TWO ENDPOINT RESULTS FOR β-SHRINKING AND β-CONVERGENT MULTIFUNCTIONS WITH APPLICATION TO AN INTEGRAL EQUATION

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We introduce β-shrinking, β-convergent and β-generalized weak contractive multifunctions, and give some results about the existence of endpoint of these classes of multifunctions. We show that our main result generalizes a recent related theorem. Finally, we provide two applications for our main results.

Keywords: β-generalized weak contractive multifunction, β-shrinking multifunction, β-convergent multifunction, approximate endpoint property, endpoint.

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1. Introduction

One of valuable recent techniques in fixed point theory is the notion of α-ψ-contractive mappings which introduced by Samet, Vetro and Vetro in 2012 ([13]). Some authors used it for some subjects in fixed point theory (see for example [5], [8], [9] and [12]). Later, it was generalized to β-ψ-contractive multifunctions (see for example [2], [3], [7] and [10]). In this paper, we introduce the new notion of β-shrinking, β-convergent and β-generalized weak contractive multifunctions and by using this notion, we generalize a recent related result in fixed point theory.

2. Preliminaries

Let (X, d) be a metric space, $CB(X)$ the collection of all nonempty bounded and closed subsets of X, $T\colon X \to 2^X$ a multifunction and H, the Hausdorff metric with respect to d, that is, $H(A, B) = \max\left\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$ for all $A, B \in CB(X)$, where $d(x, B) = \inf_{y\in B} d(x, y)$. An element $x \in X$ is said to be an *endpoint* of T whenever $Tx = \{x\}$ ([4]). We say that the multifunction T has the *approximate endpoint property* whenever $\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0$ ([4]). A function $g\colon \mathbb{R} \to \mathbb{R}$ is called *upper semi-continuous* whenever $\limsup_{n\to\infty} g(\lambda_n) \le g(\lambda)$, for all sequence $\{\lambda_n\}_{n\ge 1}$ with $\lambda_n \to \lambda$ ([1]). In 2010, Amini-Harandi proved the following result ([4]).

Theorem 2.1. *Let $\psi\colon [0, \infty) \to [0, \infty)$ be an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$, for all $t > 0$, (X, d) a complete metric*

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space and $T\colon X \to CB(X)$ *a multifunction satisfying*

$$H(Tx, Ty) \leq \psi(d(x, y)), \quad \forall\, x, y \in X.$$

Then T *has a unique endpoint if and only if* T *has the approximate endpoint property.*

Later, Moradi and Khojasteh by introducing generalized weak contractive multifunctions, improved it by providing the following result ([11]).

Theorem 2.2. *Let* $\psi\colon [0,\infty) \to [0,\infty)$ *be an upper semi-continuous function such that* $\psi(t) < t$ *and* $\liminf_{t\to\infty}(t - \psi(t)) > 0$, *for all* $t > 0$, (X, d) *a complete metric space and* $T\colon X \to CB(X)$ *a generalized weak contractive multifunction, that is, satisfying*

$$H(Tx, Ty) \leq \psi(N(x, y)), \quad \forall\, x, y \in X,$$

where $N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Then T *has a unique endpoint if and only if* T *has the approximate endpoint property.*

In this paper, we introduce β-shrinking, β-convergent and β-generalized weak contractive multifunctions and generalize Theorems 2.1 and 2.2 for the class of multifunctions.

3. Main Results

Let (X, d) be a metric space and $\beta\colon 2^X \times 2^X \to [0,\infty)$ a mapping. A multifunction $T\colon X \to 2^X$ is called *β-generalized weak contraction* whenever there exists a nondecreasing upper semi-continuous function $\psi\colon [0,+\infty) \to [0,+\infty)$ such that $\psi(t) < t$, for all $t > 0$, and

$$\beta(Tx, Ty)H(Tx, Ty) \leq \psi(N(x, y)), \quad \forall\, x, y \in X.$$

We say that the multifunction T is *β-shrinking* whenever for each sequence $\{x_n\}$ in X with $\lim\limits_{n\to\infty} \operatorname{diam}(Tx_n) = 0$, there exists a natural number N such that $\beta(Tx_n, Tx_m) \geq 1$, for all $m > n \geq N$. A multifunction T is said to be *β-convergent* whenever for each convergent sequence $\{x_n\}$, with $x_n \to x$, there exists a natural number N such that $\beta(Tx_n, Tx) \geq 1$, for all $n \geq N$.

