THE EXTENDED VERTICAL LINEAR COMPLEMENTARITY PROBLEM VIA TWO RANDOMIZED ALGORITHMS

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In this paper, inspired by the already published work (Comput. Optim. Appl., 2022, 82: 595-615), we extend two randomized algorithms, i.e., the randomized Kaczmarz algorithm and the randomized coordinate descent algorithm, to solve the extended vertical linear complementarity problem. Some convergence properties of both are presented. Numerical experiments show that these two randomized algorithms are feasible and efficient.

Keywords: randomized Kaczmarz algorithm, randomized coordinate descent algorithm, the extended vertical linear complementarity problem, convergence
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1. Introduction

The aim of extended vertical linear complementarity problem (EVLCP) is to find $x \in \mathbb{R}^n$ such that

$$\min_{1\leq s\leq \ell}\{A_s x + q_s\} = 0, \tag{1}$$

where, hereafter, "min" stands for the entry-wise minimum, $A_s \in \mathbb{R}^{n\times n}$ and $q_s \in \mathbb{R}^n$, see [5]. Clearly, the EVLCP (1) is a generalization form of the vertical linear complementarity problem (VLCP) in [3] and the classical linear complementarity problem (LCP) in [4]. Not only that, the EVLCP (1) is often used in many fields of scientific computing and engineering technology, such as volterra ecosystem [6], stochastic impulse control games [22], the discrete HJB equations [21], and others. One can see [5] for more applications.

Developing an efficient iteration algorithm to solve the EVLCP (1) has been the focus of attention. Whereas, to date, only a few iterative algorithms have been proposed to solve the EVLCP (1), such as the Newton-type algorithm in [13, 14], the projected-type algorithm [17, 12], and the modulus-type algorithm [9].

Recently, the randomized algorithm because of its economics and efficiency for solving the corresponding system has attracted much interest, like the randomized Kaczmarz (RK) algorithm [16] and its other versions, the randomized coordinate descent (RCD) algorithm [8] and its other versions. The RK algorithm and the RCD algorithm greatly enhance the convergence rate of the Kaczmarz algorithm in [7] and the coordinate descent algorithm in [15], respectively. At present, the RK algorithm and the RCD algorithm have been successfully extended to solve many practical problems, e.g., the tensor linear system [11], the tensor complementarity problem [19] and the linear least-squares problem [1]. Nevertheless, to our knowledge, there are ***no*** corresponding RK and RCD algorithms for solving the EVLCP (1), which is our main motivation. Based on this, together with the published work

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in [19], in the present paper, our main goal is to design the proper RK and RCD algorithms for solving the EVLCP (1).

The remaining part of this paper is arranged below. In Section 2, we design the proper RK and RCD algorithms for solving the EVLCP (1) and obtain their some convergence properties. In Section 3, we show the performance of the RK algorithm and the RCD algorithm by some numerical experiments. Finally, in Section 4, by using some conclusions, we end up with this paper.

2. The RK and RCD algorithms

In this section, we will establish the proper RK and RCD algorithms for solving the EVLCP (1). To achieve this aim, the following lemmas are required.

For convenience, throughout the paper, $\mathrm{tr}(\cdot)$, $(\cdot)^T$ and $(\cdot)^*$, respectively, indicate the trace, transpose and conjugate transpose of matrix. The symbol $\otimes$ denotes the Kronecker product symbol, $\mathrm{vec}(\cdot)$ stands for the vector gained by successively stacking all columns of matrix into a vector.

Lemma 2.1. [10] *For all $a_i, b_i \in \mathbb{R}^n$ $(i = 1, 2, \ldots, n)$,*

$$|\max_{1\le i\le n}\{a_i\} - \max_{1\le i\le n}\{b_i\}| \le \max_{1\le i\le n}\{|a_i - b_i|\}.$$

Lemma 2.2. [2] *Let R, S, F and G be complex matrices of proper dimensions. Then*
- $\mathrm{tr}(RS) = \mathrm{tr}(SR)$ *and* $\mathrm{tr}(R^*S) = \mathrm{vec}(R)^*\mathrm{vec}(S)$;
- $(R \otimes S)^* = R^* \otimes S^*$ *and* $\mathrm{vec}(RSG^T) = (G \otimes R)\mathrm{vec}(S)$;
- $(R \otimes S)(F \otimes G) = (RF) \otimes (SG)$.

