SOME TRIDIAGONAL DETERMINANTS RELATED TO CENTRAL DELANNOY NUMBERS, THE CHEBYSHEV POLYNOMIALS, AND THE FIBONACCI POLYNOMIALS

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In the paper, the authors give a motivation from central Delannoy numbers to a tridiagonal determinant, find a generating function for the tridiagonal determinant, prove several formulas for the tridiagonal determinant, discuss the inverse of the tridiagonal matrix, connect the tridiagonal determinant with the Chebyshev polynomials, the Fibonacci numbers and polynomials, and the Cauchy product of central Delannoy numbers, derive several formulas for the tridiagonal determinant and the second kind Chebyshev polynomials, review computation of general tridiagonal determinants, present two new formulas for computing general tridiagonal determinants, and generalize central Delannoy numbers and their Cauchy product.

Keywords: tridiagonal determinant, Cauchy product, Fibonacci number, Fibonacci polynomial, Chebyshev polynomial, central Delannoy number, inverse of tridiagonal matrix, generating function, Bell polynomial of the second kind, Faà di Bruno formula.

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1. A motivation from central Delannoy numbers

Let

$$M_k(c)=\begin{pmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{pmatrix}_{k\times k}, \quad c\in\mathbb{C}, \quad k\in\mathbb{N} \tag{1}$$

and denote the determinant $|M_k(c)|$ of the $k\times k$ tridiagonal matrix $M_k(c)$ by $D_k(c)$. From some results in [15, Theorem 1.2] for the Cauchy products of central Delannoy numbers, the explicit expression $D_k(-6)=\frac{1}{6^k}\sum_{\ell=0}^{k}(-1)^\ell 6^{2\ell}\binom{\ell}{k-\ell}$ was derived in [15, Remark 4.4], where $\binom{p}{q}=0$ for $q>p\ge0$. Hereafter, the authors guessed in [15, Remar 4.4] that the formula

$$D_k(c)=(-1)^k\sum_{\ell=0}^{k}(-1)^\ell c^{2\ell-k}\binom{\ell}{k-\ell}=\sum_{m=0}^{k}(-1)^m c^{k-2m}\binom{k-m}{m} \tag{2}$$

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should be valid for all $c \in \mathbb{C}$ and $k \in \mathbb{N}$ and claimed that this guess can be straightforwardly verified by induction on $k \in \mathbb{N}$.

In this paper, we will discover a generating function of the sequence $D_k(c)$, provide an analytic proof of the formula (2), establish a simpler formula for $D_k(c)$, find a determinantal expression for $D_k(c)$, present the inverse of $M_k(c)$, connect $D_k(c)$ with the Chebyshev polynomials and the Fibonacci numbers and polynomials, review computation of general diagonal determinants, supply two formulas for computing general diagonal determinants, generalize central Delannoy numbers, and represent the Cauchy product of the generalized central Delannoy numbers in terms of $D_k(c)$.

2. A generating function for $D_k(c)$

We now discover a generating function of the sequence $D_k(c)$.

Theorem 2.1. *Let $D_0(c) = 1$. Then the sequence $D_k(c)$ for $k \ge 0$ can be generated by $\frac{1}{t^2-ct+1} = \sum_{k=0}^{\infty} D_k(c)t^k$.*

Proof. It is clear that

$$D_1(c) = c \quad \text{and} \quad D_2(c) = c^2 - 1. \tag{3}$$

By expanding the determinant $D_k(c)$ according to the last row or column, we can obtain the recurrence relation

$$D_k(c) = cD_{k-1}(c) - D_{k-2}(c), \quad k \ge 2. \tag{4}$$

Multiplying by t^k and summing with respect to k from 2 to ∞ give

$$\sum_{k=2}^{\infty} D_k(c)t^k = ct\sum_{k=2}^{\infty} D_{k-1}(c)t^{k-1} - t^2\sum_{k=2}^{\infty} D_{k-2}(c)t^{k-2}.$$

Let $F_c(t)$ be a generating function of $D_k(c)$ for $k \ge 0$, that is,

$$F_c(t) = \sum_{k=0}^{\infty} D_k(c)t^k. \tag{5}$$

Then $F_c(t) - 1 - ct = ct[F_c(t) - 1] - t^2F_c(t)$, that is,

$$F_c(t) = \frac{1}{t^2 - ct + 1}, \tag{6}$$

which is a generating function of the infinite sequence $D_k(c)$ for $k \ge 0$. □

We now provide an analytic proof of the formula (2).