Now, we are ready to state and prove our main results.

Theorem 3.1. *Let* (X, d) *be a complete metric space,* $\beta\colon 2^X \times 2^X \to [0,\infty)$ *a mapping and* $T\colon X \to CB(X)$ *a* β*-shrinking and* β*-convergent multifunction satisfying*

$$\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y)), \quad \forall\, x, y \in X,$$

where $\psi\colon [0,\infty) \to [0,\infty)$ *is an upper semi-continuous function with* $\psi(t) < t$, *for all* $t > 0$.

Then T *has an endpoint if and only if* T *has the approximate endpoint property.*

Proof. It is clear that T has the approximate endpoint property whenever T has an endpoint.

Suppose that T has the approximate endpoint property.

Choose a sequence $\{x_n\}$ in X such that $\sup\limits_{y\in Tx_n} d(x_n, y) \to 0$. Thus, we obtain $H(\{x_n\}, Tx_n) \to 0$ and $\operatorname{diam}(Tx_n) \to 0$. Since T is β-shrinking, there exists a natural number N such that $\beta(Tx_n, Tx_m) \geq 1$ for all $m > n \geq N$. Hence for each $m > n \geq N$, we have

$$\begin{aligned} d(x_n, x_m) &\leq H(\{x_n\}, Tx_n) + H(Tx_n, Tx_m) + H(Tx_m, \{x_m\}) \\ &\leq H(\{x_n\}, Tx_n) + \beta(Tx_n, Tx_m)H(Tx_n, Tx_m) + H(Tx_m, \{x_m\}) \\ &\leq H(\{x_n\}, Tx_n) + \psi(d(x_n, x_m)) + H(Tx_m, \{x_m\}). \end{aligned}$$

Because ψ is upper semi-continuous, we get

$$\limsup_{n,m\to\infty} d(x_n, x_m) \leq \limsup_{n,m\to\infty} \psi(d(x_n, x_m)) \leq \psi(\limsup_{n,m\to\infty} d(x_n, x_m)).$$

Since $\psi(t) < t$ for all $t > 0$, $\limsup_{n,m\to\infty} d(x_n, x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence.

Choose $x_0 \in X$ such that $x_n \to x_0$.

If there exists a natural number n_0 such that $d(x_n, x_0) = 0$ for all $n \geq n_0$, then we have $x_n = x_0$ and $H(Tx_n, Tx_0) = 0$, for all $n \geq n_0$. Thus, for each $n \geq n_0$, we obtain

$$H(\{x_0\}, Tx_0) \leq d(x_0, x_n) + H(\{x_n\}, Tx_n) + H(Tx_n, Tx_0) \leq H(\{x_n\}, Tx_n)$$

and so $H(\{x_0\}, Tx_0) = 0$.

If this is not, then without loss of generality (by replacing a subsequence) we can suppose that $d(x_n, x_0) > 0$ for all n. Since $x_n \to x_0$ and T is β-convergent, there exists a natural number N_1 such that $\beta(Tx_n, Tx_0) \geq 1$, for all $n \geq N_1$. Thus, for each $n \geq N_1$ we have

$$\begin{aligned} H(\{x_0\}, Tx_0) &\leq d(x_0, x_n) + H(\{x_n\}, Tx_n) + H(Tx_n, Tx_0) \\ &\leq d(x_0, x_n) + H(\{x_n\}, Tx_n) + \beta(Tx_n, Tx_0)H(Tx_n, Tx_0) \\ &\leq d(x_0, x_n) + H(\{x_n\}, Tx_n) + \psi(d(x_n, x_0)) < 2d(x_n, x_0) + H(\{x_n\}, Tx_n). \end{aligned}$$

Hence, $H(\{x_0\}, Tx_0) = 0$. Therefore, T has an endpoint. □

Now, we add an assumption to obtain the uniqueness of endpoint. In this respect, we introduce a new notion.

Let X be a set and $\beta \colon 2^X \times 2^X \to [0, \infty)$ a map. We say that the set X *has the property* (G_β) whenever $\beta(A, B) \geq 1$, for all subsets A and B of X, with either $A \nsubseteq B$ or $B \nsubseteq A$.

Corollary 3.1. *Let (X, d) be a complete metric space, $\beta \colon 2^X \times 2^X \to [0, \infty)$ a mapping and $T \colon X \to CB(X)$ a β-shrinking and β-convergent multifunction satisfying $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$ for all x, y in X, where $\psi \colon [0, \infty) \to [0, \infty)$ is an upper semi-continuous function with $\psi(t) < t$, for all $t > 0$.*

If T has the approximate endpoint property and X has the property (G_β), then T has a unique endpoint.