Lemma 2.3. [17] *The EVLCP (1) is equivalent to looking for $x \in \mathbb{R}^n$ such that*

$$x = \max_{1\le s\le \ell}\{x - \delta\Omega(A_s x + q_s)\} \text{ with } \delta > 0,$$

in which Ω is any positive diagonal matrix.

Based on Lemma 2.3, for $\delta = 1$ and $\Omega = I$ with I being the identity matrix, together with the methodology of the RK algorithm [16], we present the proper RK algorithm for the EVLCP (1), see Algorithm 1, in which $\|A\|_F^2 = \mathrm{tr}(A^*A)$ and A^i stands for the ith row of matrix A.

Algorithm 2.1. (The RK algorithm)

Input: *A_s, q_s, x^0; iteration count N.*

Output: *x^N*

1. for $k = 0, 1, \ldots, N-1$ do

2. Select $i_k \in \{1, 2, ..., n\}$ with probability $Pr(row = i_k) = \frac{\sum_{s=1}^{\ell}\|A_s^{i_k}\|_2^2}{\sum_{s=1}^{\ell}\|A_s\|_F^2}$.

3. Set

$$x^{k+1} = \max_{1\le s\le \ell}\left\{x^k - \frac{A_s^{i_k}x^k + q_s^{i_k}}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T\right\}.$$

4. endfor

The EVLCP (1) for any block vector $\mathbf{q} = (q_1, q_2, \ldots, q_s)$ has a unique solution if and only if $\mathbf{A} = (A_1, A_2, \ldots, A_s)$ has the row $\mathcal{W}$-property, i.e.,

$$\min(A_0x, A_1x, \ldots, A_sx) \le 0 \le \max(A_0x, A_1x, \ldots, A_sx) \Rightarrow x = 0,$$

see [18]. Hence, we always assume that $\mathbf{A}$ has the row $\mathcal{W}$-property to ensure that the EVLCP (1) has a unique solution. In such case, we can present the convergence condition for Algorithm 1. To present our convergence conditions, the following lemma is required.

Lemma 2.4. *Assume that $x^\star$ is a unique solution of the EVLCP (1). The kth iteration x^k of RK and $x^\star$ satisfy*

$$|x^{k+1} - x^\star| \leq H_{i_k}|x^k - x^\star|,$$

where

$$H_{i_k} = \max_{1\leq s\leq \ell}\left\{\left|I - \frac{(A_s^{i_k})^T A_s^{i_k}}{\|A_s^{i_k}\|_F^2}\right|\right\}.$$

Proof. Since $x^\star$ is a solution of the EVLCP (1),

$$x^\star = \max_{1\leq s\leq \ell}\left\{x^\star - \frac{A_s^{i_k}x^\star + q_s^{i_k}}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T\right\}.$$

So, from Lemma 2.1 we have

$$\begin{aligned}
|x^{k+1} - x^\star| =& \left|\max_{1\leq s\leq \ell}\left\{x^k - \frac{A_s^{i_k}x^k + q_s^{i_k}}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T\right\}\right.\\
&\left. - \max_{1\leq s\leq \ell}\left\{x^\star - \frac{A_s^{i_k}x^\star + q_s^{i_k}}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T\right\}\right|\\
\leq& \max_{1\leq s\leq \ell}\left\{\left|x^k - \frac{A_s^{i_k}x^k + q_s^{i_k}}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T - \left(x^\star - \frac{A_s^{i_k}x^\star + q_s^{i_k}}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T\right)\right|\right\}\\
=& \max_{1\leq s\leq \ell}\left\{\left|x^k - x^\star - \frac{A_s^{i_k}(x^k - x^\star)}{\|A_s^{i_k}\|_F^2}(A_s^{i_k})^T\right|\right\}\\
=& \max_{1\leq s\leq \ell}\left\{\left|x^k - x^\star - \frac{(A_s^{i_k})^T A_s^{i_k}(x^k - x^\star)}{\|A_s^{i_k}\|_F^2}\right|\right\}\\
\leq& H_{i_k}|x^k - x^\star|,
\end{aligned}$$

where

$$H_{i_k} = \max_{1\leq s\leq \ell}\left\{\left|I - \frac{(A_s^{i_k})^T A_s^{i_k}}{\|A_s^{i_k}\|_F^2}\right|\right\}.$$

The proof is completed. □

In the sequel, similar to the work in [2], from the following mean squared error (MSE), i.e.,

$$\text{MSE} := \mathbb{E}\|x^N - x^\star\|_2,$$

where x^N is the Nth iteration by Algorithm 1 and $x^\star$ is a unique solution of EVLCP (1), we establish the convergence property of Algorithm 1. Concretely, we have

