APPROXIMATION, MARKOV MOMENT PROBLEM AND RELATED INVERSE PROBLEMS

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We give necessary and sufficient conditions for the existence of a unique solution of a multidimensional real classical Markov moment problem, in terms of quadratic forms. Next, we consider applications of a sufficient condition to solving geometrically nonlinear systems with infinite many equations and unknowns (inverse problems solved starting from the moments). Thus, one solves problems studied in the literature by some other methods. Our way of treating these problems works in several dimensions. In the end, one considers a problem not necessarily involving polynomials.

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1. Introduction

The moment problem is an interpolation - type problem, with one or several constraints. The construction of the solution (if it does exist and is unique), is an inverse problem, and so is solving related systems of equations. For similar problems solved by some other methods see [12]. Usually, constructing the solution requires that it is an element of a L^2 space. In this case, its Fourier coefficients are writable in terms of the moments. On the other hand, using polynomial decomposition and approximation in studying the moment problem is a natural and well-known method [1] - [5], [11] - [22]. The present paper applies the determinacy of a given measure to prove density results in L^1 spaces. The first aim of the present work is to characterize a classical multidimensional real Markov moment problem by means of quadratic forms, similarly to the one-dimensional case. A similar method of using approximation, but in solving complex moment problems appears in [16]. The proofs are completely different to those of the real case, although the statements in the latter work are in terms of (complex) quadratic forms. On the other hand, we consider applications to solving related systems of nonlinear equations with infinite many equations and unknowns (see Section 2 of the present work, [12], and also [20], Section 4). Thirdly, this work contains an application of extension of linear operators with two constraints to the Markov moment problem (Sections 3). The background of this paper is contained in [1], [4], [9], [10]. For uniqueness of the solution see [1], [5], [11], [13]. Evaluations and expansions related to generalized differential operators and PDE's are contained in [6, 7]. We start by recalling some known results on polynomial approximation on unbounded subsets. We recall that a

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determinate ($M-$ determinate) measure is, by definition, uniquely determinate by its moments [1], [5], [11], [13].

Theorem 1.1. (see [17], [19]). *Let $A \subset R^n$ be an unbounded closed subset and ν a positive M - determinate regular Borel measure on A, with finite moments of all orders. Then for any nonnegative continuous vanishing at infinity function $\psi \in (C_0(A))_+$, there is a sequence $(p_m)_m$ of polynomials on A, $p_m \geq \psi$, $p_m \to \psi$ in $L^1_\nu(A)$. In particular, one deduces*

$$\lim \int_A p_m d\nu = \int_A \psi \, d\nu,$$

the cone P_+ of positive polynomials is dense in $\left(L^1_\nu(A)\right)_+$ and P is dense in $L^1_\nu(A)$.

Theorem 1.2. *Let $\nu = \nu_1 \times \cdots \times \nu_n$ be a product of n $M-$ determinate positive regular Borel measures on R, with finite moments of all natural orders. Then any nonnegative continuous compactly supported function ψ is approximated in $L^1_\nu\left(R^n\right)$ by means of sums of tensor products $p_1 \otimes \cdots \otimes p_n$, p_j nonnegative polynomial on the real line, in variable t_j, $j = 1,\ldots,n$.*

Proof. If K is the support of a function $\psi \in \left(C_c\left(R^n\right)\right)_+$, then

$$K \subset K_1 \times \cdots \times K_n, K_j = pr_j(K), j = 1,\ldots,n.$$

In order to prove the claimed result, consider a parallelepiped

$$P = [a_1, b_1] \times \cdots \times [a_n, b_n], a_j = \inf K_j, b_j = \sup K_j, j = 1,\ldots,n,$$

containing the above Cartesian product of compacts and apply uniform approximation of ψ on P by means of Bernstein polynomials of n variables. Recall that the explicit form of these polynomials is given by

$$B_m(\psi)(t_1, \ldots, t_n) = \sum_{\substack{k_1=0,\ldots,m \\ \cdots\cdots \\ k_n=0,\ldots,m}} p_{mk_1}(t_1) \cdots p_{mk_n}(t_n) \psi\left(a_1 + (b_1 - a_1)\frac{k_1}{m}, \ldots, a_n + (b_n - a_n)\frac{k_n}{m}\right),$$

