ADDITIVE σ-RANDOM OPERATOR INEQUALITY AND RHOM-DERIVATIONS IN FUZZY BANACH ALGEBRAS

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In this paper, we solve an additive σ-random operator inequality and by the fixed point technique we get an approximation of mentioned additive σ-random operator in fuzzy Banach spaces. Also, we get an approximation of rhom-derivations in fuzzy complex Banach algebras.

Keywords: approximation; rhom-derivation in Banach algebra; additive σ-random operator; fixed point.

MSC2010: 54H12, 47H10, 39B52, 47B47, 46L57.

1. Introduction

Let $(\Omega, \mathfrak{U}, \mu)$ be a probability measure space. Assume that $(U, \mathfrak{B}_U)$ and $(V, \mathfrak{B}_V)$ are Borel measureable spaces, in which U and V are complete FN spaces and $T : \Omega \times U \to V$ is a random operator. In FN-spaces, first we solve the additive σ-random operator inequality

$$\begin{aligned} & \eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \\ \geq \; & \eta(\sigma(T(\omega, u-v) - T(\omega, u) - T(\omega, -v)), t), \end{aligned} \tag{1}$$

where $0 \neq \sigma \in \mathbb{C}$ is fixed and $|\sigma| < 1$.

By the fixed point technique, we get an approximation of the above additive σ-random operator inequality (1) in FB-spaces. Also, we get an approximation of hom-derivations in FB-algebras.

2. Preliminaries

In this paper, we let $I = [0, 1]$ and $J = (0, 1]$.

Definition 2.1. ([1, 2]) A *continuous triangular norm* (shortly, a *ct-norm*) is a continuous mapping κ from I^2 to I such that

(a) $\kappa(\varsigma, \tau) = \kappa(\tau, \varsigma)$ and $\kappa(\varsigma, \kappa(\tau, \sigma)) = \kappa(\kappa(\varsigma, \tau), \sigma)$ for all $\varsigma, \tau, \sigma \in I$;
(b) $\kappa(\varsigma, 1) = \varsigma$ for all $\varsigma \in I$;
(c) $\kappa(\varsigma, \tau) \leq \kappa(\sigma, \iota)$ whenever $\varsigma \leq \sigma$ and $\tau \leq \iota$ for all $\varsigma, \tau, \sigma, \iota \in I$.

Some examples of the *ct*-norms are as follows:

(1) $\kappa_P(\varsigma, \tau) = \varsigma\tau$;
(2) $\kappa_M(\varsigma, \tau) = \min\{\varsigma, \tau\}$;
(3) $\kappa_L(\varsigma, \tau) = \max\{\varsigma + \tau - 1, 0\}$ (: the Lukasiewicz t-norm).

Definition 2.2. ([3, 4]) Suppose that κ is a *ct*-norm, V is a linear space and η is a fuzzy set from $V \times (0, \infty)$ to J. In this case, the ordered tuple (V, η, κ) is said a *fuzzy normed space* (in short, *FN-space*) if the following conditions are satisfied:

(FN1) $\eta(v, t) = 1$ for all $t > 0$ if and only if $v = 0$;

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(FN2) $\eta(\alpha v, t) = \eta\left(v, \frac{t}{|\alpha|}\right)$ for all $v \in V$ and $\alpha \in \mathbb{C}$ with $\alpha \neq 0$;
(FN3) $\eta(u+v, t+s) \geq \kappa(\eta(u,t), \eta(v,s))$ for all $u, v \in V$ and $t, s \geq 0$.
(FN4) $\eta(v, .) : (0, \infty) \to J$ is continuous for all $v \in V$.

Example 2.3. *Consider linear normed space $(V, \|\cdot\|)$. Then*

$$\eta(v, s) = \exp(-\frac{\|v\|}{s})$$

for all $s > 0$ defines a fuzzy norm and (V, η, κ_M) is a FN-space.

Let (V, η, κ) be a FN-space. We define the *open ball* $B_v(r, t)$ with center $v \in V$ and radius $0 < r < 1$ for all $t > 0$ as follows:

$$B_v(r, t) = \{u \in V \ : \ \eta(v - u, t) > 1 - r\}.$$

In [5, 6] the authors show, every open ball $B_v(r, t)$ is open set. Now, different kinds of topologies can be introduced in a FN-space. The *(r, t)-topology* is introduced by a family of neighborhoods

$$\{B_v(r, t)\}_{v \in V, t > 0, r \in (0,1)}.$$

In fact, every fuzzy norm η on V generates a topology ((r, t)-topology) on V which has as a base the family of open sets of the form

$$\{B_v(r, t)\}_{v \in V, t > 0, r \in (0,1)}.$$

A sequence $\{v_n\}$ in V is said to be *convergent* to a point $v \in V$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\eta(v_n - v, \epsilon) > 1 - \lambda$$

whenever $n \geq N$. Also, a sequence $\{v_n\}$ in V is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\eta(v_n - v_m, \epsilon) > 1 - \lambda$$

whenever $n \geq m \geq N$. A FN-space (V, η, κ) is said to be *complete* if every Cauchy sequence in V is convergent to a point in V.

Definition 2.4. [7, 8] A *fuzzy normed algebra* (in short FN-algebra) $(V, \eta, \kappa, \kappa')$ is an FN-space (V, η, κ) with the structure of an algebra such that

(FN-5) $\eta(uv, ts) \geq \kappa'(\eta(u,t), \eta(v,s))$ for all $u, v \in V$ and all $t, s > 0$. in which κ' is a ct-norm.

