COMPUTING THE RADIO LABELING ASSOCIATED WITH ZERO DIVISOR GRAPH OF A COMMUTATIVE RING

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Let a, b be any positive integers and $\Gamma(\mathbb{Z}_a \times \mathbb{Z}_b)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_a \times \mathbb{Z}_b$. In this paper, we investigate the radio number of zero divisor graphs $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ for p, q prime numbers.

Keywords: radio labeling, radio number, distances in graph, zero divisor graph, commutative ring

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1. Introduction

The antennas propagate electromagnetic waves which have different frequencies. These waves are known as Radio waves. A specific signal can be accessed by tuning the radio receiver to a particular frequency. Every radio station must be assigned distinct channels, located within certain proximity of one another. The two radio stations are closer to each other, and then their assigned channels must have the greater difference. The task of efficiently allocating channels to transmitters is called the Channel Assignment (CA) problem.

In 1980 William Hale [12] introduced a model of the CA problem. Mostly the CA problem has been modeled as a graph coloring and labeling problem, where the transmitters are represented as the vertices of a graph. If the transmitters are significantly close to each other then two vertices are adjacent. The channels assigned to the transmitters are the labels to the vertices. For every pair of labels there is a minimum acceptable distance between two distinct vertices with assigned labels. The final aim is to locate a valid labeling such that the span (range) of the channels used is minimized.

Let G be a simple and connected graph. Two vertices are adjacent in a graph if there is an edge between them. The degree of a vertex u in G is the number of edges incident with u and it is denoted as $d_G(u)$. Let $d(u, v)$ denote the distance between two distinct vertices of a connected graph G and the maximum distance between any two vertices of G is known as diameter of G, it is denoted as $diam(G)$. A *radio labeling* or *multi-level distance labeling* [15, 14] of G is a function $\xi : V(G) \to \mathbb{N}$ for which the following condition holds for any two distinct vertices u and v:

$$d(u, v) + |\xi(u) - \xi(v)| \geq 1 + diam(G) \quad (1)$$

This condition is referred to as *radio condition.*

We denote by $S(G, \xi)$ the set of consecutive integers $\{m, m+1, ..., M\}$, where $m = \min_{u\in V(G)} \xi(u)$ and $M = \max_{u\in V(G)} \xi(u)$ is the span of ξ, denoted $span(\xi)$.

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The minimum span of a radio labeling for G is called *radio number* of G, denoted by $rn(G)$. A radio labeling ξ of G with $span(\xi) = rn(G)$ will be called optimal radio labeling for G.

Other than its inspiration by the channel task, radio labeling itself is an attractive graph labeling problem and has been considered by numerous authors. It is computationally very hard to determine the radio number on general graph. The problem is known to be NP-hard for graphs with diameter 2, yet the complication as a rule isn't known [13]. In this manner, the researcher concentrate their study in this area on special family of graphs, even for some basic classes of graphs the problem proving to be complex [14]. The radio numbers for paths and cycles were determined in [10, 9, 21], and were totally explained by Liu and Zhu [14]. Sooryanarayana and Raghunath [19] investigated the radio number for the cube of C_n for all $n \leq 20$ and for $n \equiv 0$ or 2 or 4 $\pmod 6$. They additionally proved the values of n for which this graph is radio graceful. Ahmad and Marinescu-Ghemeci [5] determined the radio numbers for some ladder related graphs. For further detail, see [8, 11, 16, 17].

Let R be a commutative ring with identity and $Z(R)$ is the set of all zero divisors of R. $G(R)$ is said to be a zero divisor graph if $x, y \in V(G(R)) = Z(R)$ and $(x, y) \in E(G(R))$ if and only if $x.y = 0$. Beck [7] introduced the notion of zero divisor graph. Anderson and Livingston [4] proved that $G(R)$ is always connected if R is commutative. Anderson and Badawi [3] introduced the total graph of R as: there is an edge between any two distinct vertices $u, v \in R$ if and only if $u + v \in Z(R)$. For a graph G, the concept of graph parameters have always a high interest. Numerous authors briefly studied the zero-divisor and total graphs from commutative rings [2, 5, 6, 18, 20].