$$p_{mk_j}(t_j) = \binom{m}{k_j}\left(\frac{t_j - a_j}{b_j - a_j}\right)^{k_j}\left(\frac{b_j - t_j}{b_j - a_j}\right)^{m-k_j}, t_j \in [a_j, b_j], \qquad j = 1, \ldots, n,$$

$$B_m(\psi) \to \psi, m \to \infty,$$

the convergence being uniform on P. Each term of such a polynomial is a tensor product $p_1 \otimes \cdots \otimes p_n$, of nonnegative polynomials in each variable, on the corresponding compact interval $[a_j, b_j], j = 1, \dots, n$. This follows from the above representation and from the hypothesis on ψ, which is nonnegative. Extend each p_j such that, by definition, it vanishes outside $[a_j, b_j]$, $j = 1, \dots, n$. Notice that such a function might have salts at the ends of the interval $[a_j, b_j]$, and equals p_j on the interval $[a_j, b_j]$. In order to avoid such discontinuities and apply theorem 1.1, we approximate the obtained compactly supported nonnegative measurable functions by compactly supported nonnegative continuous functions, in the spaces $L^1_{\nu_j}(R)$, $j = 1, \dots, n$. These approximations are done by means of Luzin's theorem, cf. [23, p.51, and p.65]. One obtains approximation by sums of tensor products of nonnegative continuous compactly supported functions, in each variable $t_j, j = 1, \dots, n$, in the spaces $L^1_{\nu_j}(R)$, $j = 1, \dots, n$. Application of Theorem 1.1 to $n = 1$, $A = R$, leads to approximation of each such function in each separate variable by dominating (positive) polynomials overall real axes in the space $L^1_{\nu_j}(R)$, $j = 1, \dots, n$. Finally, Fubini's theorem, yields an approximating process in $L^1_\nu(R^n)$. This concludes the proof. □

Remark 1.1. Notice that the preceding arguments do not use the uniform approximation on $[a_j, b_j]$ of a possible strictly positive function, by means of Bernstein polynomials, $j = 1, \dots, n$. For functions of one variable, only L^1 approximation is used, on the whole real line. On the other hand, our hypothesis on ν_j imply that $\nu_j(R) < \infty$. This yields: $\nu_j(p_j + \varepsilon) - \nu_j(p_j) = \varepsilon \nu_j(R) \to 0, \varepsilon \to 0$, so that L^1 approximation by nonnegative polynomials implies L^1 approximation by positive polynomials. Observe also that in the preceding proof we always have $K := support\psi \subseteq P$.

Example 1.1. Let $n = 2, \|(t_1, t_2)\|_1 = |t_1| + |t_2|$, $(t_1, t_2) \in R^2, B_1 = \{(t_1, t_2); \|(t_1, t_2)\|_1 \le 1 \}$,

$\psi_1(t_1, t_2) = 1 - \|(t_1, t_2)\|_1$, $\|(t_1, t_2)\|_1 < 1, \psi_1(t_1, t_2) = 0$ otherwise,

$\psi_2(t_1, t_2) = \frac{1}{3} - \left\| \left(t_1 - \frac{2}{3}, t_2 - \frac{2}{3}\right) \right\|_1$, $\left\| \left(t_1 - \frac{2}{3}, t_2 - \frac{2}{3}\right) \right\|_1 < \frac{1}{3}, \psi_2(t_1, t_2) = 0$ otherwise, $\psi := \psi_1 + \psi_2$.

Then we have

$K := support\psi = support\psi_1 \cup support\psi_2 = B_1 \cup \left(\left\{ \left(\frac{2}{3}, \frac{2}{3} \right) \right\} + \frac{1}{3} B_1 \right), P = [-1,1] \times [-1,1]$.

The paper is organized as follows. Section 2 contains applications of the above approximation results to the multidimensional Markov moment problem. It also contains a sufficient condition for the existence of the solution, and related applications to solving systems of infinitely many nonlinear equations, starting from the moments (inverse problems). One solves by another method a problem similar to that from [12]. Our proofs work in several dimensions. Section 3 is devoted to an application to the abstract L^1 spaces of a result on the abstract moment problem. Section 4 concludes the paper.