Example 2.5. *Every normed algebra $(V, \|\cdot\|)$ defines a FN-algebra $(V, \eta, \kappa_M, \kappa_P)$, where*

$$\eta(v, s) = \exp(-\frac{\|v\|}{s})$$

for all $s > 0$ if and only if

$$\|uv\| \leq \|u\|\|v\| + s\|v\| + t\|u\| \ (u, v \in V; \ t, s > 0).$$

This space is called the induced FN-algebra. A complete FN-algebra is called fuzzy Banach algebra, in short FB-algebra.

Let $(\Omega, \mathfrak{U}, \mu)$ be a probability measure space. Assume that $(U, \mathfrak{B}_U)$ and $(V, \mathfrak{B}_V)$ are Borel measureable spaces, in which U and V are complete FN spaces. A mapping $T : \Omega \times U \to V$ is said to be a random operator if $\{\omega : \ T(\omega, u) \in B\} \in \mathfrak{U}$ for all u in U and $B \in \mathfrak{B}_V$. Also, T is random operator, if $T(\omega, u) = v(\omega)$ be a V-valued random variable for every u in U. A random operator $T : \Omega \times U \to V$ is called *linear* if $T(\omega, \alpha u_1 + \beta u_2) = \alpha T(\omega, u_1) + \beta T(\omega, u_2)$ almost every where for each u_1, u_2 in U and α, β scalers, and *fuzzy random bounded* (in short FR-bounded) if there exists a nonnegative real-valued random variable $M(\omega)$ such that

$$\eta(T(\omega, u_1) - T(\omega, u_2), M(\omega)t) \geq \eta(u_1 - u_2, t),$$

almost every where for each u_1, u_2 in U and $t > 0$. The set of all linear FR-bounded random operator from U to V showed by $\mathfrak{R}(U, V)$. Also, the random operator $T : \Omega \times U \to V$ is homomorphism if $T(\omega, .)$ is homomorphism.

Mirzavaziri and Moslehian [9, 10] introduced the concept of h-derivation. Recently, Park et. al. [11], generalized the concept of h-derivation and introduced the concept of hom-derivations in a Banach algebra.

Definition 2.6. Let V be a complex FB-algebra and $\zeta : V \to V$ be a homomorphism. A $\mathbb{C}$-linear operator $\mathcal{R} : V \to V$ is called a *rhom-derivation* on V if R satisfies

$$R(\omega, uv) = R(\omega, u)\zeta(\omega, v) + \zeta(\omega, u)R(\omega, v)$$

for all $u, v \in V$ and $\omega \in \Omega$.

When we consider stability process of a random operator equation we get an approximation of random operator, the similar process done for functional equation first time introduced by Ulam [12] and solved by Hyers [13], next some mathematician got important results of this subject, Aoki [14], Rassias [15], Găvruta [16], Skof [17], Cholewa [18] and Park [19, 20] and et. al. [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

Theorem 2.7. ([37, 38]) *Consider a complete generalized metric space* (Γ, Δ) *and a strictly contractive function* $L : \Gamma \to \Gamma$ *with Lipschitz constant* $\beta < 1$. *So, for every given element* $\gamma \in \Gamma$, *either*

$$\Delta(L^m\gamma, L^{m+1}\gamma) = \infty$$

for each $m \in \mathbb{N}$ *or there is* $m_0 \in \mathbb{N}$ *such that*

(1) $\Delta(L^m\gamma, L^{m+1}\gamma) < \infty, \qquad \forall m \geq m_0$;

(2) *the fixed point* ϖ^* *of* L *is the convergent point of sequence* $\{L^m\gamma\}$;

(3) *in the set* $\Upsilon = \{\varpi \in \Gamma \mid \Delta(L^{m_0}\gamma, \varpi) < \infty\}$, ϖ^* *is the unique fixed point of* L;

(4) $(1 - \beta)\Delta(\varpi, \varpi^*) \leq \Delta(\varpi, L\varpi)$ *for every* $\varpi \in \Upsilon$.

3. Additive σ-random operator inequality: FPT

Lemma 3.1. *Let the random operator* $T : \Omega \times U \to V$ *satisfies* (1), *then* T *is additive.*

Proof. Let T satisfies (1). Replacing v by $-v$ in (1), implies that

$$\eta(T(\omega, u-v) - T(\omega, u) - T(\omega, -v), t) \geq \eta(\sigma(T(\omega, u+v) - T(\omega, u) - T(\omega, v)), t) \tag{2}$$

for all $u, v \in U$, $\omega \in \Omega$ and $t > 0$. (1) and (2) imply that

$$\eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \geq \eta\left(T(\omega, u-v) - T(\omega, u) - T(\omega, -v), \frac{t}{\sigma^2}\right)$$

and hence $T(\omega, u+v) = T(\omega, u) + T(\omega, v)$ for each $u, v \in U$ and $\omega \in \Omega$, since $|\sigma| < 1$. Thus T is additive. □

By fixed point technique we get an approximation of the additive σ-random operator inequality (1) in FB-spaces.