Let p, q be two prime numbers and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ be zero divisor graph of the commutative rings $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. In this paper, we investigate the radio number of zero divisor graphs $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ for p, q prime numbers.

2. **Results and Discussions**

Let $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ denotes the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ is defined as: For $x \in \mathbb{Z}_{p^2}$ & $y \in \mathbb{Z}_q$, $(x, y) \notin V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$ if and only if $x \neq p, 2p, 3p, \ldots, (p-1)p$ & $y \neq 0$. Let $I = \{(x, y) \notin V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) : x \neq p, 2p, 3p, \ldots, (p-1)p \ \& y \neq 0\}$, then $|I| = (p^2 - p)(q - 1)$. The vertices of the set I are the non zero divisors of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Also $(0, 0) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ is a non zero divisor. Therefore, the total number of non zero divisors are: $|I|+1 = (p^2-p)(q-1)+1 = p^2q-p^2-pq+p+1$. There are p^2q total vertices of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Hence, there are $p^2q-(p^2q-p^2-pq+p+1) = p^2+pq-p-1$ total number of zero divisors. This implies that the order of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $p^2 + pq - p - 1$ i.e $|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| = p^2 + pq - p - 1$.

In order to discuss the degree of each vertex $(x, y) \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$, we have to see four different cases.

Case 1: If $x = 0$ and $y \in \mathbb{Z}_q \setminus \{0\}$, then each such vertex $(0, y)$ is only adjacent to the vertices $(x', 0)$ for every $x' \in \mathbb{Z}_{p^2} \setminus \{0\}$. Hence the degree of each vertex $(0, y)$ is $p^2 - 1$.

Case 2: If $x \in \{p, 2p, \ldots, (p-1)p\}$ and $y \in \mathbb{Z}_q \setminus \{0\}$, then each such vertex (x, y) is only adjacent to the vertices $(x', 0)$ for every $x' \in \{p, 2p, \ldots, (p-1)p\}$. Hence the degree of each vertex (x, y) is $p - 1$.

Case 3: If $x \in \{p, 2p, \ldots, (p-1)p\}$ and $y = 0$, then each such vertex $(x, 0)$ is adjacent to the vertices $(0, y'), (x', 0)$ & (x', y') for every $y' \in \mathbb{Z}_q \setminus \{0\}$ and $x \neq x' \in \{p, 2p, \ldots, (p-1)p\}$. Hence the degree of each vertex $(x, 0)$ is $(q - 1) + (p - 2) + (pq - p - q + 1) = pq - 2$.

Case 4: If $x \in \mathbb{Z}_{p^2} \setminus \{0, p, 2p, \ldots, (p-1)p\}$ and $y = 0$, then each such vertex $(x, 0)$ is only adjacent to the vertices $(0, y')$ for every $y' \in \mathbb{Z}_q \setminus \{0\}$. Hence the degree of each vertex $(x, 0)$ is $q - 1$.

The zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ of the ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ contains $q - 1$ number of vertices of degree $p^2 - 1$; $pq - p - q + 1$ number of vertices of degree $p - 1$; $p - 1$ number of vertices of degree $pq - 2$ and $p^2 - p$ number of vertices of degree $q - 1$. By using the hand shaking lemma the number of edges of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ are $|E(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))| = \frac{1}{2}\Big\{(p^2-p)(q-1)+(q-1)(p^2-1)+(p-1)(q-1)(p-1)+(p-1)(pq-2)\Big\} = \frac{(p-1)(4pq-3p-2)}{2}$. In the following theorem, we determine the lower bound for zero divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$.

