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PDES OF HAMILTON-PFAFF TYPE VIA MULTI-TIME OPTIMIZATION PROBLEMS

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In this paper, PDEs of Hamilton-Pfaff type are derived in the sense of exterior differential via a multi-time optimal control problem. Using the control Hamiltonian 1-form associated with our basic optimal control problem and the corresponding adjoint distributions, we obtain PDEs of Hamilton-Pfaff type.

Keywords: Hamilton-Pfaff PDEs; control Hamiltonian; Lagrangian 1-form; Euler-Lagrange exterior PDEs; adjoint distributions.

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1. Introduction

Nowadays, the multi-time optimal control problems and multi-time optimization are intensively studied (see [10], [14]-[17]), both from theoretical and applied reasonings. The cost functionals of mechanical work type become very important in applications due to their physical meaning. In this direction (see [10]), let consider the following multi-time optimal control problem, formulated using as cost functional a path independent curvilinear integral with distribution-type constraints:

$$\max_{u(\cdot)} \left\{ J\left(u(\cdot)\right) = \int_{\Gamma_{t_0,t_1}} X_\beta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta \right\} \tag{1.1}$$

subject to

$$dx^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t) = X^i_\beta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta \tag{1.2}$$

$$u(t) \in \mathcal{U},\ \forall t \in \Omega_{t_0,t_1}; \quad x(t_\xi) = x_\xi,\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi} \tag{1.3}$$

$$i = \overline{1,n}, \quad \alpha_\zeta \in \{1,\ldots,m\}, \quad \zeta, j = \overline{1,s-1}, \quad \xi = 0,1.$$

The mathematical data used are: $t = (t^\alpha) \in \Omega_{t_0,t_1}$ (see $\Omega_{t_0,t_1} \subset \mathbb{R}^m_+$, the hyperparallelepiped with the diagonal opposite points $t_0 = \left(t_0^1,\ldots,t_0^m\right)$ and $t_1 = \left(t_1^1,\ldots,t_1^m\right)$) is a *multi-parameter of evolution* or a *multi-time*; Γ_{t_0,t_1} is a C^1-class curve joining the points t_0 and t_1; $x(t) = (x^i(t)),\ i = \overline{1,n}$, is a C^{s+1}-class function, called *state vector*; $u(t) = (u^a(t)),\ a = \overline{1,k}$, is a *continuous control vector*; the *running cost* $X_\beta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta$ is a *non-autonomous Lagrangian* 1-*form*; the equations in (1.2) are *distribution-type equations*; the functions $X^i_\beta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right)$ are of C^1-class; we accept the notations $x_{\alpha_1}(t) := \dfrac{\partial x}{\partial t^{\alpha_1}}(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t) := \dfrac{\partial^{s-1}x}{\partial t^{\alpha_1}\ldots\partial t^{\alpha_{s-1}}}(t)$, $\alpha_j \in \{1,2,\ldots,m\}$,

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$j = \overline{1, s-1}$. We assume summation over the repeated indices. For more details, see references [9], [10].

Consider $\hat{u}(t) \in Int\mathcal{U}$ an interior optimal solution which determines the optimal evolution $x(t)$ in (1.1), subject to (1.2) and (1.3). In a very recent work (see [10]), we proved that there exist the C^1-class co-state 1-forms, $p_r = (p_{ir})$, $r = \overline{1, s}$, defined on Ω_{t_0,t_1}, such that the relations

$$dp_{j1}(t) = -H_{x^j}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad p_{j1}(t_1) = 0 \tag{1.4}$$

$$dp_{j2}(t) = -H_{x^j_{\alpha_1}}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad p_{j2}(t_1) = 0$$

$$\vdots$$

$$dp_{js}(t) = -H_{x^j_{\alpha_1\ldots\alpha_{s-1}}}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad p_{js}(t_1) = 0$$

$$\forall t \in \Omega_{t_0,t_1};$$

$$H_{u^a}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in \Omega_{t_0,t_1}; \tag{1.5}$$