Theorem 3.2. *Let* (V, η, κ_M) *be an FB-space. Assume that* $\psi : U^2 \times (0, \infty) \to J$ *be a fuzzy set such that there exists an* $\beta < 1$ *with*

$$\psi\left(\frac{u}{2}, \frac{v}{2}, \frac{\beta t}{2}\right) \geq \psi(u, v, t) \tag{3}$$

and

$$\lim_{p\to\infty} \psi\left(\frac{u}{2^p}, \frac{v}{2^p}, \frac{t}{2^p}\right) = 1$$

for all $u, v \in U$ *and* $t > 0$. *Suppose that* $T : \Omega \times U \to V$ *is a random operator, where*

$$\begin{aligned} &\eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \\ \geq\ &\kappa_M(\eta(\sigma(T(\omega, u-v) - T(\omega, u) - T(\omega, -v)), t), \psi(u, v, t)) \end{aligned} \tag{4}$$

for each $u, v \in U$, $\omega \in \Omega$ and $t > 0$. So, there is a unique additive random operator $S :: \Omega \times U \to V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \psi\left(u, u, \frac{2(1-\beta)}{\beta} t\right)$$

almost every where for each $u \in U$ and $t > 0$.

Proof. Putting $u = v$ in (4), we have that

$$\eta(2T(\omega, u) - T(\omega, 2u), t) \geq \psi(u, u, t) \tag{5}$$

almost every where for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

On

$$\Gamma := \{H : \Omega \times U \to V, \quad H(\omega, 0) = 0\}$$

we define the following generalized metric:

$$\Delta(G, H) = \inf\{\alpha \in \mathbb{R}_+ : \eta(G(\omega, u) - H(\omega, u), \alpha t) \geq \psi(u, u, t), \ \forall u \in U, \ t > 0\}.$$

In [39], Mihet and Radu proved that (Γ, Δ) is complete (see also [40]).

Now we consider the linear mapping $L : \Gamma \to \Gamma$ such that

$$LG(\omega, u) := 2G\left(\omega, \frac{u}{2}\right)$$

almost every where for each $u \in U$ and $\omega \in \Omega$. Consider $G, H \in \Gamma$ such that $\Delta(G, H) = \varepsilon$. So,

$$\eta(G(\omega, u) - H(\omega, u), \varepsilon t) \geq \psi(u, u, t)$$

almost every where for each $u \in U$ and $t > 0$. Also,

$$\begin{aligned}
\eta(LG(\omega, u) - LH(\omega, u), \beta\varepsilon t) &= \eta\left(G\left(\omega, \frac{u}{2}\right) - H\left(\omega, \frac{u}{2}\right), \frac{\beta\varepsilon t}{2}\right) \\
&\geq \psi\left(\frac{u}{2}, \frac{u}{2}, \frac{\beta t}{2}\right) \\
&\geq \psi(u, u, t)
\end{aligned}$$

almost every where for each $u \in U$, $\omega \in \Omega$ and $t > 0$. Then, from $\Delta(G, H) = \varepsilon$ we conclude that $\Delta(LG, LH) \leq \beta\varepsilon$ and so

$$\Delta(LG, LH) \leq \beta\Delta(G, H)$$

for each $G, H \in \Gamma$.

By (5) we have that

$$\eta\left(2T\left(\omega, \frac{u}{2}\right) - T(\omega, u), \frac{\beta t}{2}\right) \geq \psi(u, u, t)$$

almost every where for each $u \in U$ and $t > 0$, which implies that $\Delta(T, LT) \leq \frac{\beta}{2}$.

Theorem 2.7 implies that, there exists a random operator $S : \Omega \times U \to V$ such that:

(1) A fixed point for function L, is S,

$$S(\omega, u) = 2S\left(\omega, \frac{u}{2}\right) \tag{6}$$

almost every where for each $u \in U$, which is unique in the set

$$\Upsilon = \{G \in \Gamma : \Delta(G, H) < \infty\};$$

(2) $\Delta(L^p T, S) \to 0$ as $p \to \infty$, which implies that

$$\lim_{p\to\infty} 2^p T\left(\omega, \frac{u}{2^p}\right) = S(\omega, u) \tag{7}$$

almost every where for each $u \in U$ and $\omega \in \Omega$;

(3) $\Delta(T, S) \leq \frac{1}{1-\beta}\Delta(T, LT)$, which implies

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \psi\left(u, u, \frac{2(1-\beta)}{\beta} t\right)$$

almost every where for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Using (4) and (7) imply that

$$\begin{aligned}
&\eta(S(\omega, u+v) - S(\omega, u) - S(\omega, v), t) \\
&= \lim_{p\to\infty} \eta\left(T\left(\omega, \frac{u+v}{2^p}\right) - T\left(\omega, \frac{u}{2^p}\right) - T\left(\omega, \frac{v}{2^p}\right), \frac{t}{2^p}\right) \\
&\geq \lim_{p\to\infty} \kappa_M\Big(\eta\left(\sigma\left(T\left(\omega, \frac{u-v}{2^p}\right) - T\left(\omega, \frac{u}{2^p}\right) - T\left(\omega, \frac{-v}{2^p}\right)\right), \frac{t}{2^p}\right) \\
&, \psi\left(\frac{u}{2^p}, \frac{v}{2^p}, \frac{t}{2^p}\right)\Big) \\
&= \eta(\sigma(S(\omega, u-v) - S(\omega, u) - S(\omega, v)), t)
\end{aligned}$$

almost every where for each $u, v \in U$, $\omega \in \Omega$ and $t > 0$. Then

$$\begin{aligned}
&\eta(S(\omega, u+v) - S(\omega, u) - S(\omega, v), t) \\
\geq\ &\eta(\sigma(S(\omega, u-v) - S(\omega, u) - S(\omega, v)), t)
\end{aligned}$$

almost every where for each $u, v \in U$, $\omega \in \Omega$ and $t > 0$. Now, Lemma 3.1, implies that S is additive random operator. □