Theorem 2.1. *Let p, q be two prime numbers with $p > q \geq 2$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. The lower bound of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is $2p^2 + 4q - 7$ i.e $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \geq 2p^2 + 4q - 7$.*

Proof. From above discussion and our convenience, suppose that

$$\begin{aligned} A =& \Big\{(x,0) : x \in \mathbb{Z}_{p^2} \setminus \{0, p, 2p, \ldots, (p-1)p\}\Big\} \\ B =& \Big\{(x,0) : x = p, 2p, \ldots, (p-1)p\Big\} \\ C =& \Big\{(0,y) : y \in \mathbb{Z}_q \setminus \{0\}\Big\} \\ D =& \Big\{(x,y) : x = p, 2p, \ldots, (p-1)p \text{ and } y \in \mathbb{Z}_q \setminus \{0\}\Big\} \end{aligned}$$

This shows that $|A| = p^2-p$, $|B| = p-1$, $|C| = q-1$ and $|D| = pq-p-q+1$. Let $d_A(u)$ denotes the degree of a vertex u in A and $d(A, B)$ denotes the distance between the vertices of two sets A and B. For any $a, a' \in A$, $b, b' \in B$, $c, c' \in C$ & $d, d' \in D$ then $d_A(a) = q-1$, $d_B(b) = pq-2$, $d_C(c) = p^2 - 1$, $d_D(d) = p - 1$ and $d(a, a') = d(a, b) = d(d, d') = d(c, c') = d(c, d) = 2$, $d(a, c) = d(c, b) = d(b, d) = d(b, b') = 1$, $d(a, d) = 3$. This implies that the diameter of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ is 3 i.e $diam(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) = 3$.

For any radio labeling ϕ of a $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ must satisfy the following the radio condition

$$d(u, v) + |\phi(u) - \phi(v)| \geq diam(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) + 1 = 4 \tag{2}$$

for any distinct vertices $u, v \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q))$. Let ϕ be an optimal radio labeling for $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$. We count the number of values need for label and add the minimum number of forbidden values for ϕ. As $p > q \geq 2$, therefore

$|A| > |D|$ and $|A| - |D| = p^2 - p - pq + p + q - 1 = p^2 - pq + q - 1$.

Since $d(a, d) = 3$, for $a \in A, d \in D$, it is possible to use consecutive labels between the vertices of sets A & D. Its mean there is no forbidden values associative with the vertices of set D. For any two distinct vertices $b, b' \in B$, such that $d(b, b') = 1$ & $d(B, D) = d(B, C) = 1$ and $d(B, A) = 2$. Therefore, $|\phi(b) - \phi(b')| \geq 3$, hence there are $2p - 2$ forbidden values associative with the vertices of the set B. For any two distinct vertices $c, c' \in C$, $d(c, c') = 2$ & $d(C, B) = d(C, A) = 1$ and $d(C, D) = 2$. Therefore, $|\phi(c) - \phi(c')| \geq 2$, there are $q - 1$ forbidden values associative with the vertices of set C and $3q - 3$ forbidden values associative for $q - 1$ vertices of set A or vice versa. Since For any two distinct vertices $a, a' \in A, d(a, a') = 2$, now p^2-pq-p vertices are left in the set A, therefore $|\phi(a)-\phi(a')| \geq 2$, so there are $p^2 - pq - p$ forbidden values. Thus the total number of minimum forbidden values are: $2p - 2 + q - 1 + 3q - 3 + p^2 - pq - p = p^2 - pq + p + 4q - 6$.

Adding the forbidden values to the number of vertices to label provide a total of $2p^2 + 4q - 7$ labels, hence $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \geq 2p^2 + 4q - 7$, for $p > q \geq 2$.

This completes the proof.