2. On Markov moment problem and related systems of equations

Let $(m_j)_{\substack{j\in N^n \\ j_k\geq 1}}$, $j=(j_1,\dots,j_n)$ be a given sequence of real numbers, $\nu=\nu_1\times\cdots\times\nu_n$ be as in Theorem 1.2. The next problem concerns characterizing the existence of a function $h\in L_\nu^\infty(R^n)$, such that

$$0\leq h(t_1,\dots,t_n)\leq 1 \quad \forall (t_1,\dots,t_n)\in R^n, \tag{2.1}$$

$$m_j = j_1\cdots j_n\cdot \int_{R^n} t_1^{j_1-1}\cdots t_n^{j_n-1}h(t_1,\dots,t_n)d\nu,\ j_k\geq 1,\ k=1,\dots,n, \tag{2.2}$$

$$j=(j_1,\dots,j_n)\in (N\setminus\{0\})^n.$$

One denotes

$\varphi_j(t) = j_1\cdots j_n\, t_1^{j_1-1}\cdots t_n^{j_n-1},\ t=(t_1,\dots,t_n)\in\mathbb{R}^n, j=(j_1,\dots,j_n)\in\mathbb{N}^n, j_k\geq 1.$

Theorem 2.1. *The following statements are equivalent:*

(a) *there exists a unique function* $h\in L_\nu^\infty(R^n)$ *with the properties* (2.1), (2.2);

(b) *for any finite subset* $J_0\subset(\mathbb{N}\setminus\{0\})^n$ *and any real numbers* $\alpha_j, j\in J_0$, *we have:*

$$\sum_{j\in J_0}\alpha_j\,\varphi_j(t)\geq 0\ \forall t\in\mathbb{R}^n \Rightarrow \sum_{j\in J_0}\alpha_j\, m_j\in\mathbb{R}_+;$$

for any finite subsets $J_1,\dots,J_n\subset\{1,2,\dots\}$ *and any* $\{\alpha_{j_k};\ j_k\in J_k\}, k=1,\dots,n,$ *the following relation holds*

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\alpha_{i_1}\alpha_{j_1}\cdots\alpha_{i_n}\alpha_{j_n}\frac{m_{i_1+j_1-1,\dots,i_n+j_n-1}}{(i_1+j_1-1)\cdots(i_n+j_n-1)}\right)\cdots\right)\leq$$

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\alpha_{i_1}\alpha_{j_1}\cdots\alpha_{i_n}\alpha_{j_n}\int_{R^n}t_1^{i_1+j_1-2}\cdots t_n^{i_n+j_n-2}d\nu\right)\cdots\right).$$

Proof. (a) ⇒ (b) is straightforward. For the converse, define the linear form F_0 on the space of polynomials, such that the moment conditions $F_0(\varphi_j) = m_j, j \in (\mathbb{N}\setminus\{0\})^n$ are true. Then using the form of positive polynomials on the real line [1], a straightforward computation shows that (b) is equivalent to $F_0(p) \geq 0$ for any nonnegative polynomial on $\mathbb{R}^n$, and

$$F_0(p_1 \otimes \cdots \otimes p_n) \leq \int_{R^n} (p_1 \otimes \cdots \otimes p_n) d\nu,\ p_j \in R[t_j],\ p_j(t_j) > 0\ \forall t_j \in R,\ j = 1,\ldots,n.$$

Let ψ be a nonnegative continuous compactly supported function. Application of theorem 1.2, leads to the existence of approximating sequence

$$\sum_{l=0}^{k(m)} p_{m,1,.l} \otimes \cdots \otimes p_{m,n,l} \to \psi,\ m \to \infty,$$

in the space $L_\nu^1(R^n)$, $p_{m,j,l}(t_j) > 0,\ \forall t_j \in R,\ j = 1,\ldots,n,\ l = 1,\ldots,k(m)$. On the other hand, the linear positive form F_0 has a linear positive extension F to the space of all integrable functions with their absolute value dominated on R^n by a polynomial (cf. [8] or [10, p. 160]). This space contains the space of continuous compactly supported functions. Hence F can be represented by a regular positive Radon measure. Moreover, using (b) and applying Fatou's lemma, one obtains:

$$0 \leq F(\psi) \leq \liminf_m F\left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes \cdots \otimes p_{m,n,j}\right) \leq$$

$$\lim_m \sum_{j=0}^{k(m)} \int_{R^n} (p_{m,1,j} \otimes \cdots \otimes p_{m,n,j}) d\nu = \int_{R^n} \psi\, d\nu,\ \ \psi \in (C_c(R^n))_+.$$

Then for an arbitrary continuous function with compact support, we have:

$$|F(\varphi)| \leq F(\varphi^+) + F(\varphi^-) \leq \int_{R^n} |\varphi|\, d\nu = \|\varphi\|_1.$$

By a standard density argument, F has a linear positive extension of norm at most one, to the space $L_\nu^1(R^n)$. This extension has a representation by a function h with the qualities mentioned at the point (a). This concludes the proof. □

Corollary 2.1. *Let* $K = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset R^n$ *be a parallelepiped,* $d\nu = \chi_K dt_1 \cdots dt_n$.