Theorem 2.2. *For $k \ge 0$ and $c \in \mathbb{C}$, the formula (2) is valid.*

Proof. In combinatorics, the second kind Bell polynomials are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \ge k \ge 0$. See [5, p. 134, Theorem A]. They satisfy the identities

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^kb^n\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{7}$$

and

$$\mathrm{B}_{n,k}(z, 1, 0, \dots, 0) = \frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}z^{2k-n} \tag{8}$$

for complex numbers $a, b, z \in \mathbb{C}$. See [5, p. 135] and [17, Theorem 4.1] respectively. The Faà di Bruno formula can be described [5, p. 139, Theorem C] by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) \mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big). \tag{9}$$

Let $u = h(t) = t^2 - ct + 1$. Then, by the Faà di Bruno formula (9) and the identities (7) and (8), we have

$$\begin{aligned}
[F_c(t)]^{(k)} &= \sum_{\ell=0}^{k} \left(\frac{1}{u}\right)^{(\ell)} \mathrm{B}_{k,\ell}\big(h'(t), h''(t), h'''(t), \dots, h^{(k-\ell+1)}\big) \\
&= \sum_{\ell=0}^{k} \frac{(-1)^\ell \ell!}{u^{\ell+1}} \mathrm{B}_{k,\ell}(2t-c, 2, 0, \dots, 0) \to \sum_{\ell=0}^{k} (-1)^\ell \ell! \mathrm{B}_{k,\ell}(-c, 2, 0, \dots, 0) \\
&= \sum_{\ell=0}^{k} (-1)^\ell \ell! 2^\ell \frac{1}{2^{k-\ell}} \frac{k!}{\ell!} \binom{\ell}{k-\ell} \left(-\frac{c}{2}\right)^{2\ell-k} = (-1)^k k! \sum_{\ell=0}^{k} (-1)^\ell \binom{\ell}{k-\ell} c^{2\ell-k}
\end{aligned}$$

as $t \to 0$. Considering (5) and (6) proves the explicit formula (2). ☐

3. A simpler formula for $D_k(c)$

We now establish a simpler formula for the tridiagonal determinant $D_k(c)$.

Theorem 3.1. *For $c \in \mathbb{C}$, $\alpha = \frac{1}{\beta} = \frac{c+\sqrt{c^2-4}}{2}$, and $k \ge 0$, we have*

$$D_k(c) = \begin{cases} \dfrac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}, & c \ne \pm 2; \\ k+1, & c = 2; \\ (-1)^k (k+1), & c = -2. \end{cases} \tag{10}$$

Proof. Usually, one looks for a solution of the recurrence relation $a_k = c_1 a_{k-1} + c_2 a_{k-2} + \dots + c_{m-1} a_{k-m+1} + c_m a_{k-m}$ by considering a sum $a_k = b_1 r_1^k + b_2 r_2^k + \dots + b_{m-1} r_{m-1}^k + b_m r_m^k$. A necessary condition on r_i is that r_i are roots of the characteristic equation $q^m = c_1 q^{m-1} + c_2 q^{m-2} + \dots + c_{m-1} q + c_m$. The constants c_i can be determined from the initial conditions of a_k. As a result, the recurrence relation (4) implies that $D_k(c) = Aa^k + Bb^k$, where a, b are roots of the characteristic equation $q^2 = cq - 1$ associated with (4), provided $c \ne \pm 2$, and A, B are constants which can be determined from identities in (3). ☐

4. A determinantal expression for $D_k(c)$

We now find a determinantal expression for $D_k(c)$ alternatively.

Theorem 4.1. *For $k \ge 1$ and $c \in \mathbb{C}$, we have*

$$D_k(c) = \frac{(-1)^k}{k!} \begin{vmatrix} -c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 2 & -2c & 1 & \cdots & 0 & 0 & 0 \\ 0 & 6 & -3c & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(k-2)c & 1 & 0 \\ 0 & 0 & 0 & \cdots & (k-1)(k-2) & -(k-1)c & 1 \\ 0 & 0 & 0 & \cdots & 0 & k(k-1) & -kc \end{vmatrix}.$$

Proof. Let $u(x)$ and $v(x) \ne 0$ be differentiable. Let $U_{(n+1)\times 1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements $v_{i,j}(x)=\begin{cases}\binom{i-1}{j-1}v^{(i-j)}(x), & i-j\ge 0\\ 0, & i-j<0\end{cases}$ for $1\le i\le n+1$ and $1\le j\le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ is the determinant of the $(n+1)\times(n+1)$ matrix $W_{(n+1)\times(n+1)}(x)=\begin{pmatrix}U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x)\end{pmatrix}$. Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[\frac{u(x)}{v(x)}\right]=(-1)^n\frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)}. \tag{11}$$

This formula (11) can be found in the paper [16, p. 94]. Applying (11) to $u(t)=1$ and $v(t)=t^2-ct+1$ and taking the limit $t\to 0$ give

$$\begin{aligned}\left(\frac{1}{t^2-ct+1}\right)^{(k)}&=\frac{(-1)^k}{(t^2-ct+1)^{k+1}}\\ &\times\begin{vmatrix}1 & t^2-ct+1 & 0 & \cdots & 0 & 0\\ 0 & \binom{1}{0}(2t-c) & t^2-ct+1 & \cdots & 0 & 0\\ 0 & \binom{2}{0}2 & \binom{2}{1}(2t-c) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \binom{k-1}{k-2}(2t-c) & t^2-ct+1\\ 0 & 0 & 0 & \cdots & \binom{k}{k-2}2 & \binom{k}{k-1}(2t-c)\end{vmatrix}\\ &\to(-1)^k\begin{vmatrix}1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & -\binom{1}{0}c & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & \binom{2}{0}2 & -\binom{2}{1}c & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \binom{k-1}{k-3}2 & -\binom{k-1}{k-2}c & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & \binom{k}{k-2}2 & -\binom{k}{k-1}c\end{vmatrix}.\end{aligned}$$

The proof of Theorem 4.1 is thus complete. □

5. The inverse of $M_k(c)$

Basing on discussions about the inverse of $M_k(c)$ in [7, Eq. (9)], we can derive the following Theorem 5.1 straightforwardly.