Theorem 2.1. *Assume that $x^\star$ is a unique solution of the EVLCP (1). For any initial guess vector $x^0 \in \mathbb{R}^n$, after N iteration steps of RK, the average error satisfies*

$$\mathbb{E}\|x^N - x^\star\|_2^2 \leq \text{vec}(I)^T H^N \text{vec}((x^0 - x^\star)(x^0 - x^\star)^T), \tag{2}$$

where

$$H = \sum_{i=1}^{n} p_i(H_i \otimes H_i) \text{ with } H_i = \max_{1\leq s\leq \ell}\left\{\left|I - \frac{(A_s^i)^T A_s^i}{\|A_s^i\|_F^2}\right|\right\}, p_i = \frac{\sum_{s=1}^{\ell}\|A_s^i\|_2^2}{\sum_{s=1}^{\ell}\|A_s\|_F^2}. \tag{3}$$

Proof. From Lemma 2.4, we know

$$|x^{k+1} - x^\star| \leq H_{i_k}|x^k - x^\star|.$$

After N iteration step, we have

$$|x^N - x^\star| \leq H_{i_{N-1}}H_{i_{N-2}}\dots H_{i_0}|x^0 - x^\star|.$$

Hence, after $N \geq 1$ steps, the MSE can be estimated below

$$\begin{aligned}\mathbb{E}\|x^N - x^\star\|_2^2 \leq & \mathbb{E}\|H_{i_{N-1}} H_{i_{N-2}} \ldots H_{i_0}(x^0 - x^\star)\|_2^2 \\ = & \mathbb{E}[(x^0 - x^\star)^T H_{i_0}^T \ldots H_{i_{N-2}}^T H_{i_{N-1}}^T I \\ & \times H_{i_{N-1}} H_{i_{N-2}} \ldots H_{i_0}(x^0 - x^\star)] \\ = & \mathbb{E}[\mathrm{tr}(H_{i_0}^T \ldots H_{i_{N-2}}^T H_{i_{N-1}}^T I H_{i_{N-1}} H_{i_{N-2}} \ldots H_{i_0} \\ & \times (x^0 - x^\star)^0 (x - x^\star)^T)] \\ = & (\mathbb{E}[\mathrm{vec}(H_{i_0}^T \ldots H_{i_{N-2}}^T H_{i_{N-1}}^T I H_{i_{N-1}} H_{i_{N-2}} \ldots H_{i_0})])^T \\ & \mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T).\end{aligned} \tag{4}$$

From Lemma 2.2, note that H_{i_k} for $k \geq 0$ is mutually independent, we have

$$\begin{aligned}&\mathbb{E}[\mathrm{vec}(H_{i_0}^T \ldots H_{i_{N-2}}^T H_{i_{N-1}}^T I H_{i_{N-1}} H_{i_{N-2}} \ldots H_{i_0})] \\ &= \mathbb{E}[(H_{i_0}^T \otimes H_{i_0}^T)\mathrm{vec}(H_{i_1}^T \ldots H_{i_{N-1}}^T I H_{i_{N-1}} \ldots H_{i_1})] \\ &= \mathbb{E}[(H_{i_0}^T \otimes H_{i_0}^T)]\mathbb{E}[\mathrm{vec}(H_{i_1}^T \ldots H_{i_{N-1}}^T I H_{i_{N-1}} \ldots H_{i_1})] \\ &= \mathbb{E}[(H_{i_0}^T \otimes H_{i_0}^T)]\mathbb{E}[(H_{i_1}^T \otimes H_{i_1}^T)] \ldots \mathbb{E}[(H_{i_{N-1}}^T \otimes H_{i_{N-1}}^T)]\mathrm{vec}(I).\end{aligned} \tag{5}$$

As

$$\mathbb{E}[(H_{i_k}^T \otimes H_{i_k}^T)] = \sum_{i_k=1}^{n} p_{i_k}(H_{i_k}^T \otimes H_{i_k}^T) = H^T,$$

we can rewrite (5) as

$$\mathbb{E}[\mathrm{vec}(H_{i_0}^T \ldots H_{i_{N-2}}^T H_{i_{N-1}}^T I H_{i_{N-1}} H_{i_{N-2}} \ldots H_{i_0})] = (H^T)^N \mathrm{vec}(I). \tag{6}$$

Substituting (6) into (4) leads to (2).