The following statements are equivalent:

(a) *there exists* $h \in L_\nu^\infty(R^n)$ *such that (2.1) and*

$$m_j = j_1 \cdots j_n \cdot \int_K t_1^{j_1-1} \cdots t_n^{j_n-1} h(t_1,\dots,t_n) d\nu,\ j_k \geq 1,\ k=1,\dots,n,$$

$$j = (j_1,\dots,j_n) \in (N \setminus \{0\})^n .$$

hold;

(b) *for any finite subset* $J_0 \subset (\mathbb{N}\setminus\{0\})^n$ *and any real numbers* $\alpha_j, j \in J_0$, *the following implication holds*

$$\sum_{j\in J_0} \alpha_j \varphi_j(t) \geq 0\ \forall t \in K \Rightarrow \sum_{j\in J_0} \alpha_j m_j \in \mathbb{R}_+;$$

for any finite subsets $J_1,\dots,J_n \subset \{1,2,\dots\}$ *and any* $\{\alpha_{j_k};\ j_k \in J_k\}$, $k=1,\dots,n$, *we have*

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n} \alpha_{i_1}\alpha_{j_1}\cdots\alpha_{i_n}\alpha_{j_n} \frac{m_{i_1+j_1-1,\dots,i_n+j_n-1}}{(i_1+j_1-1)\cdots(i_n+j_n-1)}\right)\cdots\right) \leq$$

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n} \alpha_{i_1}\alpha_{j_1}\cdots\alpha_{i_n}\alpha_{j_n} \prod_{k=1}^{n} \frac{b_k^{i_k+j_k-1} - a_k^{i_k+j_k-1}}{i_k+j_k-1}\right)\cdots\right).$$

The next result uses the following theorem on the abstract moment problem [18]. For the classical formulation, see [14].

Theorem 2.2. (Theorem 4 [18]). *Let* X *be an ordered vector space,* Y *an order complete vector lattice,* $\{x_j\}_{j\in J} \subset X, \{y_j\}_{j\in J} \subset Y$ *given families and* $F_1, F_2 \in L(X,Y)$ *two linear operators. The following statements are equivalent:*

(a) *there is a linear operator* $F \in L(X,Y)$ *such that*

$$F_1(x) \leq F(x) \leq F_2(x)\ \forall x \in X_+,\ F(x_j) = y_j\ \forall j \in J;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R$, *we have:*

$$\left(\sum_{j\in J_0} \lambda_j x_j = \psi_2 - \psi_1,\ \psi_1,\psi_2 \in X_+\right) \Rightarrow \sum_{j\in J_0} \lambda_j y_j \leq F_2(\psi_2) - F_1(\psi_1).$$

Theorem 2.3. *Let* $K = [a_1,b_1]\times\cdots\times[a_n,b_n]$, $\nu = dt_1\cdots dt_n{}_{|K}$. *Consider the following statements*

(a) *there exists a unique* $h \in L_\nu^\infty(K)$, $0 \leq h(t_1,\dots,t_n) \leq 1$ *a.e., such that*

$$m_j = j_1 \cdots j_n \cdot \int_K t_1^{j_1-1} \cdots t_n^{j_n-1} h(t_1,\dots,t_n) dt_1 \cdots dt_n,\ j_k \geq 1,\ k=1,\dots,n;$$

(b) *for any finite subset* $S \subset \{1,2,..\} \times \cdots \times \{1,2,..\}$ *and any* $\{\lambda_j;\, j \in S\} \subset R$, *we have*

$$\sum_{j \in S} \lambda_j m_j \leq \sum_{j \in S} \lambda_j \left(b_1^{j_1} - a_1^{j_1}\right) \cdots \left(b_n^{j_n} - a_n^{j_n}\right).$$

Then $(b) \Rightarrow (a)$.