$$dx^i_{\alpha_1\ldots\alpha_{r-1}}(t) = \frac{\partial H}{\partial p_{ir}}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad \forall t \in \Omega_{t_0,t_1} \tag{1.6}$$

$$x(t_0) = x_0,\ x_{\alpha_1\ldots\alpha_j}(t_0) = \tilde{x}_{\alpha_1\ldots\alpha_j 0},\ j = \overline{1, s-1},\ i = \overline{1, n},\ r = \overline{1, s}$$

$$\left(see\ dx^i_{\alpha_1\ldots\alpha_0}(t) := dx^i(t)\right)$$

are fulfilled (see H as the associated *control Hamiltonian* 1*-form*).

The aim of this paper is to introduce PDEs of Hamilton-Pfaff type (often used in Mechanics) using the *simplified multi-time maximum principle* (see (1.4), (1.5), (1.6)). For other different ideas connected to this subject, see [1]-[8], [12], [13], [18].

Section 1 motivates the study and provides, for a better coherence of this paper, the main ingredients used in previous works (see [10], [11], [14]-[17]). Section 2 includes the main result, while Section 3 contains the conclusions of this study.

2. **Hamilton-Pfaff PDEs**

Let consider the following path independent curvilinear integral functional

$$J\left(u(\cdot)\right) = \int_{\Gamma_{t_0,t_1}} X_\beta\left(x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta$$

subject to

$$dx^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t) = u^i_\beta(t) dt^\beta$$

$$t \in \Omega_{t_0,t_1} \subset R^m_+; \quad x(t_0) = x_0,\ x_{\alpha_1\ldots\alpha_j}(t_0) = \tilde{x}_{\alpha_1\ldots\alpha_j 0}$$

$$i = \overline{1, n}, \quad \alpha_\zeta \in \{1, \ldots, m\}, \quad \zeta, j = \overline{1, s-1}.$$

Here, the running cost $X_\beta\left(x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta$ is a C^2-class *autonomous Lagrangian* 1*-form*, the control vector $u(t) = \left(u^i_\beta(t)\right)$ is a continuous vector function and Γ_{t_0,t_1} is a C^1-class curve joining the points t_0 and t_1 from R^m_+.

For solving the associated basic control problem, we need the *control Hamiltonian* 1*-form*

$$H\left(v_1(t), v_2(t), \ldots, v_s(t), u(t), p_1(t), \ldots, p_s(t)\right)$$

$$= X_\beta\left(v_1(t), v_2(t), \ldots, v_s(t), u(t)\right) dt^\beta + p_{i1}(t) v^i_{1\alpha_1}(t) dt^{\alpha_1}$$

$$+ \ldots + p_{is-1}(t) v^i_{s-1\alpha_{s-1}}(t) dt^{\alpha_{s-1}} + p_{is}(t) u^i_\beta(t) dt^\beta,$$

where $\{v_1(t), \dots, v_s(t)\}$ are *auxiliary variables*, defined as follows:

$$x(t) := v_1(t),\ (x_{\alpha_1}(t)) := v_2(t),\ \cdots,\ \big(x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t)\big) := v_s(t),$$

$$dv^i_s(t) = u^i_\beta(t)dt^\beta,\ i = \overline{1,n},\ \beta = \overline{1,m},$$

or, equivalently,

$$(v_{1\alpha_1}(t)) = v_2(t), \quad (v_{2\alpha_2}(t)) = v_3(t),\ \cdots,\ \big(v_{s-1\alpha_{s-1}}(t)\big) = v_s(t)$$

$$v^i_{s\beta}(t) = u^i_\beta(t), \quad i = \overline{1,n},$$

(we have denoted $v_{\gamma\eta}(t) := \dfrac{\partial v_\gamma}{\partial t^\eta}(t),\ \gamma = \overline{1,s},\ \eta \in \{1,\dots,m\}$), and the *co-state* 1-*forms* $p_\gamma(t) = p_{i\gamma}(t)dv^i_\gamma,\ \gamma = \overline{1,s}$, satisfying the following *adjoint distributions*:

$$dp_{j1}(t) = -\frac{\partial X_\beta}{\partial x^j}\big(x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta \tag{2.1}$$

$$dp_{j2}(t) = -p_{j1}(t)dt^{\alpha_1} - \frac{\partial X_\beta}{\partial x^j_{\alpha_1}}\big(x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta$$

$$\vdots$$

$$dp_{js}(t) = -p_{js-1}(t)dt^{\alpha_{s-1}} - \frac{\partial X_\beta}{\partial x^j_{\alpha_1\dots\alpha_{s-1}}}\big(x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta.$$