Theorem 5.1. *For $k\in\mathbb{N}$, $c\in\mathbb{C}$, and $\alpha=\frac{1}{\beta}=\frac{c+\sqrt{c^2-4}}{2}$, the inverse $M_k^{-1}(c)$ of the tridiagonal matrix $M_k(c)$ can be computed by $M_k^{-1}(c)=\left(R_{ij}\right)_{k\times k}$, where*

$$R_{ij}=\begin{cases}(-1)^{i+j}\dfrac{\left(\alpha^i-\beta^i\right)\left(\alpha^{k-j+1}-\beta^{k-j+1}\right)}{(\alpha-\beta)\left(\alpha^{k+1}-\beta^{k+1}\right)}, & c\ne\pm2\\ (-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c=2\\ -\dfrac{i(k-j+1)}{k+1}, & c=-2\end{cases}$$

for $i<j$ and $R_{ij}=R_{ji}$ for $i>j$.

6. Relations between $D_n(c)$ and the Chebyshev polynomials

The Chebyshev polynomials of the second kind $U_n(x)$ for $n\ge 0$ can be generated by

$$\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n$$

for $|x| < 1$ and $|t| < 1$. By Theorem 2.1, it follows immediately that

$$U_n(x) = D_n(2x), \quad n \ge 0. \tag{12}$$

This recovers the first result in [20, Lemma 5]. Hence, by the definition of $D_k(c)$ and Theorem 4.1, we obtain the determinantal expressions

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2x \end{vmatrix}_{n\times n}$$

and

$$U_n(x) = \frac{(-1)^n}{n!} \begin{vmatrix} -2x & 1 & 0 & \cdots & 0 & 0 \\ 2 & -4x & 1 & \cdots & 0 & 0 \\ 0 & 6 & -6x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2(n-1)x & 1 \\ 0 & 0 & 0 & \cdots & n(n-1) & -2nx \end{vmatrix}.$$

The Rodrigues representation for $U_n(x)$ is

$$U_n(x) = \frac{(-1)^n (n+1)\sqrt{\pi}}{2^{n+1}(n+1/2)!(1-x^2)^{1/2}} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\Big[\big(1-x^2\big)^{n+1/2}\Big],$$

see [10]. By the same argument as in the proof of Theorem 2.2, we can obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\Big[\big(1-x^2\big)^{n+1/2}\Big] = n! \frac{\big(1-x^2\big)^{n+1/2}}{(2x)^n} \sum_{\ell=0}^{n} \left\langle n+\frac{1}{2}\right\rangle_\ell \frac{(-1)^\ell}{\ell!} \binom{\ell}{n-\ell} \frac{(2x)^{2\ell}}{(1-x^2)^\ell},$$

where $\langle x\rangle_n$ is the falling factorial defined by

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1; \\ 1, & n = 0. \end{cases}$$

Accordingly, it follows that

$$U_n(x) = (-1)^n (n+1)! \frac{\big(1-x^2\big)^n}{(2x)^n} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(2\ell)!!(2n-2\ell+1)!!} \binom{\ell}{n-\ell} \frac{(2x)^{2\ell}}{(1-x^2)^\ell}$$

and

$$D_n(x) = (-1)^n (n+1)! \frac{\big(4-x^2\big)^n}{(4x)^n} \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(2\ell)!!(2n-2\ell+1)!!} \binom{\ell}{n-\ell} \frac{(2x)^{2\ell}}{(4-x^2)^\ell}.$$

In [10], it was stated that the polynomials $U_n(x)$ can be expressed as

$$U_n(x) = \sum_{r=0}^{\lfloor n/2\rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-2r} = \sum_{m=0}^{\lfloor n/2\rfloor} \binom{n+1}{2m+1} x^{n-2m} \big(x^2-1\big)^m,$$

where $\lfloor x\rfloor$ is the floor function whose value equals the largest integer less than or equal to x. The first equality above is equivalent to the last one in (2). From the second formula above, it follows that

$$D_n(x) = \frac{x^n}{2^n} \sum_{m=0}^{\lfloor n/2\rfloor} \binom{n+1}{2m+1} \left(1-\frac{4}{x^2}\right)^m.$$

In [10], it was given that

$$U_n(x)=\frac{\bigl(x+\sqrt{x^2-1}\,\bigr)^{n+1}-\bigl(x-\sqrt{x^2-1}\,\bigr)^{n+1}}{2\sqrt{x^2-1}}$$

$$=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2k-(n+1)}{k}(2x)^{n-2k}=\sum_{k=0}^{n}2^k\binom{n+k+1}{2k+1}(x-1)^k.$$