Proof. We denote
$\varphi_j(t_1,...,t_n) = j_1 \cdot t_1^{j_1 - 1} \cdots j_n \cdot t_n^{j_n - 1},\ j_k \geq 1,\ k = 1,...,n, (t_1,...,t_n) \in K.$ Then using the hypothesis (b), the following implications hold:

$$\sum_{j \in S} \lambda_j \varphi_j = \psi_2 - \psi_1, \psi_j \in \left(L_\nu^1(K)\right)_+,\ j = 1,2 \Rightarrow$$

$$\sum_{j \in S} \lambda_j m_j \leq \sum_{(j,k) \in S} \lambda_j \left(b_1^{j_1} - a_1^{j_1}\right) \cdots \left(b_n^{j_n} - a_n^{j_n}\right) =$$

$$\int_K \sum_{j \in S} \lambda_j \varphi_j dt_1 \cdots dt_n \leq \int_K \psi_2 dt_1 \cdots dt_n = F_2(\psi_2) - F_1(\psi_1),\ F_1 := 0.$$

Hence the implication from (b), Theorem 2.2 is accomplished. Application of the latter theorem leads to the existence of a linear functional F on $X = L_\nu^1(K)$, such that $0 \leq F(\psi) \leq \int_K \psi \cdot dt_1 \cdots dt_n, \quad \forall \psi \in X_+.$

The functional F has a representation by means of a function h, that has all the properties mentioned at point (a) by measure theory arguments. □

The preceding theorem 2.3 suggests the following algorithm in solving the system of equations (2.3) from below.

(i)*Step 1*. Assume that the moments verify condition (b) of Theorem 2.3. Find an approximation of the solution h in terms of the moments $m_j,\ j_k \geq 1.$ To this end, since $h \in L^\infty(K) \subset L^2(K)$, it has a Fourier expansion with respect to the Hilbert base $(\psi_j)_{j_k \geq 1}$ associated following Gram-Schmidt procedure to the complete system of linearly independent polynomials $(\varphi_j)_{j_k \geq 1}$. The Fourier coefficients $\langle h, \psi_j \rangle$ are given by: $\langle h, \psi_j \rangle = \sum_{\substack{l_k \leq j_k, \\ k=1,...,n}} \alpha_l \langle h, \varphi_l \rangle = \sum_{\substack{l_k \leq j_k, \\ k=1,...,n}} \alpha_l m_l,$

where α_l are given by the Gram-Schmidt procedure, so that we know h in terms of the moments. Recall that there exists a subsequence of the sequence of Fourier partial sums, which converges pointwise to h. Hence, we can write: $h \approx \tilde{h}$, where

$\widetilde{h}$ is a partial sum of the Fourier series of h. Note that all these partial sums are polynomials, so that they are continuous.

Step 2. (i) Let $\widetilde{h}$ be a partial sum of the Fourier series with respect to the orthogonal polynomials $(\psi_j)_{j_k \geq 1}$. By using Schwarz inequality, one deduces

$$m_j \approx \int_K \phi_j \tilde{h} dt_1 \cdots dt_n \approx \int_K \phi_j \left(\sum_{p,q} c_{p,q} \cdot \chi_{D_{p,q}} (t_1,...,t_n) \right) dt_1 \cdots dt_n =$$

$$\sum_{p,q} c_{p,q} \cdot \left(\sum_{m\in N} \int_K \phi_j \cdot \chi_{[x_{1,m,p,q}}, y_{1m,p,q})(t_1) \cdots \chi_{[x_{n,m,p,q}}, y_{n,m,p,q})(t_n) dt_1 \cdots dt_n \right) =$$

$$\sum_{p,q} c_{p,q} \left(\sum_{m\in \mathrm{N}} \left(y_{1,m,p,q}^{j_1} - x_{1,m,p,q}^{j_1} \right) \cdots \left(y_{n,m,p,q}^{j_n} - x_{n,m,p,q}^{j_n} \right) \right),$$

$$D_{p,q} = \bigcup_{m\in\mathrm{N}} [x_{1,m,p,q}, y_{1,m,p,q}) \times \cdots \times [x_{n,m,p,q}, y_{n,m,p,q}) \supset$$

$$= \left\{ (t_1,...,t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1,...,t_n) < \frac{m_q + 1}{2^p} \right\}, dt(D_{p,q})$$

$$\approx dt\left(\left\{ (t_1,...,t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1,...,t_n) < \frac{m_q + 1}{2^p} \right\} \right).$$