According to (1.4) and (1.5), we have

$$H_{x^j_{\alpha_1\alpha_2\dots\alpha_\eta}}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t), p_1(t), \dots, p_s(t)\big) = -dp_{j\eta+1}(t) \tag{2.2}$$

$$\eta = \overline{0, s-1}, \quad t \in \Omega_{t_0,t_1}, \quad p_{jr}(t_1) = 0,\ r = \overline{1,s}$$

$$H_{u^j_\beta}\big(x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t), p_1(t), \dots, p_s(t)\big) = 0, \quad \forall t \in \Omega_{t_0,t_1}.$$

By a direct computation, we get

$$H_{x^i}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t), p_1(t), \dots, p_s(t)\big)$$
$$= X_{\beta x^i}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta;$$
$$H_{x^i_{\alpha_1\alpha_2\dots\alpha_\eta}}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t), p_1(t), \dots, p_s(t)\big)$$
$$= X_{\beta x^i_{\alpha_1\alpha_2\dots\alpha_\eta}}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta + p_{i\eta}(t)dt^{\alpha_\eta}, \quad \eta = \overline{1, s-1};$$
$$H_{u^i_\beta}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t), p_1(t), \dots, p_s(t)\big)$$
$$= X_{\beta u^i_\beta}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta + p_{is}(t)dt^\beta,$$

or, equivalently, (see (2.2))

$$X_{\beta x^i}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta + dp_{i1}(t) = 0 \tag{2.3}$$

$$X_{\beta x^i_{\alpha_1\dots\alpha_\eta}}\big(x(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta + p_{i\eta}(t)dt^{\alpha_\eta} + dp_{i\eta+1}(t) = 0$$

$$\eta = \overline{1, s-1}$$

$$X_{\beta u^i_\beta}\big(x(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t), u(t)\big)\,dt^\beta + p_{is}(t)dt^\beta = 0.$$

The previous relations give us

$$p_{i\eta}dt^{\alpha_\eta} = -dp_{i\eta+1} - \frac{\partial X_\beta}{\partial x^i_{\alpha_1\alpha_2\dots\alpha_\eta}}dt^\beta, \quad \eta = \overline{1, s-1} \tag{2.4}$$

$$dp_{i1} = -\frac{\partial X_\beta}{\partial x^i} dt^\beta, \quad p_{is} dt^\beta = -\frac{\partial X_\beta}{\partial u^i_\beta} dt^\beta.$$

From the relations (2.4), we find

$$dp_{i\eta} \wedge dt^{\alpha_\eta} = -d\left(\frac{\partial X_\beta}{\partial x^i_{\alpha_1\alpha_2\ldots\alpha_\eta}}\right) \wedge dt^\beta, \quad \eta = \overline{1, s-1}$$

$$dp_{is} \wedge dt^\beta = -d\left(\frac{\partial X_\beta}{\partial u^i_\beta}\right) \wedge dt^\beta.$$