Corollary 3.3. *Let (V, η, κ_M) be a FB-space, $\rho > 1$ and $\tau > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator, where*

$$\begin{aligned}
&\eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \qquad (8)\\
\geq\ &\kappa_M\Big(\eta(\sigma(T(\omega, u-v) - T(\omega, u) - T(\omega, -v)), t), \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}\Big),
\end{aligned}$$

in which $|\sigma| < 1$. So, there is a unique additive random operator $S : \Omega \times U \to V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \frac{(1 - 2^{1-\rho})t}{(1 - 2^{1-\rho})t + \tau 2^{1-\rho}\|u\|^\rho}$$

for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. In Theorem 3.2 put $\psi(u, v, t) = \frac{t}{t+\tau(\|u\|^\rho+\|v\|^\rho)}$ for each $u \in U$ and $t > 0$ and $\beta = 2^{1-\rho}$. □

Theorem 3.4. *Let (V, η, κ_M) be an FB-space. Assume that $\psi : U^2 \times (0, \infty) \to J$ be a fuzzy set such that there exists an $\beta < 1$ with*

$$\psi(u, v, 2\beta t) \geq \psi\left(\frac{u}{2}, \frac{v}{2}, t\right) \qquad (9)$$

and

$$\lim_{p\to\infty} \psi(2^p u, 2^p v, 2^p t) = 1$$

for all $u, v \in U$ and $t > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator, where satisfies in (4). So, there is a unique additive random operator $S :: \Omega \times U \to V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \psi(u, u, 2(1-\beta)t)$$

almost every where for each $u \in U$ and $t > 0$.

Proof. Consider the generalized metric space (Γ, Δ) defined in the proof of Theorem 3.2. Now we consider the linear mapping $L : \Gamma \to \Gamma$ such that

$$LG(\omega, u) := \frac{1}{2} G(\omega, 2u)$$

almost every where for all $u \in U$. It follows from (5) that

$$\eta\left(\frac{T(\omega, 2u)}{2} - T(\omega, u), t\right) \geq \psi\left(\frac{u}{2}, \frac{u}{2}, \frac{t}{\beta}\right)$$

almost every where for each $u \in U$ and $t > 0$.

The proof of Theorem 3.2 leads the rest of the proof. □

Corollary 3.5. *Let (V,η,κ_M) be a FB-space, $\rho<1$ and $\tau>0$. Suppose that $T:\Omega\times U\to V$ be a random operator satisfies (8). So, there is a unique additive random operator $S:\Omega\times U\to V$ such that*

$$\eta(T(\omega,u)-S(\omega,u),t)\geq\frac{(1-2^{1-\rho})t}{(1-2^{1-\rho})t+\tau\|u\|^{\rho}}$$

for each $u\in U$ and $t>0$.

Proof. In Theorem 3.4 put $\psi(u,u,t)=\frac{t}{t+\tau(\|u\|^{\rho}+\|v\|^{\rho})}$ for each $u\in U$ and $t>0$ and $\beta=2^{\rho-1}$. □

4. Additive σ-random operator inequality: DT

By direct technique we get an approximation of the additive σ-random operator inequality (1) in FB-spaces.

Theorem 4.1. *Let (V,η,κ_M) be a FB-space. Let $\varphi:U^2\times(0,\infty)\to J$ be a fuzzy map such that such that there exists an $\beta<1$ with*

$$\psi\left(\frac{u}{2},\frac{v}{2},\frac{\beta t}{2}\right)\geq\psi(u,v,t) \tag{10}$$

and

$$\lim_{p\to\infty}\psi\left(\frac{u}{2^p},\frac{v}{2^p},\frac{t}{2^p}\right)=1 \tag{11}$$

for all $u,v\in U$ and $t>0$. Suppose that $T:\Omega\times U\to V$ is a random operator, where satisfies in (4). So, there is a unique additive random operator $S:\Omega\times U\to V$ such that

$$\eta(T(\omega,u)-S(\omega,u),t)\geq\psi\left(u,u,\frac{2(1-\beta)}{\beta}t\right) \tag{12}$$

almost every where for each $u\in U$, $\omega\in\Omega$ and $t>0$.