Proof. By using Theorem 3.1, T has a endpoint. If T has two distinct endpoints x^* and y^*, then $\beta(Tx^*, Ty^*) = \beta(\{x^*\}, \{y^*\}) \geq 1$ because X has the property (G_β). Hence,

$$\begin{aligned} d(x^*, y^*) &\leq H(Tx^*, Ty^*) \leq \beta(Tx^*, Ty^*)H(Tx^*, Ty^*) \\ &\leq \psi(N(x^*, y^*)) < N(x*, y*) = d(x^*, y^*) \end{aligned}$$

which is a contradiction. Thus, T has a unique endpoint. □

If we consider $\beta\colon 2^X \times 2^X \to [0,+\infty)$, $\beta(A,B)=1$, for all $A,B\subseteq X$, then every multifunction is β-shrinking and β-convergent. Also, X has the property (G_β). Thus, Theorem 2.1 is a consequence of Theorem 3.1.

In 2011, Haghi, Rezapour and Shahzad showed that there are some fixed point generalizations which are not real generalizations ([6]). Next example shows that Theorem 3.1 is a real generalization of Theorem 2.1.

Example 3.1. Let $X=[0,1]\cup[\frac{3}{2},\infty)$ and $d(x,y)=|x-y|$. Now, define

$$T\colon X\to CB(X),\quad Tx=\begin{cases}[\frac{x}{4},\frac{x}{2}], & x\in[0,1]\\ \\ \{1\}, & x\in[\frac{3}{2},\infty).\end{cases}$$

Put $x=1$ and $y=\frac{3}{2}$. Then,

$$H(Tx,Ty)=H(T1,T\frac{3}{2})=H([\frac{1}{4},\frac{1}{2}],\{1\})=\frac{3}{4}>\psi(\frac{1}{2})=\psi(d(x,y)),$$

where $\psi\colon[0,\infty)\to[0,\infty)$ is an arbitrary upper semi-continuous function, with $\psi(t)<t$, for all $t>0$. Thus, the condition of Theorem 2.1 does not hold.

Now, we show that the conditions of Theorem 3.1 hold for this multifunction. In this respect, define $\psi(t)=\frac{7}{8}t$ for all $t\geq 0$ and $\beta\colon 2^X\times 2^X\to[0,\infty)$ by $\beta(A,B)=0$ whenever $A\subseteq(\frac{1}{8},\frac{1}{2}]$, and $B=\{1\}$ and $\beta(A,B)=1$ otherwise.

If $0\leq x\leq y\leq 1$, then $\beta(Tx,Ty)=1$ and so

$$\beta(Tx,Ty)H(Tx,Ty)=H([\frac{x}{4},\frac{x}{2}],[\frac{y}{4},\frac{y}{2}])=\frac{1}{2}d(x,y)\leq\frac{7}{8}d(x,y)=\psi(d(x,y)).$$

If $0\leq x\leq\frac{1}{2}$ and $y\geq\frac{3}{2}$, then $\beta(Tx,Ty)=1$ and so

$$\begin{aligned}\beta(Tx,Ty)H(Tx,Ty)&=H(Tx,Ty)=H([\frac{x}{4},\frac{x}{2}],\{1\})=1-\frac{x}{4}\\&=\frac{3}{8}+\frac{5}{8}-\frac{x}{4}\leq\frac{3}{8}+\frac{5}{8}d(x,y)-\frac{x}{4}=\frac{2}{8}.\frac{3}{2}+\frac{5}{8}y-\frac{5}{8}x-\frac{x}{4}\\&\leq\frac{2}{8}y+\frac{5}{8}y-\frac{5}{8}x-\frac{2}{8}x=\frac{7}{8}y-\frac{7}{8}x=\frac{7}{8}d(x,y)=\psi(d(x,y)).\end{aligned}$$

If $\frac{1}{2}<x\leq 1$ and $y\geq\frac{3}{2}$, then $Tx\subseteq(\frac{1}{8},\frac{1}{2}]$ and $Ty=\{1\}$. Hence, $\beta(Tx,Ty)=0$ and so $\beta(Tx,Ty)H(Tx,Ty)=0\leq\psi(d(x,y))$.