□

Theorem 2.2. *Let p, q be two prime numbers with $p > q \geq 2$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Then $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \leq 2p^2 + 4q - 7$.*

Proof. We shall provide a radio labeling of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ with span $2p^2 + 4q - 7$, which implies $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \leq 2p^2 + 4q - 7$. From Theorem 2.1 and our convenience we define the following:

$$\begin{aligned}
A =& \Big\{(x,0) : x \in \mathbb{Z}_{p^2} \setminus \{0, p, 2p, \ldots, (p-1)p\}\Big\} = \Big\{a_i : 1 \leq i \leq p^2 - p\Big\} \\
B =& \Big\{(x,0) : x = p, 2p, \ldots, (p-1)p\Big\} = \Big\{b_i : 1 \leq i \leq p-1\Big\} \\
C =& \Big\{(0,y) : y \in \mathbb{Z}_q \setminus \{0\}\Big\} = \Big\{c_i : 1 \leq i \leq q-1\Big\} \\
D =& \Big\{(x,y) : x = p, 2p, \ldots, (p-1)p \text{ and } y \in \mathbb{Z}_q \setminus \{0\}\Big\} \\
=& \Big\{d_i : 1 \leq i \leq pq - p - q + 1\Big\}
\end{aligned}$$

The radio labeling $\phi : V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \to \mathbb{Z}^+$ is defined the following:

$$\phi(a_i) = \begin{cases} 2i - 1; & 1 \leq i \leq pq - p - q + 2 \\ 4i - 2pq + 2p + 2q - 5; & pq - p - q + 3 \leq i \leq pq - q + 1 \\ 6i - 4pq + 2p + 4q - 7; & pq - q + 2 \leq i \leq pq \\ 2i + 2p + 4q - 7; & pq + 1 \leq i \leq p^2 - p. \end{cases}$$

$\phi(b_i) = 4i + 2pq - 2p - 2q + 1$, for $1 \leq i \leq p-1$; $\phi(c_i) = 6i + 2pq + 2p - 2q - 4$, for $1 \leq i \leq q-1$ and $\phi(d_i) = 2i$, for $1 \leq i \leq pq - p - q + 1$. It is easy to see that the span of ϕ is equal to $2p^2 + 4q - 7$.

Claim: The labeling ϕ is a valid radio labeling.

We must show that the radio condition

$$d(u,v) + |\phi(u) - \phi(v)| \geq diam(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) + 1 = 4 \tag{3}$$

holds for all pairs of vertices $u, v \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$, where $u \neq v$.

1: Consider the pairs (a_i, a_j) with $i \neq j$, note that $d(a_i, a_j) = 2$ for $i \neq j$ and $|\phi(a_i) - \phi(a_j)| \geq 2$ for all $a_i \neq a_j$. Hence, the radio condition (3) is satisfied.

2: Consider the pairs (b_i, b_j) with $i \neq j$, $1 \leq i, j \leq p-1$, we have $d(b_i, b_j) = 1$ and the label difference for these pairs are $|\phi(b_i) - \phi(b_j)| = 4|i - j| \geq 4$, so the condition (3) is satisfied.

3: Consider the pairs (c_i, c_j) with $i \neq j$, $1 \leq i, j \leq q-1$, we have $d(c_i, c_j) = 2$ and $|\phi(c_i) - \phi(c_j)| = 6|i - j| \geq 6$, so the radio condition (3) is satisfied.

4: As $|\phi(d_i) - \phi(d_j)| = 2|i - j| \geq 2$ for pairs (d_i, d_j) with $i \neq j$ and $d(d_i, d_j) = 2$. This shows that radio condition (3) is satisfied.

5: Consider the pairs (b_i, c_j) with $1 \leq i \leq p-1$, $1 \leq j \leq q-1$ and $d(b_i, c_j) = 1$, also $|\phi(b_i) - \phi(c_j)| = |6j - 4i + 4p - 5| \geq 5$. This satisfied the radio condition (3).

6: Consider the pairs (b_i, d_j) with $1 \leq i \leq p-1$, $1 \leq j \leq pq - p - q + 1$ and $d(b_i, d_j) = 1$, $|\phi(b_i) - \phi(d_j)| = |4i - 2j + 2pq - 2p - 2q + 1| \geq 3$, so the condition (3) is satisfied.