Proof. Putting $u=v$ in (4), we have that

$$\eta(2T(\omega,u)-T(\omega,2u),t)\geq\psi(u,u,t) \tag{13}$$

and

$$\eta\left(2T\left(\omega,\frac{u}{2}\right)-T(\omega,u),t\right)\geq\psi\left(\frac{u}{2},\frac{u}{2},t\right) \tag{14}$$

and so

$$\eta\left(2T\left(\omega,\frac{u}{2}\right)-T(\omega,u),\frac{\beta}{2}t\right)\geq\psi(u,u,t) \tag{15}$$

almost every where for each $u\in U$, $\omega\in\Omega$ and $t>0$. Replacing u by $\frac{u}{2^\ell}$ in (15) and applying (10) we get

$$\eta\left(2^{\ell+1}T\left(\omega,\frac{u}{2^{\ell+1}}\right)-2^{\ell}T\left(\omega,\frac{u}{2^{\ell}}\right),\frac{\beta^{\ell+1}}{2}t\right)\geq\psi(u,u,t) \tag{16}$$

which implies that

$$\eta\left(2^{\ell}T\left(\omega,\frac{u}{2^{\ell}}\right)-T(\omega,u),\sum_{k=1}^{\ell}\frac{\beta^k}{2}t\right)\geq\psi(u,u,t) \tag{17}$$

Replacing u by $\frac{u}{2^m}$ in (17) we get

$$\eta\left(2^{\ell+m}T\left(\omega,\frac{u}{2^{\ell+m}}\right)-2^{m}T\left(\omega,\frac{u}{2^{m}}\right),t\right)\geq\psi\left(u,u,\frac{t}{\sum_{k=m+1}^{\ell+m}\frac{\beta^k}{2}}\right) \tag{18}$$

which tend to 1 when m, ℓ tend to ∞ and so the sequence $\left\{2^\ell T\left(\omega, \frac{u}{2^\ell}\right)\right\}$ is Cauchy in the FB-space (V, η, κ_M) and converges to a point $S(x) \in V$. Now, for every $\varsigma > 0$ we have that,

$$\begin{aligned} & \eta(T(\omega, u) - S(\omega, u), t + \varsigma) \\ \geq\; & \kappa_M\left(\eta(T(\omega, u) - 2^\ell T\left(\omega, \frac{u}{2^\ell}\right), t), \eta(S(\omega, u) - 2^\ell T\left(\omega, \frac{u}{2^\ell}\right), \varsigma)\right) \\ \geq\; & \kappa_M\left(\psi\left(u, u, \frac{t}{\sum_{k=1}^{\ell} \frac{\beta^k}{2}}\right), \eta(S(\omega, u) - 2^\ell T\left(\omega, \frac{u}{2^\ell}\right), \varsigma)\right) \end{aligned} \tag{19}$$

when when ℓ tend to ∞ in (19) we have that

$$\eta(T(\omega, u) - S(\omega, u), t + \varsigma) \geq \psi\left(u, u, \frac{2(1-\beta)}{\beta} t\right). \tag{20}$$

Since $\varsigma > 0$ is arbitrary in (20) we have that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \psi\left(u, u, \frac{2(1-\beta)}{\beta} t\right). \tag{21}$$

Replacing u and v by $\frac{u}{2^m}$ and $\frac{v}{2^m}$ in (4) and using (11) implies that S satisfies Lemma 3.1 and hence is additive. Now, let S' be another additive satisfies (12). For a arbitrary $u \in U$ and $\omega \in \Omega$, we have that $2^m S\left(\omega, \frac{u}{2^m}\right) = S(\omega, u)$ and $2^m S'\left(\omega, \frac{u}{2^m}\right) = S'(\omega, u)$ for each natural element m. Using (12), we have that,

$$\begin{aligned} & \eta(S(\omega, u) - S'(\omega, u), t) \\ =\; & \lim_{m\to\infty} \eta\left(2^m S\left(\omega, \frac{u}{2^m}\right) - 2^m S'\left(\omega, \frac{u}{2^m}\right), t\right) \\ \geq\; & \lim_{m\to\infty} \kappa_M\left(\eta\left(2^m S\left(\omega, \frac{u}{2^m}\right) - 2^m T\left(\omega, \frac{u}{2^m}\right), \frac{t}{2}\right),\right. \\ & \left.\eta\left(2^m T\left(\omega, \frac{u}{2^m}\right) - 2^m S'\left(\omega, \frac{u}{2^m}\right), \frac{t}{2}\right)\right) \\ \geq\; & \lim_{m\to\infty} \psi\left(\frac{u}{2^m}, \frac{u}{2^m}, \frac{1-\beta}{\beta} t\right) \\ \geq\; & \lim_{m\to\infty} \psi\left(u, u, \frac{2^m}{\beta^m} \frac{1-\beta}{\beta} t\right) \\ \to\; & 1, \end{aligned}$$

which implies that $S(\omega, u) = S'(\omega, u)$ shows the uniqueness.

□

Corollary 4.2. *Let (V, η, κ_M) be a FB-space, $\rho > 1$ and $\tau > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator, hold in (8). So, there is a unique additive random operator $S : \Omega \times U \to V$ such that*

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \frac{(2 - 2^{2-\rho})t}{(2 - 2^{2-\rho})t + \tau 2^{2-\rho}\|u\|^\rho}$$

for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. In Theorem 4.1 put $\psi(u, u, t) = \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}$ for each $u \in U$ and $t > 0$ and $\beta = 2^{1-\rho}$. □