In addition, when $N = 0$ in (2), it becomes an exact equality. This is because from Lemma 2.2 we have

$$\begin{aligned}\mathbb{E}\|x^0 - x^\star\|_2^2 &= \mathrm{tr}((x^0 - x^\star)^T(x^0 - x^\star)) \\ &= \mathrm{tr}((x^0 - x^\star)(x^0 - x^\star)^T) \\ &= \mathrm{vec}(I)^T \mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T).\end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let σ_1 be the largest singular value of matrix H in (3), and let $x^\star$ be a unique solution of the EVLCP (1). For any initial guess vector $x^0 \in \mathbb{R}^n$, after N iteration steps of RK, the average error satisfies*

$$\mathbb{E}\|x^N - x^\star\|_2^2 \leq \sigma_1^N \sqrt{n}\|x^0 - x^\star\|_2^2.$$

Proof. As is known, the matrix $H \in \mathbb{R}^{n^2 \times n^2}$ is written as $H = U\Omega V^*$, where $U \in \mathbb{R}^{n^2 \times n^2}$ and $V \in \mathbb{R}^{n^2 \times n^2}$ are two unitary matrices, and

$$\Omega = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{n^2}) \in \mathbb{R}^{n^2 \times n^2} \text{with } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n^2} \geq 0$$

is a diagonal matrix with σ_i $(i = 1, 2, \ldots, n^2)$ being the singular value of H.

Based on (2), we have

$$\mathbb{E}\|x^N - x^\star\|_2^2 \leq \mathrm{vec}(I)^T (U\Omega V^*)^N \mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T).$$

Let

$$\mathrm{vec}(I)^T (U\Omega V^*)^N = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_{n^2}) \text{ and } \mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T) = (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{n^2})^T.$$

With the aid of the Hölder inequality, we obtain

$$\begin{aligned}\mathbb{E}\|x^N - x^\star\|_2^2 &\le \sum_{i=1}^{n^2} \bar{q}_i\hat{q}_i \le \sum_{i=1}^{n^2} |\bar{q}_i\hat{q}_i| \le \left(\sum_{i=1}^{n^2} |\bar{q}_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n^2} |\hat{q}_i|^2\right)^{\frac{1}{2}} \\ &= \|\mathrm{vec}(I)^T(U\Omega V^*)^N\|_2\|\mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T)\|_2 \\ &\le \|\mathrm{vec}(I)^T\|_2\|(U\Omega V^*)^N\|_2\|\mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T)\|_2 \\ &\le \sigma_1^N\sqrt{n}\|x^0 - x^\star\|_2^2,\end{aligned}$$

which completes the proof. □

Similarly, based on the methodology of the RCD algorithm [8], we present the proper RCD algorithm for the EVLCP (1), see Algorithm 2.

Algorithm 2.2. (The RCD algorithm)

Input: *A_s, q_s, x^0; iteration count N.*

Output: *x^N*

1. for $k = 0, 1, \ldots, N-1$ do

2. Select $j_k \in \{1, 2, ..., n\}$ with probability Pr(column $= j_k$) $= \frac{\sum_{s=1}^{\ell}\|(A_s)_{j_k}\|_2^2}{\sum_{s=1}^{\ell}\|A_s\|_F^2}$.

3. Set

$$x^{k+1} = \max_{1\le s\le \ell}\left\{x^k - \frac{(A_s)_{j_k}^T(A_s x^k + q_s)}{\|(A_s)_{j_k}\|_2^2}e_{j_k}\right\}.$$

4. endfor

For Algorithm 2, similar to Lemma 2.4, Theorems 2.1 and 2.2, we have Lemma 2.5 and Theorem 2.3.

Lemma 2.5. *Assume that $x^\star$ is a unique solution of the EVLCP (1). The kth iteration x^k of RCD and $x^\star$ satisfy*

$$|x^{k+1} - x^\star| \le G_{j_k}|x^k - x^\star|,$$

where

$$G_{j_k} = \max_{1\le s\le \ell}\left\{\left|I - \frac{e_{j_k}(A_s)_{j_k}^T A_s}{\|(A_s)_{j_k}\|_2^2}\right|\right\}.$$