Consequently, we deduce the formula (10) and

$$D_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2k-(n+1)}{k}x^{n-2k}=\sum_{k=0}^{n}\binom{n+k+1}{2k+1}(x-2)^k.$$

In [10], it was mentioned that the Chebyshev polynomials of the second kind $U_n(x)$ can be expressed in terms of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{1}{(1-x)^\alpha(1+x)^\beta}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\bigl[(1-x)^{\alpha+n}(1+x)^{\beta+n}\bigr]$$

for $\alpha,\beta>-1$ as $U_n(x)=(n+1)\frac{P_n^{(1/2,1/2)}(x)}{P_n^{(1/2,1/2)}(1)}$. Since

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\bigl[(1-x)^{\alpha+n}(1+x)^{\beta+n}\bigr]=\sum_{\ell=0}^{n}\binom{n}{\ell}\bigl[(1-x)^{\alpha+n}\bigr]^{(\ell)}\bigl[(1+x)^{\beta+n}\bigr]^{(n-\ell)}$$

$$=(1-x)^{\alpha+n}(1+x)^\beta\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell\langle\alpha+n\rangle_\ell\langle\beta+n\rangle_{n-\ell}\biggl(\frac{1+x}{1-x}\biggr)^\ell,$$

the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed explicitly as

$$P_n^{(\alpha,\beta)}(x)=\frac{(x-1)^n}{2^n n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\langle\alpha+n\rangle_\ell\langle\beta+n\rangle_{n-\ell}\biggl(\frac{x+1}{x-1}\biggr)^\ell.$$

As a result, we have

$$U_n(x)=D_n(2x)=\frac{1}{2}\biggl(\frac{x-1}{2}\biggr)^n\sum_{\ell=0}^{n}\binom{2(n+1)}{2\ell+1}\biggl(\frac{x+1}{x-1}\biggr)^\ell.$$

Finally, we recall from [5, p. 50] that $U_n(\cos\theta)=\frac{\sin[(n+1)\theta]}{\sin\theta}$ which is equivalent to $D_n(2\cos\theta)=\frac{\sin[(n+1)\theta]}{\sin\theta}$. This result was discussed in [7, p. 1512]. The other related results discussed and used in [7, p. 1512] are

$$D_n(\pm2\cosh\theta)=(-1)^{(1\mp1)k/2}\frac{\sinh[(k+1)\theta]}{\sinh\theta}.$$

7. Relations between $D_n(c)$ and the Fibonacci polynomials

The Fibonacci numbers $F_n=\frac{(1+\sqrt5\,)^n-(1-\sqrt5\,)^n}{2^n\sqrt5}$ for $n\in\mathbb{N}$ form a sequence of integers and satisfy the linear recurrence relation $F_n=F_{n-1}+F_{n-2}$ with $F_1=F_2=1$. The Fibonacci numbers F_n can be viewed as a particular case $F_n(1)$ of the Fibonacci polynomials

$$F_n(s)=\frac{1}{2^n}\frac{\bigl(s+\sqrt{4+s^2}\,\bigr)^n-\bigl(s-\sqrt{4+s^2}\,\bigr)^n}{\sqrt{4+s^2}} \tag{13}$$

which can be generated by

$$\frac{t}{1-ts-t^2}=\sum_{n=1}^{\infty}F_n(s)t^n=t+st^2+\bigl(s^2+1\bigr)t^3+\bigl(s^3+2s\bigr)t^4+\cdots, \tag{14}$$

see the monograph [6] and related references therein. In [9, p. 215, Example 1], it was deduced that

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}_{(n-1)\times(n-1)}, \quad n \in \mathbb{N}. \tag{15}$$

In [8], among other things, it was listed that

$$F_{r+1}(x) = \begin{vmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ -1 & x & 1 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix}_{r\times r}, \quad r \in \mathbb{N}. \tag{16}$$

In [3, p. 224, Example 3], it was obtained that $D_n(\pm 3) = (\pm 1)^n F_{2(n+1)}$ for $n \in \mathbb{N}$.

Let $i = \sqrt{-1}$ denote the imaginary unit. Taking $t = ix$ on both sides of the formal expansion (14) and simplifying yield $\frac{1}{1-(is)x+x^2} = \sum_{n=0}^{\infty} i^n F_{n+1}(s) x^n$ which implies, due to Theorem 2.1, the relation

$$D_n(is) = i^n F_{n+1}(s), \quad n \ge 0. \tag{17}$$

Consequently, we can express the Fibonacci polynomials $F_n(x)$ for $n \in \mathbb{N}$ in terms of symmetric tridiagonal determinants by

$$F_n(s) = \frac{D_{n-1}(is)}{i^{n-1}} = \frac{U_{n-1}\left(\frac{i}{2}s\right)}{i^{n-1}} = \begin{vmatrix} s & \pm i & 0 & 0 & \cdots & 0 & 0 & 0 \\ \pm i & s & \pm i & 0 & \cdots & 0 & 0 & 0 \\ 0 & \pm i & s & \pm i & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \pm i & s & \pm i \\ 0 & 0 & 0 & 0 & \cdots & 0 & \pm i & s \end{vmatrix}.$$

By Theorems 2.2 and 3.1 and from the relation (17), we can recover the formula (13) and the explicit formula

$$F_n(s) = \sum_{m=0}^{n-1} \binom{n-m-1}{m} s^{n-2m-1}$$

.