Theorem 4.3. *Let (V, η, κ_M) be an FB-space. Assume that $\psi : U^2 \times (0, \infty) \to J$ be a fuzzy set hold in (9) and*

$$\lim_{p\to\infty} \psi(2^p u, 2^p v, 2^p t) = 1$$

for all $u, v \in U$ and $t > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator, where satisfies in (4). So, there is a unique additive random operator $S :: \Omega \times U \to V$ such that

$$\eta(T(\omega,u) - S(\omega,u),t) \geq \psi\,(u,u,2(1-\beta)t)$$

almost every where for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. Putting $u = v$ in (4), we have that

$$\eta\left(T\,(\omega,u) - \frac{T(\omega,2u)}{2}, t\right) \geq \psi\,(u,u,2t) \tag{22}$$

and so

$$\eta\left(T\,(\omega,u) - \frac{T(\omega,2u)}{2}, \beta t\right) \geq \psi\left(\frac{u}{2},\frac{u}{2},t\right) \tag{23}$$

almost every where for each $u \in U$, $\omega \in \Omega$ and $t > 0$. Replacing u by $\frac{u}{2^\ell}$ in (23) and applying (10) we get

$$\eta\left(\frac{T(\omega,u)}{2^\ell} - \frac{T(\omega,2u)}{2^{\ell+1}}), \frac{\beta^\ell}{2}t\right) \geq \psi\,(u,u,t) \tag{24}$$

which implies that

$$\eta\left(\frac{T(2^\ell u)}{2^\ell} - T(u), \sum_{k=0}^{\ell-1}\frac{\beta^k}{2}t\right) \geq \psi\,(u,u,t)\,. \tag{25}$$

The rest of the proof is similar to the proof of Theorem 4.1. □

5. Approximation of rhom-derivations in FB-algebras

By fixed point technique, we get an approximation of rhom-derivations in FB-algebras, associated to the additive σ-random operator inequality (1).

Theorem 5.1. *Let $(V, \eta, \kappa_M, \kappa_M)$ be a FB-algebra. Let $\varphi : V^2 \times (0,\infty) \to J$ be a fuzzy map such that such that there exists an $\beta < 1$ with*

$$\psi\left(\frac{u}{2},\frac{v}{2},\frac{\beta t}{2}\right) \geq \psi\left(\frac{u}{2},\frac{v}{2},\frac{\beta t}{4}\right) \geq \psi\,(u,v,t) \tag{26}$$

and

$$\lim_{p\to\infty}\psi\left(\frac{u}{2^p},\frac{v}{2^p},\frac{t}{2^p}\right) = \lim_{p\to\infty}\psi\left(\frac{u}{2^p},\frac{v}{2^p},\frac{t}{4^p}\right) = 1 \tag{27}$$

for all $u, v \in U$ and $t > 0$. Suppose that $T, S : \Omega \times V \to V$ are odd random operator, where satisfies in

$$\begin{aligned} &\eta(T(\omega,c(u+v)) - c(T(\omega,u) - T(\omega,v)),t) \\ \geq\ &\kappa_M(\eta(\sigma(T(\omega,u-v) - T(\omega,u) - T(\omega,-v)),t),\psi\,(u,v,t)), \end{aligned} \tag{28}$$

$$\begin{aligned} &\eta(S(\omega,c(u+v)) - c(S(\omega,u) - S(\omega,v)),t) \\ \geq\ &\kappa_M(\eta(\sigma(S(\omega,u-v) - S(\omega,u) - S(\omega,-v)),t),\psi\,(u,v,t)), \end{aligned} \tag{29}$$

$$\eta(S(\omega,uv) - S(\omega,u)S(\omega,v),t) \quad\geq\quad \psi\,(u,v,t)\,, \tag{30}$$

$$\eta(T(\omega,uv) - T(\omega,u)S(\omega,v) - S(\omega,u)T(\omega,v),t) \quad\geq\quad \psi\,(u,v,t)\,, \tag{31}$$

almost every where for each $u, v \in V$, $\omega \in \Omega$ and $t > 0$ and for all $c \in \mathbb{T}^1 := \{d \in \mathbb{C} : |d| = 1\}$.

So, there is a unique random homomorphism $\zeta : \Omega \times V \to V$ and a unique rhom-derivation $R : \Omega \times V \to V$ such that

$$\eta(T(\omega, u) - R(\omega, u), t) \geq \psi\left(u, u, \frac{2(1-\beta)}{\beta} t\right), \tag{32}$$

$$\eta(S(\omega, u) - \zeta(\omega, u), t) \geq \psi\left(u, u, \frac{2(1-\beta)}{\beta} t\right), \tag{33}$$

$$R(\omega, uv) = R(\omega, u)\zeta(\omega, v) + \zeta(\omega, u)R(\omega, v), \tag{34}$$

almost every where for each $u, v \in V$, $\omega \in \Omega$, $t > 0$.

Proof. Put $c = 1$ in (28) and (29). According proof of Theorem 3.2, there are unique random operators $\zeta, R : \Omega \times V \to V$ hold in (32) and (33), respectively, where made by

$$\begin{aligned} \zeta(\omega, u) &= \lim_{n\to\infty} 2^n S\left(\omega, \frac{u}{2^n}\right), \\ R(\omega, u) &= \lim_{n\to\infty} 2^n T\left(\omega, \frac{u}{2^n}\right) \end{aligned}$$

for each $u \in V$, $\omega \in \Omega$.