7: Consider the pairs (c_i, d_j) with $1 \leq i \leq q-1$, $1 \leq j \leq pq - p - q + 1$ and $d(c_i, d_j) = 2$, the difference between any pair is $|\phi(c_i) - \phi(d_j)| = |6i - 2j + 2pq + 2p - 2q - 4| \geq 4$, hence the condition (3) is satisfied.

8: Finally, consider the pairs (u,v), where $u \in A = \Big\{a_i : 1 \le i \le p^2 - p\Big\}$ and $v \in B = \Big\{b_i : 1 \le i \le p-1\Big\}$ or $v \in C = \Big\{c_i : 1 \le i \le q-1\Big\}$ or $v \in D = \Big\{d_i : 1 \le i \le pq-p-q+1\Big\}$. This implies that $\phi(u) \in \Big\{1,3,5,\dots,2pq-2p-2q+3\Big\} \cup \Big\{2pq-2p-2q+7, 2pq-2p-2q+11,\dots,2pq+2p-2q-1\Big\} \cup \Big\{2pq+2p-2q+5, 2pq+2p-2q+11,\dots,2pq+2p+4q-7\Big\} \cup \Big\{2pq+2p+4q-5, 2pq+2p+4q-3,\dots,2p^2+4q-7\Big\}$ and $\phi(v) \in \Big\{2,4,6,\dots,2p^2-2p-2q+2\Big\}$ or $\phi(v) \in \Big\{2pq-2p-2q+5, 2pq-2p-2q+9,\dots,2pq+2p-2q-3\Big\}$ or $\phi(v) \in \Big\{2pq+2p-2q+2, 2pq+2p-2q+8,\dots,2pq+2p+4q-10\Big\}$.

- If $d(u,v)=1$, then $\phi(v) \in \Big\{2pq+2p-2q+2, 2pq+2p-2q+8,\dots,2pq+2p+4q-10\Big\}$ and $|\phi(u)-\phi(v)| \ge 3$.
- If $d(u,v)=2$, then $\phi(v) \in \Big\{2pq-2p-2q+5, 2pq-2p-2q+9,\dots,2pq+2p-2q-3\Big\}$ and $|\phi(u)-\phi(v)| \ge 2$.
- If $d(u,v)=3$, then $\phi(v) \in \Big\{2,4,6,\dots,2p^2-2p-2q+2\Big\}$ and $|\phi(u)-\phi(v)| \ge 1$.

It follows that the radio condition (3) is satisfied for these pairs. These eight cases establish the claim that ϕ is a radio labeling of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$. Thus $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \le span(\phi) \le 2p^2+4q-7$. This completes the proof.

□

Theorem 2.3. *Let p,q be two prime numbers with $p > q \ge 2$ and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Then $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = 2p^2+4q-7$.*

Proof. Theorem 2.1 shows $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \ge 2p^2+4q-7$ for $p > q \ge 2$ and Theorem 2.2 shows $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) \le 2p^2+4q-7$ for $p > q \ge 2$. Therefore, $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = 2p^2+4q-7$ for $p > q \ge 2$. □

Theorem 2.4. *Let $p \ge 2$ be a prime number. Then $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \ge 2p^2+2p-4$.*

Proof. Let $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. The graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ contains $p-1$ number of vertices of degree p^2-1; $p-1$ number of vertices of degree p^2-2 and $(p^2-p)+(p^2-2p+1)$ number of vertices of degree $p-1$ among of p^2-p number of vertices are adjacent with vertices of degree p^2-1 & p^2-2p+1 number of vertices are adjacent with vertices of degree p^2-2. For our convenience, we partition the set of vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ into disjoint subsets as: $V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = A \cup B \cup C \cup D$, where