8. **Computation of general tridiagonal determinants**

Generally, a tridiagonal determinant is defined by

$$D_n(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{vmatrix},$$

where $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$, $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$, and $\boldsymbol{c} = (c_1, c_2, \dots, c_n)$ are complex multiples. Lemma 1 in [4] reads that the general tridiagonal determinant $D_n(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ satisfies the initial values $D_1(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = a_1$ and $D_2(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = a_1a_2 - b_1c_1$ and satisfies the recurrence equation $D_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = a_nD_{n-1}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) - b_{n-1}c_{n-1}D_{n-2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$.

In linear algebra, the determinant $D_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is also called a Jacobi determinant which is different from the determinant of the Jacobi matrix (or say, the Jacobian) in analysis.

Let us examine some special cases of $D_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$.

By induction on n, one can prove straightforwardly that

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{n\times n} = \begin{cases} \dfrac{a^{n+1}-b^{n+1}}{a-b}, & a\ne b; \\ (n+1)a^n, & a=b. \end{cases} \tag{18}$$

The formula (10) is the special case $ab = 1$ and $c = a + b$ of the equality (18).

By the same method as in the proof of Theorem 3.1, one can prove that

$$D_n = |M_n| = \begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}_{n\times n}$$
$$= \begin{cases} \dfrac{\left(a+\sqrt{a^2-4bc}\right)^{n+1} - \left(a-\sqrt{a^2-4bc}\right)^{n+1}}{2^{n+1}\sqrt{a^2-4bc}}, & a^2\ne 4bc; \\ (n+1)\left(\dfrac{a}{2}\right)^n, & a^2 = 4bc. \end{cases} \tag{19}$$

The formulas (10) and (18) are special cases of the equality (19).

Example 7.2.5 in [11, pp. 514–516] states that the eigenvalues and eigenvectors of the Toeplitz matrix M_n in (19) with $a \ne 0 \ne c$ are given by

$$\lambda_j = b + 2a\sqrt{\frac{c}{a}}\cos\frac{j\pi}{n+1} \quad \text{and} \quad \boldsymbol{x}_j = \begin{pmatrix} \left(\frac{c}{a}\right)^{1/2}\sin\frac{1j\pi}{n+1} \\ \left(\frac{c}{a}\right)^{2/2}\sin\frac{2j\pi}{n+1} \\ \vdots \\ \left(\frac{c}{a}\right)^{n/2}\sin\frac{nj\pi}{n+1} \end{pmatrix}$$

for $1 \le j \le n$ and, consequently, the Toeplitz matrix M_n is diagonalizable. As a result, we conclude that

$$D_n = |M_n| = \prod_{j=1}^{n}\left(b + 2a\sqrt{\frac{c}{a}}\cos\frac{j\pi}{n+1}\right)$$

.

The tridiagonal determinant

$$\begin{vmatrix} x_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & x_2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & x_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & x_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & x_n \end{vmatrix}$$

is called as a continuant $K_n(x_1, x_2, \ldots, x_n)$ which is defined recursively by $K_0 = 1$, $K_1(x_1) = x_1$, and $K_n(x_1, x_2, \ldots, x_n) = x_n K_{n-1}(x_1, x_2, \ldots, x_{n-1}) + K_{n-2}(x_1, x_2, \ldots, x_{n-2})$. Comparing with (15) and (16), it is clear the Fibonacci numbers and polynomials F_n and $F_n(s)$ satisfy $F_n = K_{n-1}(1, 1, \ldots, 1)$ and $F_n(s) = K_{n-1}(s, s, \ldots, s)$.

Tridiagonal matrices play an essential role in the theory of orthogonal polynomials. See the book [1]. Let

$$A_n = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 & 0 \\ 1 & a_1 & b_2 & \cdots & 0 & 0 \\ 0 & 1 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-1} \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}.$$

When expanding the characteristic polynomial $q_n(x) \triangleq \det(xI - A_n)$ with respect to the last row, we obtain $q_n(x) = (x - a_{n-1})q_{n-1}(x) - b_{n-1}q_{n-2}(x)$ with $q_{-1}(x) = 0$ and $q_0(x) = 1$. These polynomials q_n are orthogonal and each monic orthogonal polynomial satisfies the above recurrence relation. This has already been showed by Favard in 1935 and in another form earlier by Stieltjes. One of the simplest and oldest examples are the Chebyshev polynomials $U_n(x)$ or equivalently the Fibonacci polynomials $F_{n+1}(x)$ with $a_n = 0$ and $b_n = -1$. Since the sequence $h(n) = (n+1)!F_{n+1}(s)$ satisfies $h(n) = (n+1)sh(n-1) + n(n+1)h(n-2)$ with $h(-1) = 0$ and $h(0) = 1$, it is clear that $h(n)$ for $n \ge 1$ is the $n \times n$ determinant of the matrix

$$\begin{pmatrix} 2s & 2\cdot 3 & 0 & \cdots & 0 & 0 \\ -1 & 3s & 3\cdot 4 & \cdots & 0 & 0 \\ 0 & -1 & 4s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & ns & n(n+1) \\ 0 & 0 & 0 & \cdots & 0 & (n+1)s \end{pmatrix}_{n\times n}.$$

In [2, pp. 3658–3659, Remark 1.1], it was mentioned that the matrix

$$\begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}, \quad b_i > 0$$

has generalized eigenfunctions $p_j(x)$ which are orthonormal polynomials and satisfy $b_{j-1}p_{j-1}(x) + a_j p_j(x) + b_j p_{j+1}(x) = xp_j(x)$ for $j \ge 1$ and $b_0 \equiv 0$.