Putting $v = 0$ in (28), implies that

$$\eta(T(\omega, cu) - cT(\omega, u), t) \geq \psi(u, 0, t),$$

for each $u \in V$, $\omega \in \Omega$ and $t > 0$ and for all $c \in \mathbb{T}^1 := \{d \in \mathbb{C} : |d| = 1\}$. Then

$$\begin{aligned} \eta(R(\omega, cu) - cR(\omega, u), t) &= \eta\left(2^p T\left(\omega, c\frac{u}{2^p}\right) - 2^p cT\left(\omega, \frac{u}{2^p}\right), t\right) \\ &= \eta\left(T\left(\omega, c\frac{u}{2^p}\right) - cT\left(\omega, \frac{u}{2^p}\right), \frac{t}{2^p}\right) \\ &\geq \psi\left(\frac{u}{2^p}, 0, \frac{t}{2^p}\right) \to 1, \end{aligned}$$

and so $R(\omega, cu) = cR(\omega, u)$ for each $u \in V$, $\omega \in \Omega$ and $t > 0$ and for all $c \in \mathbb{T}^1 := \{d \in \mathbb{C} : |d| = 1\}$. By the same reasoning as in the proof of [41, Theorem 2.1], the random operator $R : \Omega \times V \to V$ is $\mathbb{C}$-linear.

By similar method we can prove that the additive random operator $\zeta : \Omega \times V \to V$ is $\mathbb{C}$-linear.

From inequality (30) we have that

$$\begin{aligned} \eta(\zeta(\omega, uv) - \zeta(\omega, u)\zeta(\omega, v), t) &= \eta\left(4^p S\left(\omega, \frac{uv}{4^p}\right) - 4^p S\left(\omega, \frac{u}{2^p}\right) S\left(\omega, \frac{v}{2^p}\right), t\right) \\ &= \eta\left(S\left(\omega, \frac{uv}{4^p}\right) - S\left(\omega, \frac{u}{2^p}\right) S\left(\omega, \frac{v}{2^p}\right), \frac{t}{4^p}\right) \\ &\geq \psi\left(\frac{u}{2^p}, 0, \frac{t}{4^p}\right), \end{aligned}$$

$$\begin{aligned} &\eta(R(\omega, uv) - R(\omega, u)\zeta(\omega, v) - \zeta(\omega, u)R(\omega, v), t) \\ = \;& \eta\left(4^p T\left(\omega, \frac{uv}{4^p}\right) - 4^p T\left(\omega, \frac{u}{2^p}\right) S\left(\omega, \frac{v}{2^p}\right) - 4^p S\left(\omega, \frac{u}{2^p}\right) T\left(\omega, \frac{v}{2^p}\right), t\right) \\ = \;& \eta\left(T\left(\omega, \frac{uv}{4^p}\right) - T\left(\omega, \frac{u}{2^p}\right) S\left(\omega, \frac{v}{2^p}\right) - S\left(\omega, \frac{u}{2^p}\right) T\left(\omega, \frac{v}{2^p}\right), \frac{t}{4^p}\right) \\ \geq \;& \psi\left(\frac{u}{2^p}, 0, \frac{t}{4^p}\right) \to 1, \end{aligned}$$

for each $u, v \in V$, $\omega \in \Omega$ and $t > 0$ and for all $c \in \mathbb{T}^1 := \{d \in \mathbb{C} : |d| = 1\}$. Then, $\zeta(\omega, uv) = \zeta(\omega, u)\zeta(\omega, v)$ for each $u, v \in V$, $\omega \in \Omega$. Therefore the $\mathbb{C}$-linear random operator $\zeta : \Omega \times V \to V$ is a random homomorphism hold in (33).

From inequality (31) we have that

$$\begin{aligned}
&\eta(R(\omega,uv)-R(\omega,u)\zeta(\omega,v)-\zeta(\omega,u)R(\omega,v),t)\\
=&\ \eta\left(4^pT\left(\omega,\frac{uv}{4^p}\right)-4^pT\left(\omega,\frac{u}{2^p}\right)S\left(\omega,\frac{v}{2^p}\right)-4^pS\left(\omega,\frac{u}{2^p}\right)T\left(\omega,\frac{v}{2^p}\right),t\right)\\
=&\ \eta\left(T\left(\omega,\frac{uv}{4^p}\right)-T\left(\omega,\frac{u}{2^p}\right)S\left(\omega,\frac{v}{2^p}\right)-S\left(\omega,\frac{u}{2^p}\right)T\left(\omega,\frac{v}{2^p}\right),\frac{t}{4^p}\right)\\
\geq&\ \psi\left(\frac{u}{2^p},0,\frac{t}{4^p}\right)\to 1,
\end{aligned}$$

for each $u,v\in V$, $\omega\in\Omega$ and $t>0$ and for all $c\in\mathbb{T}^1:=\{d\in\mathbb{C}:|d|=1\}$. Then, $R(\omega,uv)=R(\omega,u)\zeta(\omega,v)-\zeta(\omega,u)R(\omega,v)$ for each $u,v\in V$, $\omega\in\Omega$. Thus the $\mathbb{C}$-linear random operator $R:\Omega\times V\to V$ is a rhom-derivation hold in (32) and (34). □