If $x,y\geq\frac{3}{2}$, then $\beta(Tx,Ty)=1$ and so

$$\beta(Tx,Ty)H(Tx,Ty)=H(Tx,Ty)=H(\{1\},\{1\})=0\leq\psi(d(x,y)).$$

Thus, $\beta(Tx,Ty)H(Tx,Ty)\leq\psi(d(x,y))$ for all $x,y\in X$.

Now, we show that T is β-shrinking.

Suppose that $\{x_n\}$ is a sequence in X with $diam Tx_n\to 0$.

If $x_n\in[0,1]$ for all $n\geq N$, then $\beta(Tx_n,Tx_m)\geq 1$ for all $m>n\geq N$.

If $x_n\in[\frac{3}{2},\infty)$ for all $n\geq N$, then $\beta(Tx_n,Tx_m)\geq 1$, for all $m>n\geq N$.

If there exist subsequences $\{x_{n_k}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_n\}=\{x_{n_k}\}\cup\{x_{n_i}\}$, $x_{n_k}\in[0,1]$ and $x_{n_i}\in[\frac{3}{2},\infty)$, for all k and i, then there exist natural numbers N_1 such that $x_{n_k}\in[0,\frac{1}{8}]$ and $x_{n_i}\in[\frac{3}{2},\infty)$ for all $k,i\geq N_1$. Thus, it is easy to see that $\beta(Tx_n,Tx_m)\geq 1$ for all $m>n\geq N_1$. Hence, T is β-shrinking.

Now, we show that T is β-convergent.

Suppose that $\{x_n\}$ is a sequence in X with $x_n\to x$.

If $x \in [0,1]$, then there exists a natural number N_1 such that $x_n \in [0,1]$ for all $n \geq N_1$. Hence, $\beta(Tx_n, Tx) \geq 1$ for all $n \geq N_1$.

If $x \in [\frac{3}{2}, \infty)$, then there exists a natural number N_2 such that $x_n \in [\frac{3}{2}, \infty)$ for all $n \geq N_2$. Hence, $\beta(Tx_n, Tx) \geq 1$ for all $n \geq N_2$. Thus, T is β-convergent.

Finally, note that $\sup_{y \in T0} d(0,y) = 0$ and so $\inf_{x\in X} \sup_{y\in Tx} d(x,y) = 0$. Hence, T has the approximate endpoint property. Also, $T0 = \{0\}$.

Corollary 3.2. *Let (X,d) be a complete metric space, $\beta\colon 2^X \times 2^X \to [0,\infty)$ a mapping, $k \in [0,1)$ and $T\colon X \to CB(X)$ a β-shrinking and β-convergent multifunction satisfying $\beta(Tx,Ty)H(Tx,Ty) \leq kd(x,y)$ for all x, y in X.*

Then T has an endpoint if and only if T has the approximate endpoint property.

If T has the approximate endpoint property and X has the property (G_β), then T has a unique endpoint x_0 and $\operatorname{Fix}(T) = \{x_0\}$.

Proof. Define $\psi\colon [0,\infty) \to [0,\infty)$, $\psi(t) = kt$. Then by using Theorem 3.1, T has an endpoint if and only if T has the approximate endpoint property.

Now, suppose that T has the approximate endpoint property and X has the property (G_β). Then by using Corollary 3.1, T has a unique endpoint such x_0.

Let y be a fixed point of T. We have to show that $y = x_0$.

If $Tx_0 = Ty$, then $x_0 = y$.

If $Tx_0 \neq Ty$, then $\beta(Tx_0, Ty) \geq 1$ because X has the property (G_β). Therefore, we obtain $d(x_0, y) \leq H(Tx_0, Ty) \leq \beta(Tx_0, Ty)H(Tx_0, Ty) \leq kd(x_0, y)$, and we get $d(x_0, y) = 0$. □

Next corollary shows us the role of a point in the existence of endpoints.

Corollary 3.3. *Let (X,d) be a complete metric space, $x^* \in X$ a fixed element and $T\colon X \to CB(X)$ a multifunction such that $H(Tx,Ty) \leq \psi(d(x,y))$ for all $x, y \in X$ with $x^* \in Tx \cap Ty$, where $\psi\colon [0,\infty) \to [0,\infty)$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$.*

Suppose that for each sequence $\{x_n\}$ in X with diam$(Tx_n) \to 0$, there exists a natural number N_1 such that $x^* \in Tx_n \cap Tx_m$ for all $m > n \geq N_1$.