Proposition 4 in [21] can be reformulated as follows. For a n-tuple $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $x_k \in \mathbb{C}$, define

$$\mathfrak{F}(\boldsymbol{x}) = 1 + \sum_{m=1}^{n} (-1)^m \sum_{k_1=1}^{n-1} \sum_{k_2=k_1+2}^{n-1} \cdots \sum_{k_m=k_{m-1}+2}^{n-1} \prod_{\ell=1}^{m} x_{k_\ell} x_{k_\ell+1},$$

where by convention an empty sum is equal to 0, or equivalently

$$\mathfrak{F}(\boldsymbol{x}) = 1 + \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^m \sum_{k_1=1}^{n-2m+1} \sum_{k_2=k_1+2}^{n-2m+3} \cdots \sum_{k_m=k_{m-1}+2}^{n-1} \prod_{\ell=1}^{m} x_{k_\ell} x_{k_\ell+1}.$$

Let

$$\gamma_{2k-1}^{+}=\prod_{j=1}^{k-1}\frac{b_{2j}}{c_{2j-1}},\quad \gamma_{2k}^{+}=b_1\prod_{j=1}^{k-1}\frac{b_{2j+1}}{c_{2j}},\quad \gamma_{2k-1}^{-}=\prod_{j=1}^{k-1}\frac{c_{2j}}{b_{2j-1}},\quad \gamma_{2k}^{-}=c_1\prod_{j=1}^{k-1}\frac{c_{2j+1}}{b_{2j}}$$

for $k\in\mathbb{N}$, or equivalently and recursively, $\gamma_1^{\pm}=1$, $\gamma_{k+1}^{+}=\frac{b_k}{\gamma_k^{-}}$, and $\gamma_{k+1}^{-}=\frac{c_k}{\gamma_k^{+}}$ for $k\ge 1$. Then the equality

$$\begin{vmatrix} a_1-z & b_1 & 0 & \cdots & 0 & 0\\ c_1 & a_2-z & b_2 & \cdots & 0 & 0\\ 0 & c_2 & a_3-z & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1}-z & b_{n-1}\\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n-z \end{vmatrix} = \left[\prod_{k=1}^{n}(a_k-z)\right]\mathfrak{F}\left(\frac{\gamma_1^{+}\gamma_1^{-}}{a_1-z},\frac{\gamma_2^{+}\gamma_2^{-}}{a_2-z},\dots,\frac{\gamma_n^{+}\gamma_n^{-}}{a_n-z}\right)$$

holds for all $z\in\mathbb{C}$. This is an explicit and closed-form formula for computing a general tridiagonal determinant (a Jacobi determinant).

The following result can be found in [12]. The tridiagonal determinant of the form

$$J_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=\begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ -c_2 & a_2 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & -c_3 & a_3 & b_3 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & -c_{n-1} & a_{n-1} & b_{n-1}\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -c_n & a_n \end{vmatrix},$$

where $\boldsymbol{a}=(a_1,a_2,\dots,a_n)$, $\boldsymbol{b}=(b_1,b_2,\dots,b_n)$, and $\boldsymbol{c}=(c_1,c_2,\dots,c_n)$ are complex multiples, equals the sum of its leading term $a_1a_2\dots a_n$ and all the terms obtained from this product by replacing one or several pairs of neighbour product a_ja_{j+1} by b_jc_{j+1}. For example, when $n=5$, it has the form

$$\begin{aligned} a_1a_2a_3a_4a_5&+(b_1c_2a_3a_4a_5+a_1b_2c_3a_4a_5+a_1a_2b_3c_4a_5+a_1a_2a_3b_4c_5)\\ &+(b_1c_2b_3c_4a_5+b_1c_2a_3b_4c_5+a_1b_2c_3b_4c_5). \end{aligned}$$

By the above description, we summarize the following theorems.

