get $x_n \to z$ as $n \to \infty$.
Similarly, for case 2, we follow the proof lines of Theorem 3.1 and obtain the required result. □

4. Application and numerical example

Using above results, we solve the Convex Feasibility Problem (CFP), that is, finding an element

$$\bar{x} \in S := \bigcap_{i=1}^{m} C_i,$$

where C_i, $i = 1, 2, \ldots, m$, are closed and convex sets in a real Hilbert space H.

Let $f\colon H \to]-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential

$$\partial f(x) = \{y \in H : f(x) + \langle z - x, y\rangle \le f(z),\ \forall z \in H\},$$

for all $x \in H$. By Rockafellar theorem [24], ∂f is a maximal monotone operator of H into itself.

Let C be a closed convex subset of H and i_C be the indicator function of C, that is

$$i_C x = \begin{cases} 0, & x \in C \\ \infty, & x \notin C. \end{cases}$$

Also recall, the normal cone for C at a point $x \in C$ is defined by

$$N_C(x) = \{y \in H : \langle y, z - x\rangle \le 0,\ \forall z \in H\}.$$

Since $i_C\colon H \to]-\infty, \infty]$ is a proper lower semicontinuous convex function, ∂i_C is a maximal monotone operator. Also, it is known that $\partial i_C = N_C$ (see [4, Ex. 16.12]). So, from Theorem 3.2 and by using the equality

$$(I + r\partial i_C)^{-1} = (I + rN_C)^{-1} = P_C,$$

for all closed convex subset C in H and for all $r > 0$, we solve the CFP as follows:

Theorem 4.1. *Let C_i, $i = 1, 2, \ldots, m$, be closed and convex sets in a real Hilbert space H such that $S := \bigcap_{i=1}^{m} C_i \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_{i,n}\}$ be the sequences of positive real numbers satisfying the following conditions:*

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\beta_{i,n}\} \subset (0, 2)$, $0 < \liminf_{n\to\infty} \beta_{i,n} \le \limsup_{n\to\infty} \beta_{i,n} < 2$, *for all* $i = 1, 2, \ldots, m$;

then the sequence $\{x_n\}$ defined by x_0, $u \in H$ and

$$\begin{cases} y_n^0 = x_n, & n \ge 0, \\ y_n^i = (1 - \beta_{i,n}) y_n^{i-1} + \beta_{i,n} P_{C_i} y_n^{i-1}, & i = 1, 2, \ldots, m,\ n \ge 0, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n^m, & n \ge 0, \end{cases} \tag{27}$$

converges strongly to some $x^* \in S$.

Proof. Put $A_i = N_{C_i} (i = 1, 2, ..., m)$ in Theorem 3.2, where N_{C_i} is normal cone of closed and convex set C_i, we obtain the desired result. □

As illustration to our algorithm, let us solve a CFP by applying Theorem 4.1. Let C_1 and C_2 be two solid cuboids in $\mathbb{R}^3$ defined as

$$\begin{aligned} C_1 &= \{(x, y, z) : 0 \le x \le 2\ ; 0 \le y \le 1\ ; 0 \le z \le 1.5\}, \\ C_2 &= \{(x, y, z) : 1 \le x \le 4\ ; 0.5 \le y \le 2\ ; 1 \le z \le 3\}. \end{aligned}$$

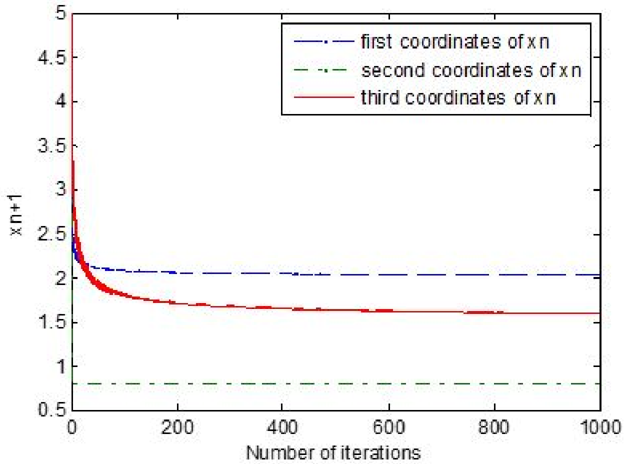
Clearly $S = C_1 \cap C_2 \neq \emptyset$. Applying the iterative algorithm (27) with $x_1 = (4, 3.6, 5)$ and $u = (2.5, 0.8, 3.5)$ (arbitrary chosen), $\beta_{1,n} = 1$, $\beta_{2,n} = \dfrac{3}{2}$ and $\{\alpha_n\}$ is given by $\alpha_{2n} = (n+1)^{-\frac{1}{2}}$ and $a_{2n-1} = (n + 1)^{-\frac{1}{2}} + (n + 1)^{-1}$, $n \in \mathbb{N}$. Note that we relax condition $\beta_{i,n} \in (0, 1)$

to $\beta_{i,n} \in (0,2)$ and also, α_n does not satisfy extra condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Table 1: Numerical experiment of the algorithm (27)

n	x_{n+1}
0	$(4, 3.6, 5)$
1	$(2.60355, 0.758579, 3.91421)$
2	$(2.35355, 0.787868, 2.91421)$
3	$(2.45534, 0.798916, 3.32137)$
4	$(2.28868, 0.799542, 2.6547)$
5	$(2.375, 0.799886, 3)$
6	$(2.25, 0.799943, 2.5)$
7	$(2.32361, 0.79998, 2.79443)$
8	$(2.22361, 0.799989, 2.39443)$
15	$(2.22222, 0.8, 2.38889)$
100	$(2.07001, 0.8, 1.78006)$
1000	$(2.02234, 0.8, 1.58935)$

Figure 1: The exact solution is $x^* = P_S u = (2, 0.8, 1.5)$



5. Conclusions

In this paper, we establish strong convergence theorems using iterative algorithms (4) and (2) for finding common zeros of finite accretive operators and remove many superfluous conditions on the parameters which were used in the results of Kim and Tuyen [16, 17]. Further, by using our main results, we solve convex feasibility problem. The results presented in this paper improve the previous work from the current existing literature (see, for example, [14, 16, 17, 18, 22] and many others).

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