Also, assume that for each convergent sequence $\{x_n\}$ with $x_n \to x$, there exists a natural number N_2 such that $x^* \in Tx_n \cap Tx$ for all $n \geq N_2$.

Then T has an endpoint if and only if T has the approximate endpoint property.

Proof. It is sufficient to define $\beta\colon 2^X \times 2^X \to [0,\infty)$ by $\beta(A,B) = 1$ whenever $x^* \in A \cap B$ and $\beta(A,B) = 0$ otherwise, and then we use Theorem 3.1. □

Let $(X, d, \leq)$ be an ordered metric space. Define the order $\preceq$ on arbitrary subsets A and B of X by $A \preceq B$ if and only if for each $a \in A$ there exists $b \in B$ such that $a \leq b$. It is easy to check that $(CB(X), \preceq)$ is a partially ordered set.

Corollary 3.4. *Let $(X, d, \leq)$ be a complete ordered metric space and T a closed and bounded valued multifunction on X such that $H(Tx,Ty) \leq \psi(d(x,y))$, for all $x, y \in X$ with $Tx \preceq Ty$, where $\psi\colon [0,+\infty) \to [0,+\infty)$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$.*

Suppose that for each sequence $\{x_n\}$ in X with diam$(Tx_n) \to 0$, there exists a natural number N_1 such that $Tx_n \preceq Tx_m$ for all $m > n \geq N_1$.

Also, assume that for each convergent sequence $\{x_n\}$ with $x_n \to x$, there exists a natural number N_2 such that $Tx_n \preceq Tx$ for all $n \geq N_2$.

Then T has an endpoint if and only if T has the approximate endpoint property.

Proof. It is sufficient to define $\beta(A,B)=1$ whenever $A \preceq B$ and $\beta(A,B)=0$ otherwise, and then we use Theorem 3.1. □

Theorem 3.2. *Let (X,d) be a complete metric space, $\beta\colon 2^X\times 2^X\to[0,\infty)$ a mapping and $T\colon X\to CB(X)$ a β-shrinking, β-convergent and β-generalized weak contractive multifunction.*

Then T has an endpoint if and only if T has the approximate endpoint property.

Proof. It is clear that T has the approximate endpoint property whenever T has an endpoint.

Suppose that T has the approximate endpoint property.

Choose a sequence $\{x_n\}$ in X such that $\sup_{y\in Tx_n} d(x_n,y)\to 0$. Thus, we obtain that $H(\{x_n\},Tx_n)\to 0$ and $\operatorname{diam}(Tx_n)\to 0$. But, we have

$$\begin{aligned} N(x_n,x_m) &= \max\{d(x_n,x_m), d(x_n,Tx_n), d(x_m,Tx_m), \\ &\qquad \frac{d(x_n,Tx_m)+d(x_m,Tx_n)}{2}\} \\ &\le d(x_n,x_m)+H(\{x_n\},Tx_n)+H(\{x_m\},Tx_m) \\ &= d(x_n,x_m)-H(\{x_n\},Tx_n)-H(\{x_m\},Tx_m)+2H(\{x_n\},Tx_n)+2H(\{x_m\},Tx_m) \\ &\le H(Tx_n,Tx_m)+2H(\{x_n\},Tx_n)+2H(\{x_m\},Tx_m) \end{aligned}$$

for all $m,n\ge 1$. Since T is β-shrinking, there exists a natural number N such that $\beta(Tx_n,Tx_m)\ge 1$ for all $m>n\ge N$.

For each $m>n\ge N$, we have

$$\begin{aligned} N(x_n,x_m) &\le H(Tx_n,Tx_m)+2H(\{x_n\},Tx_n)+2H(Tx_m,\{x_m\}) \\ &\le \beta(Tx_n,Tx_m)H(Tx_n,Tx_m)+2H(\{x_n\},Tx_n)+2H(Tx_m,\{x_m\}) \\ &\le \psi(N(x_n,x_m))+2H(\{x_n\},Tx_n)+2H(Tx_m,\{x_m\}). \end{aligned}$$

Because ψ is upper semi-continuous, we get

$$\limsup_{n,m\to\infty} N(x_n,x_m)\le \limsup_{n,m\to\infty}\psi(N(x_n,x_m))\le \psi(\limsup_{n,m\to\infty} N(x_n,x_m)).$$

Since $\psi(t)<t$ for all $t>0$, $\limsup_{n,m\to\infty} N(x_n,x_m)=0$. This implies that $\limsup_{n,m\to\infty} d(x_n,x_m)=0$. Hence, $\{x_n\}$ is a Cauchy sequence.