Theorem 8.1. *The determinant* $J_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ *for* $n\in\mathbb{N}$ *can be computed by*

$$J_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=\left(1+\sum_{m=1}^{\lfloor n/2\rfloor}\sum_{\ell_1=2m-1}^{n-1}\sum_{\ell_2=2m-3}^{\ell_1-2}\cdots\sum_{\ell_{m-1}=3}^{\ell_{m-2}-2}\sum_{\ell_m=1}^{\ell_{m-1}-2}\prod_{k=1}^{m}\frac{b_{\ell_k}c_{\ell_k+1}}{a_{\ell_k}a_{\ell_k+1}}\right)\prod_{\ell=1}^{n}a_\ell.$$

Proof. Expanding with respect to the last row of J_n gives

$$J_n=a_nJ_{n-1}+b_{n-1}c_nJ_{n-2}. \tag{20}$$

Note that $J_1=a_1$ and $J_2=a_1a_2+b_1c_2$. In general,

$$J_n=\prod_{\ell=1}^{n}a_\ell+\sum_{m=1}^{\lfloor n/2\rfloor}\sum_{(l_1,l_1+1)\prec\cdots\prec(l_m,l_m+1)}\prod_{\ell=1}^{n}a_\ell\prod_{k=1}^{m}\frac{b_{l_k}c_{l_k+1}}{a_{l_k}a_{l_k+1}}, \tag{21}$$

where the pair of neighbor indices $(a,a+1)$ precedes the pair $(b,b+1)$, that is, $(a,a+1)\prec(b,b+1)$, if $a+1<b$. In other words, in the right-hand side of (21), we have the term $\prod_{\ell=1}^{n}a_\ell$ and the other summands which can be obtained as follows: we substitute $b_{l_k}c_{l_k+1}$

instead of $a_{l_k}a_{l_k+1}$ in the term $\prod_{\ell=1}^{n} a_\ell$ for each pair (l_k, l_k+1) of the ordered collection of pairs $(l_1, l_1+1) \prec \cdots \prec (l_m, l_m+1)$. The proof by induction on n is not hard in view of the recurrence formula (20). Note that the number of summands in the the right-hand side of (21) is equal to the nth Fibonacci number F_n. Also note that the sum over all ordered collections of pairs $(l_1, l_1+1) \prec \cdots \prec (l_m, l_m+1)$ can be rewritten as

$$\sum_{(l_1,l_1+1)\prec\cdots\prec(l_m,l_m+1)} = \sum_{l_1=1}^{l_2-2}\sum_{l_2=3}^{l_3-2}\cdots\sum_{l_m=2m-1}^{n-1}.$$

Hence, we obtain the formula

$$J_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \left(\prod_{l=1}^{n} a_l\right)\left(1 + \sum_{m=1}^{\lfloor n/2\rfloor}\sum_{l_1=1}^{l_2-2}\sum_{l_2=3}^{l_3-2}\cdots\sum_{l_m=2m-1}^{n-1}\prod_{k=1}^{m}\frac{b_{l_k}c_{l_k+1}}{a_{l_k}a_{l_k+1}}\right).$$

The proof of Theorem 8.1 is thus complete. □

Theorem 8.2. *For $n \in \mathbb{N}$, the determinant $J_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ can be computed by*

$$J_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \sum_{i=1}^{2^{3n}} \chi(w_i)\prod_{k=1}^{n} a_k^{\alpha_{i_k}} b_k^{\beta_{i_k}} c_k^{\gamma_{i_k}}, \tag{22}$$

where $\chi : \{0,1\}^{3n} \to \{0,1\}$ is the characteristic function of the set

$$W_{3n} = \big\{w_i = (\alpha_{i_1}, \beta_{i_1}, \gamma_{i_1}; \alpha_{i_2}, \beta_{i_2}, \gamma_{i_2}; \dots; \alpha_{i_n}, \beta_{i_n}, \gamma_{i_n}) \in \{0,1\}^{3n} :$$
$$\gamma_{i_1} = \beta_{i_n} = 0, \gamma_{i_{\ell+1}} = \beta_{i_\ell}, \alpha_{i_k} + \beta_{i_k} + \gamma_{i_k} = 1, 1 \le \ell \le n-1, 1 \le k \le n\big\}.$$

Proof. For $n = 1, 2$, it is easy to verify the formula (22).

Suppose the formula (22) is valid for all $n \le m$. Then for $n = m+1$

$$\begin{aligned}\sum_{i=1}^{2^{3(m+1)}} \chi(w_i)\prod_{k=1}^{m+1} a_k^{\alpha_{i_k}} b_k^{\beta_{i_k}} c_k^{\gamma_{i_k}} &= \sum_{i=1}^{2^{3m}} \chi(w_i) a_{m+1}^1 b_{m+1}^0 c_{m+1}^0 \prod_{k=1}^{m} a_k^{\alpha_{i_k}} b_k^{\beta_{i_k}} c_k^{\gamma_{i_k}} \\ &+ \sum_{i=1}^{2^{3(m-1)}} \chi(w_i) a_m^0 b_m^1 c_m^0 a_{m+1}^0 b_{m+1}^0 c_{m+1}^1 \prod_{k=1}^{m-1} a_k^{\alpha_{i_k}} b_k^{\beta_{i_k}} c_k^{\gamma_{i_k}} \\ = a_{m+1}\sum_{i=1}^{2^{3m}} \chi(w_i)\prod_{k=1}^{m} a_k^{\alpha_{i_k}} b_k^{\beta_{i_k}} c_k^{\gamma_{i_k}} &+ b_m c_{m+1} \sum_{i=1}^{2^{3(m-1)}} \chi(w_i)\prod_{k=1}^{m-1} a_k^{\alpha_{i_k}} b_k^{\beta_{i_k}} c_k^{\gamma_{i_k}} \\ &= a_{m+1}J_m + b_m c_{m+1} J_{m-1} = J_{m+1}.\end{aligned}$$