$$
\begin{aligned}
A =& \Big\{(x,0) : x \in \mathbb{Z}_{p^2} \ \&\ x \ne 0, p, 2p, \dots, (p-1)p\Big\} = \Big\{a_i : 1 \le i \le p^2-p\Big\}\\
B =& \Big\{(x,0) : x = p, 2p, \dots, (p-1)p\Big\} = \Big\{b_i : 1 \le i \le p-1\Big\}\\
C =& \Big\{(0,y) : y \in \mathbb{Z}_p \setminus \{0\}\Big\} = \Big\{c_i : 1 \le i \le p-1\Big\}\\
D =& \Big\{(x,y) : x = p, 2p, \dots, (p-1)p \text{ and } y \in \mathbb{Z}_p \setminus \{0\}\Big\}\\
=& \Big\{d_i : 1 \le i \le p^2-2p+1\Big\}
\end{aligned}
$$

This shows that $|A| = p^2-p$, $|B| = |C| = p-1$, $|D| = p^2-2p+1$ and the number of vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ are $2p^2-p-1$. Let $d(A,B)$ denotes the distance between the vertices of two sets A and B. For any $a, a' \in A$, $b, b' \in B$, $c, c' \in C$ & $d, d' \in D$ then

$d(a,c) = d(c,b) = d(b,d) = d(b,b') = 1$, $d(a,a') = d(a,b) = d(d,d') = d(c,c') = d(c,d) = 2$ and $d(a,d) = 3$. The maximum distance between among the vertices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is 3. Therefore the diameter of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ is 3 i.e $diam(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p) = 3$.

For any radio labeling ψ of a $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ must satisfy the following the radio condition

$$d(u,v) + |\psi(u) - \psi(v)| \geq diam(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p) + 1 = 4 \tag{4}$$

for any distinct vertices $u, v \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p))$. Let ψ be an optimal radio labeling for $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$. To obtain the radio number for ψ, we count the minimum number of forbidden values and add these into the number of values need for label. For $p \geq 2$,

$|A| > |D|$ and $|A| - |D| = p^2 - p - (p^2 - 2p + 1) = p - 1$.

Since the distance between the vertices of two sets A & D is 3 and among the vertices of sets is 2. Therefore, it is possible to use consecutive labels between the vertices of sets A & D. This means that there is no forbidden values associative with the vertices of set D. The distance between the vertices of the set B with the vertices of the sets C, D and A is 1,1 and 2, respectively and the distance among different vertices of set B is also 1. For any $b, b' \in B$ must satisfy $|\psi(b) - \psi(b')| \geq 3$, hence there are $2p - 2$ forbidden values corresponding to the the vertices of the set A & D. Finally, the distance among two distinct vertices of the set C is 2 and the distance between the vertices of set C with the vertices of the sets A, B & D is 1,1 & 2, respectively. For any $c, c' \in C$, must satisfy $|\psi(c) - \psi(c')| \geq 2$ therefore there are $p - 1$ forbidden values. Hence there are total number of minimum forbidden values are: $2p - 2 + p - 1 = 3p - 3$.

By adding the forbidden values and the number of vertices to label provide a total of $2p^2 + 2p - 4$ labels, hence $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \geq 2p^2 + 2p - 4$, for $p \geq 2$.

This completes the proof.

□

Theorem 2.5. *Let $p \geq 2$ be a prime number. Then $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \leq 2p^2 + 2p - 4$.*

Proof. Consider the sets A, B, C and D defined in Theorem 2.4. We define a radio labeling ψ of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ with span $2p^2 + 2p - 4$, which implies $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \leq 2p^2 + 2p - 4$. The radio labeling $\psi : V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \to \mathbb{Z}^+$ is defined in the following way:

$$\psi(a_i) = \begin{cases} 2i - 1; & 1 \leq i \leq p^2 - 2p + 2 \\ 4i - 2p^2 + 4p - 5; & p^2 - 2p + 3 \leq i \leq p^2 - p. \end{cases}$$

$\psi(b_i) = 4i + 2p^2 - 4p + 1$, for $1 \leq i \leq p - 1$; $\psi(c_i) = 2p^2 + 2i - 2$, for $1 \leq i \leq p - 1$ and $\psi(d_i) = 2i$, for $1 \leq i \leq p^2 - 2p + 1$. It is easy to see that the span of ψ is equal to $2p^2 + 2p - 4$.