Corollary 5.2. *Let $(V,\eta,\kappa_M,\kappa_M)$ be a FB-algebra, $\rho>1$ and $\tau>0$. Suppose that $T,S:\Omega\times V\to V$ is a random operators, where hold in*

$$\begin{aligned}
&\eta(T(\omega,c(u+v))-c(T(\omega,u)-T(\omega,v)),t) \qquad (35)\\
\geq&\ \kappa_M\left(\eta(\sigma(T(\omega,u-v)-T(\omega,u)-T(\omega,-v)),t),\frac{t}{t+\tau(\|u\|^\rho+\|v\|^\rho)}\right),
\end{aligned}$$

$$\begin{aligned}
&\eta(S(\omega,c(u+v))-c(S(\omega,u)-S(\omega,v)),t) \qquad (36)\\
\geq&\ \kappa_M\left(\eta(\sigma(S(\omega,u-v)-S(\omega,u)-S(\omega,-v)),t),\frac{t}{t+\tau(\|u\|^\rho+\|v\|^\rho)}\right),
\end{aligned}$$

$$\eta(S(\omega,uv)-S(\omega,u)S(\omega,v),t)\geq\frac{t}{t+\tau(\|u\|^\rho+\|v\|^\rho)}, \qquad (37)$$

$$\eta(T(\omega,uv)-T(\omega,u)S(\omega,v)-S(\omega,u)T(\omega,v),t)\geq\frac{t}{t+\tau(\|u\|^\rho+\|v\|^\rho)}, \qquad (38)$$

almost every where for each $u,v\in V$, $\omega\in\Omega$ and $t>0$ and for all $c\in\mathbb{T}^1:=\{d\in\mathbb{C}:|d|=1\}$.

So, there is a unique random homomorphism $\zeta:\Omega\times V\to V$ and a unique rhom-derivation $R:\Omega\times V\to V$ such that

$$\eta(T(\omega,u)-R(\omega,u),t)\geq\frac{(2-2^{2-\rho})t}{(2-2^{2-\rho})t+\tau 2^{2-\rho}\|u\|^\rho}, \qquad (39)$$

$$\eta(S(\omega,u)-\zeta(\omega,u),t)\geq\frac{(2-2^{2-\rho})t}{(2-2^{2-\rho})t+\tau 2^{2-\rho}\|u\|^\rho}, \qquad (40)$$

$$R(\omega,uv)=R(\omega,u)\zeta(\omega,v)+\zeta(\omega,u)R(\omega,,v), \qquad (41)$$

almost every where for each $u,v\in V$, $\omega\in\Omega$, $t>0$.

Proof. In Theorem 5.1 put $\psi(u,u,t)=\frac{t}{t+\tau(\|u\|^\rho+\|v\|^\rho)}$ for each $u\in U$ and $t>0$ and $\beta=2^{1-\rho}$. □

Theorem 5.3. *Let $(V,\eta,\kappa_M,\kappa_M)$ be a FB-algebra. Let $\varphi:V^2\times(0,\infty)\to J$ be a fuzzy map such that such that there exists an $\beta<1$ with*

$$\psi(u,v,4\beta t)\geq\psi(u,v,2\beta t)\geq\psi\left(\frac{u}{2},\frac{v}{2},t\right) \qquad (42)$$

and

$$\lim_{p\to\infty}\psi(2^pu,2^pv,2^pt)=\lim_{p\to\infty}\psi(2^pu,2^pv,4^pt)=1 \qquad (43)$$

for all $u, v \in U$ and $t > 0$. Suppose that $T, S : \Omega \times V \to V$ are odd random operator, where satisfies in (28), (29), (30) *and* (31). *So, there is a unique random homomorphism $\zeta : \Omega \times V \to V$ and a unique rhom-derivation $R : \Omega \times V \to V$ such that*

$$\eta(T(\omega, u) - R(\omega, u), t) \geq \psi(u, u, 2(1 - \beta)t), \tag{44}$$

$$\eta(S(\omega, u) - \zeta(\omega, u), t) \geq \psi(u, u, 2(1 - \beta)t), \tag{45}$$

$$R(\omega, uv) = R(\omega, u)\zeta(\omega, v) + \zeta(\omega, u)R(\omega, v), \tag{46}$$

almost every where for each $u, v \in V$, $\omega \in \Omega$, $t > 0$.

Proof. By similar method used in the proof of Theorem 5.1, we can get the results. □

Corollary 5.4. *Let $(V, \eta, \kappa_M, \kappa_M)$ be a FB-algebra, $\rho < 1$ and $\tau > 0$. Suppose that $T, S : \Omega \times V \to V$ is a random operators, where hold in* (35), (36), (37) *and* (38).

So, there is a unique random homomorphism $\zeta : \Omega \times V \to V$ and a unique rhom-derivation $R : \Omega \times V \to V$ such that

$$\eta(T(\omega, u) - R(\omega, u), t) \geq \frac{(1 - 2^{1-\rho})t}{(1 - 2^{1-\rho})t + \tau\|u\|^{\rho}}, \tag{47}$$

$$\eta(S(\omega, u) - \zeta(\omega, u), t) \geq \frac{(1 - 2^{1-\rho})t}{(1 - 2^{1-\rho})t + \tau\|u\|^{\rho}}, \tag{48}$$

$$R(\omega, uv) = R(\omega, u)\zeta(\omega, v) + \zeta(\omega, u)R(\omega, v), \tag{49}$$

almost every where for each $u, v \in V$, $\omega \in \Omega$, $t > 0$.

Proof. In Theorem 5.3 put $\psi(u, u, t) = \frac{t}{t + \tau(\|u\|^{\rho} + \|v\|^{\rho})}$ for each $u \in U$ and $t > 0$ and $\beta = 2^{\rho - 1}$. □

Acknowledgements

The authors are thankful to the anonymous referee for giving valuable comments and suggestions which helped to improve the final version of this paper.

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