Now, using (2.4) and the above equalities, we get

$$-\frac{\partial X_\beta}{\partial x^i} dt^\beta \wedge dt^{\alpha_1} + d\left(\frac{\partial X_\beta}{\partial x^i_{\alpha_1}}\right) \wedge dt^\beta = 0$$

$$\sum_{r=1}^{s-2} \left\{ \left[-p_{ir} dt^{\alpha_r} - \frac{\partial X_\beta}{\partial x^i_{\alpha_1\ldots\alpha_r}} dt^\beta \right] \wedge dt^{\alpha_{r+1}} + d\left(\frac{\partial X_\beta}{\partial x^i_{\alpha_1\ldots\alpha_{r+1}}}\right) \wedge dt^\beta \right\} = 0$$

$$\left[-p_{is-1} dt^{\alpha_{s-1}} - \frac{\partial X_\beta}{\partial x^i_{\alpha_1\ldots\alpha_{s-1}}} dt^\beta \right] \wedge dt^\beta + d\left(\frac{\partial X_\beta}{\partial u^i_\beta}\right) \wedge dt^\beta = 0$$

or, equivalently,

$$\left[\frac{\partial X_\lambda}{\partial x^i} \delta^{\alpha_1}_\beta - \frac{\partial}{\partial t^\lambda}\left(\frac{\partial X_\beta}{\partial x^i_{\alpha_1}}\right) \right] dt^\lambda \wedge dt^\beta = 0$$

(*Euler-Lagrange exterior equations*), and

$$\left[\left(p_{ir} \delta^{\alpha_r}_\lambda + \frac{\partial X_\lambda}{\partial x^i_{\alpha_1\ldots\alpha_r}} \right) \delta^{\alpha_{r+1}}_\beta - \frac{\partial}{\partial t^\lambda}\left(\frac{\partial X_\beta}{\partial x^i_{\alpha_1\ldots\alpha_{r+1}}}\right) \right] dt^\lambda \wedge dt^\beta = 0, \ r = \overline{1, s-2}$$

$$\left[\left(p_{is-1} \delta^{\alpha_{s-1}}_\lambda + \frac{\partial X_\lambda}{\partial x^i_{\alpha_1\ldots\alpha_{s-1}}} \right) - \frac{\partial}{\partial t^\lambda}\left(\frac{\partial X_\beta}{\partial u^i_\beta}\right) \right] dt^\lambda \wedge dt^\beta = 0.$$

Suppose that the critical point condition (see (2.3), the third relation)

$$p_{is}(t) dt^\alpha = -X_{\beta u^i_\alpha}\left(x(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta$$

admits a unique solution that satisfies

$$u^i_\beta(t) dt^\beta = u^i_\beta\left(x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), p_1(t), \ldots, p_s(t)\right) dt^\beta = dx^i_{\alpha_1\ldots\alpha_{s-1}}(t).$$

Using a path independent curvilinear integral, we have

$$x^i_{\alpha_1\ldots\alpha_{s-1}}(t) = x^i_{\alpha_1\ldots\alpha_{s-1}}(t_0) + \int_{\Gamma_{t_0 t}} u^i_\beta\left(x(l), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(l), p_1(l), \ldots, p_s(l)\right) dl^\beta,$$

where $\Gamma_{t_0 t}$ is a piecewise C^1-class curve included in $\Gamma_{t_0 t_1}$. Let remark that the previous critical point condition defines the co-state p_{is} as a *moment* along the curve $\Gamma_{t_0 t}$. Now, we establish the main result of our study.

Theorem 2.1. ***(PDEs of Hamilton-Pfaff type)*** *Under the above assumptions, consider the control Hamiltonian* 1*-form*

$H\left(x(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), u\left(x(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), p_1(t), \ldots, p_s(t)\right), p_1(t), \ldots, p_s(t)\right).$

Then, the associated PDEs of Hamilton-Pfaff type are given by

$$dx^i_{\alpha_1 \ldots \alpha_{r-1}} = \frac{\partial H}{\partial p_{ir}}, \quad t \in \Omega_{t_0, t_1}, \quad x(t_0) = x_0, \quad x_{\alpha_1 \ldots \alpha_j}(t_0) = \tilde{x}_{\alpha_1 \ldots \alpha_j 0}$$

$$j = \overline{1, s-1}, \quad i = \overline{1, n}, \quad r = \overline{1, s}, \quad \left(see\ dx^i_{\alpha_1 \ldots \alpha_0}(t) := dx^i(t)\right)$$

and

$$-\frac{\partial H}{\partial x^i} = dp_{i1}, \quad -\frac{\partial H}{\partial x^i_{\alpha_1}} = dp_{i2}, \ \cdots, \ -\frac{\partial H}{\partial x^i_{\alpha_1 \ldots \alpha_{s-1}}} = dp_{is}.$$