Theorem 2.3. *Assume that $x^\star$ is a unique solution of the EVLCP (1). For any initial guess vector $x^0 \in \mathbb{R}^n$, after N iteration steps of RCD, the average error satisfies*

$$\mathbb{E}\|x^N - x^\star\|_2^2 \le \mathrm{vec}(I)^T G^N \mathrm{vec}((x^0 - x^\star)(x^0 - x^\star)^T),$$

where $G = \sum_{j=1}^n p_j(G_j \otimes G_j)$ with $p_j = \frac{\sum_{s=1}^{\ell}\|(A_s)_j\|_2^2}{\sum_{s=1}^{\ell}\|A_s\|_F^2}$ and $G_j = \max_{1\le s\le \ell}\left\{\left|I - \frac{e_j(A_s)_j^T A_s}{\|(A_s)_j\|_2^2}\right|\right\}$. Let δ_1 be the largest singular value of matrix G. Then

$$\mathbb{E}\|x^N - x^\star\|_2^2 \le \delta_1^N\sqrt{n}\|x^0 - x^\star\|_2^2.$$

3. Numerical verification

In this section, a simple randomized example is provided to evaluate the performance of the proposed RK and RCD algorithms. For convenience, in our computations, we take $\ell = 2$, and construct test problems with $q_1 = u^\star - A_1\mathbf{1}$ and $q_2 = v^\star - A_2\mathbf{1}$ with $\mathbf{1} = (1, 1, \ldots, 1)^T$, where $u^\star$ and $v^\star$ are given by

$$u^\star = (1, 0, 1, 0 \ldots, 1, 0, \ldots)^T,\ v^\star = (0, 1, 0, 1 \ldots, 0, 1, \ldots)^T.$$

The stopping criterion is RES $:= \frac{\|x^k - x^\star\|}{\|x^\star\|} \le 10^{-4}$ with $x^\star = \mathbf{1}$.

Example 3.1 Consider the EVLCP (1) from the view of the random matrix, in which A_1 and A_2 are given by

$$(A_1)_{ij} = S_{ij}, i \neq j,\ (A_1)_{ii} = 4 + \sum_{i \neq j}^{n} |S_{ij}|;$$

$$(A_2)_{ij} = T_{ij}, i \neq j, \text{and } (A_2)_{ii} = 4 + \sum_{i \neq j}^{n} |T_{ij}|,$$

where $S = \mu \cdot rand(n,n) - 1$, and $T = \nu \cdot rand(n,n) - 1$. Clearly, A_1 and A_2 are full random matrices.

Since A_1 and A_2 are two strict diagonally dominant in Example 3.1, from Lemma 3.4 in [20], we know that the corresponding $\mathbf{A}$ has row $\mathcal{W}$-property such that the corresponding EVLCP (1) has a unique solution, i.e., $\mathbf{1}$ constructed by us.

Table 1. Numerical results of RK and RCD for Example 3.1 with $\mu = 3$ and $\nu = 2$.

	m	600	800	1000	1200
	IT	9	11	10	10
RK	CPU	0.1157	0.2688	0.3441	0.5262
	RES	6.5891e-5	1.3030e-5	1.5519e-5	2.3385e-5
	IT	7	7	6	6
RCD	CPU	0.1878	0.3104	0.5367	0.9338
	RES	7.0879e-5	9.8252e-5	8.4438e-5	3.8140e-5

Table 2. Numerical results of RK and RCD for Example 3.1 with $\mu = 3$ and $\nu = 5$.

	m	600	800	1000	1200
	IT	9	9	9	8
RK	CPU	0.1119	0.2004	0.3224	0.4389
	RES	1.0352e-5	3.0849e-5	1.6279e-5	3.4870e-5
	IT	9	7	6	6
RCD	CPU	0.2239	0.3522	0.4852	0.8775
	RES	4.8578e-5	1.7820e-5	6.0824e-5	1.8343e-5

In our computations, we consider two aspects: (1) $\mu = 3$ and $\nu = 2$; (2) $\mu = 3$ and $\nu = 5$. For these two cases, Tables 1 and 2 in turn display the numerical results of IT, CPU and RES by RK and RCD for solving Example 3.1 with various dimensions, where 'IT', 'CPU' in order stand for the iteration numbers and the elapsed CPU times (second). From the numerical results in Tables 1 and 2, we can drive a simple conclusion that the presented RK and RCD algorithms are feasible and effective when both are adopted to solve the EVLCP (1).

4. **Conclusion**

In this paper, we have established the proper RK algorithm and the proper RCD algorithm for solving the EVLCP (1), and discussed their convergence properties under the mild conditions. The performance of these proposed algorithms are confirmed by a simple randomized example.

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