Claim: The labeling ψ is a valid radio labeling.

We must show that the radio condition

$$d(u,v) + |\phi(u) - \phi(v)| \geq diam(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p) + 1 = 4 \tag{5}$$

holds for all pairs of vertices $u, v \in V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$, where $u \neq v$.

1: For $i \neq j$, $d(a_i, a_j) = 2$ and $|\psi(a_i) - \psi(a_j)| \geq 2$, similarly $|\psi(b_i) - \psi(b_j)| \geq 4$, $|\psi(c_i) - \psi(c_j)| \geq 2$, $|\psi(d_i) - \psi(d_j)| \geq 2$, here $b_i, b_j \in B$, $c_i, c_j \in C$, $d_i, d_j \in D$ with $i \neq j$ and $d(c_i, c_j) = 2$, $d(b_i, b_j) = 1$, $d(d_i, d_j) = 2$. Hence, the radio condition (5) is satisfied.

2: Consider the pairs (b_i, c_j) for $1 \leq i, j \leq p - 1$ and $|\psi(b_i) - \psi(c_j)| \geq 3$ with $d(b_i, c_j) = 1$. This satisfied the radio condition (5).

3: Consider the pairs (b_i, d_j) for $1 \leq i \leq p-1$, $1 \leq j \leq p^2-2p+1$ and $|\psi(b_i) - \psi(d_j)| \geq 3$ with $d(b_i, d_j) = 1$, so the condition (5) is satisfied.

4: Consider the pairs (c_i, d_j) for $1 \leq i \leq p-1$, $1 \leq j \leq p^2-2p+1$ and $|\psi(c_i) - \psi(d_j)| = |2p^2 - 2 + 2(i - j)| \geq 4p - 2$ with $d(c_i, d_j) = 2$, so the radio condition (5) is satisfied.

5: Finally, consider the pairs (u, v), where $u \in A = \Big\{a_i : 1 \le i \le p^2 - p\Big\}$ and $v \in B = \Big\{b_i : 1 \le i \le p-1\Big\}$ or $v \in C = \Big\{c_i : 1 \le i \le p-1\Big\}$ or $v \in D = \Big\{d_i : 1 \le i \le p^2 - 2p + 1\Big\}$. This implies that $\phi(u) \in \Big\{1, 3, 5, \ldots, 2p^2 - 4p + 3\Big\} \cup \Big\{2p^2 - 4p + 7, 2p^2 - 4p + 11, \ldots, 2p^2 - 5\Big\}$ and $\phi(v) \in \Big\{2, 4, 6, \ldots, 2p^2 - 4p + 2\Big\}$ or $\phi(v) \in \Big\{2p^2, 2p^2 + 2, \ldots, 2p^2 + 2p - 4\Big\}$ or $\phi(v) \in \Big\{2p^2 - 4p + 5, 2p^2 - 4p + 9, \ldots, 2p^2 - 3\Big\}$.

- If $d(u, v) = 1$, then $\psi(v) \in \Big\{2p^2, 2p^2 + 2, \ldots, 2p^2 + 2p - 4\Big\}$ and $|\psi(u) - \psi(v)| \ge 3$.
- If $d(u, v) = 2$, then $\psi(v) \in \Big\{2p^2 - 4p + 5, 2p^2 - 4p + 9, \ldots, 2p^2 - 3\Big\}$ and $|\psi(u) - \psi(v)| \ge 2$.
- If $d(u, v) = 3$, then $\psi(v) \in \Big\{2, 4, 6, \ldots, 2p^2 - 4p + 2\Big\}$ and $|\psi(u) - \psi(v)| \ge 1$.

It follows that the radio condition (5) is satisfied for these pairs. These five cases establish the claim that ψ is a radio labeling of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$. Thus $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \le span(\phi) \le 2p^2 + 2p - 4$. This completes the proof.