Now, choose $x_0\in X$ such that $x_n\to x_0$.

If there exists a natural number n_0 such that $N(x_n,x_0)=0$ for all $n\ge n_0$, then $d(x_n,x_0)=0$ for all $n\ge n_0$. Hence, $x_n=x_0$ and $H(Tx_n,Tx_0)=0$ for all $n\ge n_0$. Thus, for each $n\ge n_0$ we have

$$H(\{x_0\},Tx_0)\le d(x_0,x_n)+H(\{x_n\},Tx_n)+H(Tx_n,Tx_0)\le H(\{x_n\},Tx_n)$$

and so $H(\{x_0\},Tx_0)=0$.

If this is not, then without loss of generality (by replacing a subsequence) we can suppose that $N(x_n,x_0)>0$ for all n.

Since $x_n\to x_0$ and T is β-convergent, there exists a natural number N_1 such that $\beta(Tx_n,Tx_0)\ge 1$ for all $n\ge N_1$. Thus, for each $n\ge N_1$ we have

$$\begin{aligned} H(\{x_n\},Tx_0)-H(\{x_n\},Tx_n) &\le H(Tx_n,Tx_0) \\ &\le \beta(Tx_n,Tx_0)H(Tx_n,Tx_0)\le\psi(N(x_n,x_0))<N(x_n,x_0) \end{aligned}$$

$$\leq d(x_n, x_0) + H(\{x_n\}, Tx_n) + H(\{x_0\}, Tx_0).$$

Hence, $N(x_n, x_0) \to H(\{x_0\}, Tx_0)$.

Since ψ is upper semi-continuous, we get $\limsup_{n\to\infty} \psi(N(x_n, x_0)) \leq \psi(H(\{x_0\}, Tx_0))$.

Now, from last inequalities, we conclude that $H(\{x_0\}, Tx_0) \leq \psi(H(\{x_0\}, Tx_0))$. Thus, $H(\{x_0\}, Tx_0) = 0$. Therefore, T has an endpoint. □

Corollary 3.5. *Let (X, d) be a complete metric space, $\beta\colon 2^X \times 2^X \to [0, \infty)$ a mapping and $T\colon X \to CB(X)$ a β-shrinking, β-convergent and β-generalized weak contractive multifunction.*

If T has the approximate endpoint property and X has the property (G_β), then T has a unique endpoint.

It is easy to check that Theorem 2.2 is a consequence of Theorem 3.2. Moreover, Example 3.1 shows us that Theorem 3.2 is a real generalization of Theorem 2.2. In fact, one can easily check that the multifunction T in Example 3.1 is a β-generalized weak contractive multifunction while is not a generalized weak contractive multifunction.

Corollary 3.6. *Let (X, d) be a complete metric space, $\beta\colon 2^X \times 2^X \to [0, \infty)$ a mapping, $k \in [0, 1)$ and $T\colon X \to CB(X)$ a β-shrinking and β-convergent multifunction satisfying $\beta(Tx, Ty)H(Tx, Ty) \leq kN(x, y)$ for all x, y in X.*

Then T has an endpoint if and only if T has the approximate endpoint property.

If T has the approximate endpoint property and X has the property (G_β), then T has a unique endpoint x_0 and $\mathrm{Fix}\,(T) = \{x_0\}$.

Corollary 3.7. *Let (X, d) be a complete metric space, $x^* \in X$ a fixed element and $T\colon X \to CB(X)$ a multifunction such that $H(Tx, Ty) \leq \psi(N(x, y))$ for all $x, y \in X$ with $x^* \in Tx \cap Ty$, where $\psi\colon [0, \infty) \to [0, \infty)$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$.*

Suppose that for each sequence $\{x_n\}$ in X with $\mathrm{diam}\,(Tx_n) \to 0$*, there exists a natural number N_1 such that $x^* \in Tx_n \cap Tx_m$ for all $m > n \geq N_1$.*

Also, assume that for each convergent sequence $\{x_n\}$ with $x_n \to x$, there exists a natural number N_2 such that $x^* \in Tx_n \cap Tx$ for all $n \geq N_2$.

Then T has an endpoint if and only if T has the approximate endpoint property.