The proof is thus complete. □

9. **Going back to central Delannoy numbers**

Central Delannoy numbers $D(n)$ have the generating function

$$\frac{1}{\sqrt{1-6x+x^2}} = \sum_{k=0}^{\infty} D(k)x^k = 1 + 3x + 13x^2 + 63x^3 + \cdots. \tag{23}$$

Squaring on both sides of (23) results in

$$\frac{1}{1-6x+x^2} = \left[\sum_{k=0}^{\infty} D(k)x^k\right]^2 = \sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k} D(\ell)D(k-\ell)\right]x^k.$$

On the other hand, making use of Theorem 2.1 gives

$$\frac{1}{1-6x+x^2}=\sum_{k=0}^{\infty}D_k(6)x^k=\sum_{k=0}^{\infty}(-1)^kD_k(-6)x^k.$$

Consequently, we obtain

$$D_k(6)=(-1)^kD_k(-6)=\sum_{\ell=0}^{k}D(\ell)D(k-\ell)$$

and, by the relation (12) between $D_n(c)$ and the Chebyshev polynomials of the second kind $U_n(x)$,

$$U_k(3)=(-1)^kU_k(-3)=\sum_{\ell=0}^{k}D(\ell)D(k-\ell).$$

In other words, the Cauchy products of central Delannoy numbers $D(k)$ are special values $U_k(3)$ of the Chebyshev polynomials of the second kind $U_n(x)$.

In [15, Theorem 1.3], by virtue of the Cauchy integral formula, central Delannoy numbers $D(k)$ were represented by

$$D(k)=\frac{1}{\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{1}{\sqrt{\bigl(t-3+2\sqrt{2}\,\bigr)\bigl(3+2\sqrt{2}\,-t\bigr)}}\frac{1}{t^{k+1}}\,\mathrm{d}\,t,\quad k\ge0.$$

The central Delannoy numbers $D(k)$ were generalized in [19] as

$$D_{a,b}(k)=\frac{1}{\pi}\int_a^b\frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{t^{k+1}}\,\mathrm{d}\,t,\quad k\ge0,\quad b>a>0 \tag{24}$$

and, by [15, Lemma 2.4], we find that $D_{a,b}(k)$ can be generated by

$$\frac{1}{\sqrt{(x+a)(x+b)}}=\sum_{k=0}^{\infty}D_{a,b}(k)x^k. \tag{25}$$

By virtue of conclusions in [13, Section 2.4] and [18, Remark 4.1], the generalized central Delannoy numbers $D_{a,b}(k)$ for $k\ge0$ can be computed by the explicit formulas

$$D_{a,b}(k)=\frac{1}{G(a,b)}\frac{(-1)^k}{[2A(a,b)]^k}\sum_{\ell=0}^{k}(-1)^\ell2^{2\ell}\frac{(2\ell-1)!!}{(2\ell)!!}\binom{\ell}{k-\ell}\biggl[\frac{A(a,b)}{H(a,b)}\biggr]^\ell$$

and

$$D_{a,b}(k)=\frac{1}{\sqrt{ab}}\frac{1}{b^k}\sum_{\ell=0}^{k}\frac{(2\ell-1)!!}{(2\ell)!!}\frac{[2(k-\ell)-1]!!}{[2(k-\ell)]!!}\Bigl(\frac{b}{a}\Bigr)^\ell,$$

where the quantities $A(a,b)=\frac{a+b}{2}$, $G(a,b)=\sqrt{ab}$, and $H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$ are respectively the arithmetic, geometric, and harmonic means of $a,b>0$.

Squaring on both sides of (25) reveals that the Cauchy product of the generalized central Delannoy numbers $D_{a,b}(k)$ can be generated by

$$\frac{1}{(t+a)(t+b)}=\frac{1}{t^2+(a+b)t+ab}=\sum_{k=0}^{\infty}D_{a,b}(k)t^k.$$

As a result, by some arguments carried out in previous sections, we can see that the Cauchy product of the generalized central Delannoy numbers $D_{a,b}(k)$ can be computed by

$$\sum_{\ell=0}^{k}D_{a,b}(\ell)D_{a,b}(k-\ell)=\frac{1}{(a+b)^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{\ell}{k-\ell}\frac{(a+b)^{2\ell}}{(ab)^{\ell+1}}=\frac{D_k\Bigl(\frac{a+b}{\sqrt{ab}}\Bigr)}{(ab)^{k/2+1}}$$

for $k \geq 0$, where D_k is the diagonal determinant of the diagonal matrix M_k defined by the equation (1).

The definition (24) and the generating function (25) can be extended to $x \in \mathbb{C}$ and $a, b \in \mathbb{C}$ such that the straight segment between $a \in \mathbb{C}$ and $b \in \mathbb{C}$ does not pass through the origin $0 \in \mathbb{C}$.

Remark 9.1. This paper is a revised and shortened version of the preprint [14].

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