□

Theorem 2.6. *Let $p \ge 2$ be a prime number and $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Then $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = 2p^2 + 2p - 4$.*

Proof. Theorem 2.4 shows $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \ge 2p^2 + 2p - 4$ for $p \ge 2$ and Theorem 2.5 shows $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) \le 2p^2 + 2p - 4$ for $p \ge 2$. Therefore, $rn(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)) = 2p^2 + 2p - 4$ for $p \ge 2$. □

REFERENCES

[1] *A. Ahmad, R. Marinescu-Ghemeci*, Radio labeling of some Ladder-related Graphs, Mathematical Reports, **19(69) No. 1,**, 1(2014), 107–119.

[2] *S. Akbari, A. Mohammadian*, On the zero-divisor graph of a commutative ring, J. Algebra, **274**(2004), 847–855.

[3] *D.F. Anderson, A. Badawi*, On the Zero-Divisor Graph of a Ring, Communications in Algebra, **36** (8)(2008), 3073–3092.

[4] *D.F. Anderson, P.S. Livingston*, The Zero-divisor Graph of Commutative Ring, Journal of Algebra, **217**(1999), 434–447.

[5] *D.F. Anderson, S.B. Mulay*, On the diameter and girth of a zero-divisor graph, J. Pure Appl. Algebra, **210**(2)(2008), 543–550.

[6] *T. Asir, T. Tamizh Chelvam*, On the total graph and its complement of a commutative ring, Comm. Algebra, **41**(10)(2013), 3820–3835.

[7] *I. Beck*, Coloring of a commutative ring, J. Algebra, **116** (1988), 208–226.

[8] *G. Chartrand, P. Zhang*, Radio colorings of graphsa survey, Int. J. Comput. Appl. Math., **2 (3)**(2007), 237–252.

[9] *G. Chartrand, D. Erwin, and P. Zhang*, A graph labeling problem suggested by FM channel restrictions, Bull. Inst. Combin. Appl., **43**(2005), 43–57.

[10] *G. Chartrand, D. Erwin, F. Harary and P. Zhang*, Radio labelings of graphs, Bull. Inst. Combin. Appl., **33**(2001), 77–85.

[11] *J. A. Gallian*, A Dynamic Survey of Graph Labeling, The electronic journal of combinatorics, **20**(2017), #DS6.

[12] *W. K. Hale*, Frequency assignment: theory and applications, Proc. IEEE, **68**(1980), 1497–1514.

[13] *M. Kchikech, R. Khennoufa and O. Togni*, Linear and cyclic radio k-labelings of trees, Discussiones Mathematicae Graph Theory, **130**(2007)(1), 105–123.

[14] *D. Liu and X. Zhu*, Multi-level distance labelings for paths and cycles, SIAM J. Disc. Math., **19(3)**(2005), 610–621.

[15] *D. Liu*, Radio number for trees, Disc. Math., **308**(2008), 1153–1164.

[16] *D. Liu, M. Xie*, Radio number for square paths, Ars Combin., **90**(2009), 307–319.

[17] *P. Martinez, J. Ortiz, M. Tomova, C. Wyels*, Radio numbers for generalized prism graphs, Discuss. Math. Graph Theory, **31 (1)**(2011), 45–62.

[18] *K. Samei*, The zero-divisor graph of a reduced ring, Journal of Pure and Applied Algebra, **209**(2007), 813–821.

[19] *B. Sooryanarayana and P. Raghunath*, Radio labeling of cube of a cycle, Far East J. Appl. Math., **29**(2007) 113–147.

[20] *T. Tamizh Chelvam, T. Asir*, A note on total graph of $\mathbb{Z}_n$, J. Discrete Math. Sci. Cryptogr., **14**(1)(2011), 1–7.

[21] *P. Zhang*, Radio labelings of cycles, Ars. Combin., **65**(2002), 21–32.