4. **Applications**

Let L be a positive real number and $I = [0, L]$. Denote the set of all real valued continuous functions on I by $C(I)$. If we endow this set with the uniform distance, $d(u, v) = \sup_{t\in I} |u(t) - v(t)|$, then $(C(I), d)$ becomes a complete metric space. Suppose that $K\colon I \times I \times \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. Consider the integral equation

$$u(t) = \int_0^L K(t, s, u(s))ds + g(t), \quad \forall\, t \in I.$$

Now, let X be a set and φ a selfmap on X. Define the multifunction $T_\varphi\colon X \to 2^X$, $T_\varphi x = \{\varphi x\}$. In this case, it is easy to check that $H(T_\varphi x, T_\varphi y) = d(\varphi x, \varphi y)$ for all $x, y \in X$.

Theorem 4.1. *Suppose that* $K\colon I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist two non-negative maps* $\beta\colon 2^{C(I)} \times 2^{C(I)} \to [0,\infty)$ *and* $\alpha\colon \mathbb{R} \times \mathbb{R} \to [0,\infty)$ *such that* $\sup_{t\in I} \alpha(u(t),v(t)) = \beta(\{u\},\{v\}) \geq 1$, *for all* $u,v \in C(I)$.

Assume that, there exists a continuous function $G\colon I \times I \to \mathbb{R}$ *such that*

$$|K(t,s,x) - K(t,s,y)| \leq G(t,s)\frac{|x-y|}{2}, \quad \forall\ x,y \in \mathbb{R}, \quad t,s \in I.$$

Also, suppose that

$$\inf_{u\in C(I)} \sup_{t\in I} \Big| u(t) - \int_0^L K(t,s,u(s))ds - g(t) \Big| = 0,$$

and

$$\sup_{t\in I} \Big(\int_0^L G^2(t,s)ds)^{\frac{1}{2}},\ \alpha\Big(\int_0^L K(t,s,u(s))ds + g(t), \int_0^L K(t,s,v(s))ds + g(t) \Big) \Big) \leq \frac{1}{\sqrt{L}}.$$

Then the integral equation has a solution.

Proof. Define $\varphi\colon C(I) \to C(I)$, $\varphi u(t) = \displaystyle\int_0^L K(t,s,u(s))ds + g(t)$, for all $t \in I$. Then, we have

$$\begin{aligned}|\varphi u(t) - \varphi v(t)| &\leq \int_0^L |K(t,s,u(s)) - K(t,s,v(s))|ds \\ &\leq \int_0^L G(t,s)\frac{|u(s)-v(s)|}{2}ds.\end{aligned}$$

By using the Cauchy-Schwartz inequality, we obtain

$$|\varphi u(t) - \varphi v(t)| \leq (\int_0^L G^2(t,s)ds)^{\frac{1}{2}} (\int_0^L (\frac{|u(s)-v(s)|}{2})^2 ds)^{\frac{1}{2}}$$

for all $t \in I$ and $u,v \in C(I)$.

Hence,

$$\begin{aligned}&\alpha(\varphi u(t), \varphi v(t))|\varphi u(t) - \varphi v(t)| \\ &\leq \alpha(\varphi u(t), \varphi v(t))(\int_0^L G^2(t,s)ds)^{\frac{1}{2}} (\int_0^L (\frac{|u(s)-v(s)|}{2})^2 ds)^{\frac{1}{2}} \\ &\leq \sup_{t\in I} \alpha(\varphi u(t), \varphi v(t))(\int_0^L G^2(t,s)ds)^{\frac{1}{2}} (\int_0^L (\frac{|u(s)-v(s)|}{2})^2 ds)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{L}} (\int_0^L (\frac{|u(s)-v(s)|}{2})^2 ds)^{\frac{1}{2}} \leq \frac{d(u,v)}{2},\end{aligned}$$

for all $t \in I$ and $u,v \in C(I)$.

This implies that

$$\beta(T_\varphi u, T_\varphi v) H(T_\varphi u, T_\varphi v) \leq \psi(d(u,v))$$

for all $u,v \in C(I)$, where $\psi(t) = \frac{t}{2}$, for all $t \geq 0$.

It is easy to check that T_φ a β-shrinking and β-convergent multifunction.