Proof. Computing the partial derivative of

$H\left(x(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), u\left(x(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), p_1(t), \ldots, p_s(t)\right), p_1(t), \ldots, p_s(t)\right)$

with respect to p_{ir}, $r = \overline{1, s}$, we get *the first part of Hamilton-Pfaff equations*

$$dx^i_{\alpha_1 \ldots \alpha_{r-1}}(t) = \frac{\partial H}{\partial p_{ir}}\left(x(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), u(t), p_1(t), \ldots, p_s(t)\right)$$

$$t \in \Omega_{t_0, t_1},\ x(t_0) = x_0,\ x_{\alpha_1 \ldots \alpha_j}(t_0) = \tilde{x}_{\alpha_1 \ldots \alpha_j 0},\ j = \overline{1, s-1},\ i = \overline{1, n},\ r = \overline{1, s}$$

$$\left(see\ dx^i_{\alpha_1 \ldots \alpha_0}(t) := dx^i(t)\right).$$

Let illustrate two computations:

$$\frac{\partial H}{\partial p_{j1}} = \frac{\partial X_\beta}{\partial u^i_\alpha} dt^\beta \frac{\partial u^i_\alpha}{\partial p_{j1}} + dx^j + p_{is} \frac{\partial u^i_\alpha}{\partial p_{j1}} dt^\alpha = dx^j$$

$$\frac{\partial H}{\partial p_{js}} = \frac{\partial X_\beta}{\partial u^i_\alpha} dt^\beta \frac{\partial u^i_\alpha}{\partial p_{js}} + u^j_\alpha dt^\alpha + p_{is} \frac{\partial u^i_\alpha}{\partial p_{js}} dt^\alpha = dx^j_{\alpha_1 \alpha_2 \ldots \alpha_{s-1}}.$$

Now, by a direct computation and using the relations in (2.1), we obtain *the second part of Hamilton-Pfaff equations*

$$-\frac{\partial H}{\partial x^j} = -\left[\frac{\partial X_\beta}{\partial x^j} dt^\beta + \frac{\partial X_\beta}{\partial u^i_\alpha} dt^\beta \frac{\partial u^i_\alpha}{\partial x^j}\right] - p_{is} \frac{\partial u^i_\beta}{\partial x^j} dt^\beta = dp_{j1}$$

$$-\frac{\partial H}{\partial x^j_{\alpha_1}} = -\left[\frac{\partial X_\beta}{\partial x^j_{\alpha_1}} dt^\beta + \frac{\partial X_\beta}{\partial u^i_\alpha} dt^\beta \frac{\partial u^i_\alpha}{\partial x^j_{\alpha_1}}\right] - p_{j1} dt^{\alpha_1} - p_{is} \frac{\partial u^i_\beta}{\partial x^j_{\alpha_1}} dt^\beta = dp_{j2}$$

$$\vdots$$

$$-\frac{\partial H}{\partial x^j_{\alpha_1 \ldots \alpha_{s-1}}} = -\left[\frac{\partial X_\beta}{\partial x^j_{\alpha_1 \ldots \alpha_{s-1}}} dt^\beta + \frac{\partial X_\beta}{\partial u^i_\alpha} dt^\beta \frac{\partial u^i_\alpha}{\partial x^j_{\alpha_1 \ldots \alpha_{s-1}}}\right] - p_{js-1} dt^{\alpha_{s-1}}$$

$$-p_{is} \frac{\partial u^i_\beta}{\partial x^j_{\alpha_1 \ldots \alpha_{s-1}}} dt^\beta = dp_{js},$$

and the proof is complete. □

3. Conclusion

In this paper, we introduced a study of a multi-time optimal control problem subject to distribution-type constraints. Using a geometrical language and variational calculus techniques (under simplified hypotheses), we formulated the main result of this paper (see Theorem 2.1), that is, the form of Hamilton-Pfaff PDEs.

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