Since

$$\inf_{u\in C(I)} d(u,T_\varphi u) = \inf_{u\in C(I)} \sup_{t\in I} \Big| u(t) - \int_0^L K(t,s,u(s))ds - g(t)\Big| = 0,$$

T_φ has the approximate fixed point property. Thus, by using Theorem 3.1, T_φ has a fixed point u^*, which is a solution for the integral equation. □

Now, let $f\colon \mathbb{R}\to\mathbb{R}$ and $\psi\colon C(I)\to C(I)$ be two functions. Put

$$J(x,y) = \max\Big\{|x-y|, |x-fx|, |y-fy|, \frac{|x-fy|+|y-fx|}{2}\Big\},$$

for all $x,y\in\mathbb{R}$ and

$$N_\psi(u,v) = \max\Big\{d(u,v), d(u,\psi u)d(v,\psi v), \frac{d(v,\psi u)+d(u,\psi v)}{2}\Big\},$$

for all $u,v\in C(I)$. Note that, $N_\psi(u,v) = \sup_{t\in I} J(u(t),v(t))$.

By using a similar proof of Theorem 4.1 and Theorem 3.2, one can prove next result.

Theorem 4.2. *Suppose that $K\colon I\times I\times\mathbb{R}\to\mathbb{R}$ and $g\colon\mathbb{R}\to\mathbb{R}$ are continuous functions and there exist two non-negative maps $\beta\colon 2^{C(I)}\times 2^{C(I)}\to[0,\infty)$ and $\alpha\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$ such that $\sup_{t\in I}\alpha(u(t),v(t)) = \beta(\{u\},\{v\})\geq 1$ for all $u,v\in C(I)$.*

Assume that, there exists a continuous function $G\colon I\times I\to\mathbb{R}$ such that

$$|K(t,s,x)-K(t,s,y)| \leq G(t,s)\frac{J(x,y)}{2}, \quad \forall\, x,y\in\mathbb{R}, \quad t,s\in I.$$

Also, suppose that

$$\inf_{u\in C(I)} \sup_{t\in I} \Big| u(t) - \int_0^L K(t,s,u(s))ds - g(t)\Big| = 0,$$

and

$$\sup_{t\in I}\Big(\int_0^L G^2(t,s)ds)^{\frac{1}{2}},\ \alpha\Big(\int_0^L K(t,s,u(s))ds + g(t), \int_0^L K(t,s,v(s))ds + g(t)\Big)\Big) \leq \frac{1}{\sqrt{L}}.$$

Then the integral equation has a solution.

REFERENCES

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Springer, New york (2009).

[2] H. Alikhani, V. Rakocevic, Sh. Rezapour, N. Shahzad *Some fixed point results for β-ψ-contractive multifunctions*, J. Nonlinear Conv. Anal. 16 (2015) No. 12, 2491–2497.

[3] H. Alikhani, Sh. Rezapour, N. Shahzad *Fixed points of a new type contractive mappings and multifunctions*, Filomat 27 (2013) No. 7, 1315–1319.

[4] A. Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal. 72 (2010) 132–134.

[5] D. Baleanu, H. Mohammadi, Sh. Rezapour, *Some existence results on nonlinear fractional differential equations*, Philos. Trans. of the Royal Society A 371 (2013) 20120144.

[6] R. H. Haghi, Sh. Rezapour, N. Shahzad, *Some fixed point generalizations are not real generalizations*, Nonlinear Anal. 74 (2011) 1799–1803.

[7] J. Hasanzade Asl, Sh. Rezapour, N. Shahzad, *On fixed points of α-ψ-contractive multifunctions*, Fixed Point Theory Appl. (2012) 2012:212.

[8] M. Jleli, B. Samet, *Best proximity points for α-ψ-proximal contractive type mappings and applications*, Bull. Sci. Math. 137 (2013) 977–995.

[9] M. A. Miandaragh, M. Postolache, Sh. Rezapour, *Some approximate fixed point results for generalized α-contractive mappings*, U. Politeh. Buch. Sci. Bull. Ser. A 75 (2013) No. 2, 3-10.

[10] B. Mohammadi, Sh. Rezapour, N. Shahzad, *Some results on fixed points of α-ψ-Ciric generalized multifunctions*, Fixed Point Theory Appl. (2013) 2013:24.

[11] S. Moradi, F. Khojasteh, *Endpoints of multi-valued generalized weak contraction mappings*, Nonlinear Anal. 74 (2011) 2170–2174.

[12] Sh. Rezapour, J. Hasanzade Asl, *A simple method for obtaining coupled fixed points of α-ψ-contractive type mappings*, Int. J. Anal. (2013) Article ID 438029, 7 pages.

[13] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. 75 (2012) 2154–2165.