The above computation contains terms involving the approximation of the polynomial $\widetilde{h}$ by simple functions [23]

$$\widetilde{h}(t_1,...,t_n) \approx \sum_{p,q\leq M} c_{p,q} \chi_{D_{p,q}}(t_1,...,t_n),$$

where p is large and m_q is suitable chosen for approximating $\widetilde{h}$. For the one-dimensional case and for a finite number of equations see [20], Remark 29. The conclusion is that we can determine (approximating) the “unknowns” $y_{k,m,p,q}, x_{k,m,p,q}, k = 1,...,n$ by means of the cell decomposition of the subsets $D_{p,q}$ associated to the known polynomial $\widetilde{h}$. Namely, the “unknowns” are the coordinates of the vertices of the cells from the cell – decomposition of the open subsets $D_{p,q}$. The basic relations can be summarized as

$$m_j = \int_K \varphi_j h \, dt_1 \cdots dt_n \approx \int_K \varphi_j \widetilde{h} dt_1 \cdots dt_n \approx$$

$$\sum_{p,q\leq M} c_{p,q} \left(\sum_{m\in\mathrm{N}} \left(y_{1,m,p,q}^{j_1} - x_{1,n,p,q}^{j_1} \right) \cdots \left(y_{n,m,p,q}^{j_n} - x_{n,m,p,q}^{j_n} \right) \right), \tag{2.3}$$

$j_k \geq 1, k = 1,..., n.$

where m_j are given, $c_{p,q}$ are known from Step 1, and the unknowns can be determined in terms of the cell decomposition of the intersection of sublevel and upper level sets of the known polynomial $\widetilde{h}$. The numbers $c_{p,q}$ are the values of $\widetilde{h}$ at some points in $\left\{(t_1,\ldots,t_n);\ \frac{m_q}{2^p} \le \widetilde{h}(t_1,\ldots,t_n) < \frac{m_q+1}{2^p}\right\}$. For a similar problem solved by using other methods see [12].

(ii) Let consider the functions

$$\varphi_j(t_1,\ldots,t_n) = \exp(-j_1t_1 - \cdots - j_nt_n),\ (t_1,\ldots,t_n) \in K = \prod_{k=1}^{n}[a_k, b_k],$$
$$0 < a_k < b_k,\ k = 1,\ldots,n,\ j = (j_1,\ldots,j_n) \in \mathrm{N}^n.$$

Then it is easy to see that the condition:

$$\sum_{j\in J_0}\lambda_j m_j \le \sum_{j\in J_0}\lambda_j \prod_{k=1}^{n}\frac{\exp(-j_k a_k) - \exp(-j_k b_k)}{j_k}$$

for any finite subset $\{\lambda_j\}_{j\in J_0} \subset R$, implies the existence of function

$$\exists h \in L^{\infty}_{dt}(K),\ 0 \le h \le 1,\ m_j = \int_K h\varphi_j\, dt_1 \cdots dt_n,\ \forall j \in \mathrm{N}^n.$$

following a similar proof to that of Theorem 2.3. Using the notations of the Step 2 (i), we have:

$$m_j \approx \int_K \widetilde{h}\varphi_j\, dt_1 \cdots dt_n =$$
$$\sum_{p,q\le M} c_{p,q}\left(\sum_{m\in \mathrm{N}}\prod_{k=1}^{n}\frac{\exp\left(-j_k x_{k,m,p,q}\right) - \exp\left(-j_k y_{k,m,p,q}\right)}{j_k}\right) =$$
$$\sum_{p,q\le M} c_{p,q}\left(\sum_{m\in \mathrm{N}}\prod_{k=1}^{n}\exp\left(-j_k y_{k,m,p,q}\right)\cdot\frac{\exp\left(j_k\left(y_{k,m,p,q} - x_{k,m,p,q}\right)\right) - 1}{j_k}\right) \approx$$
$$\sum_{p,q\le M} c_{p,q}\left(\sum_{m\in \mathrm{N}}\prod_{k=1}^{n}\exp\left(-j_k y_{k,m,p,q}\right)\cdot\left(y_{k,m,p,q} - x_{k,m,p,q}\right)\right).$$

The “unknowns” are the coordinates of the vertices of the cells from the cell – decomposition of the open subsets $D_{p,q}$ mentioned at point (i), Step 2. Notice that we can assume that $\sup\{y_{k,m,p,q} - x_{k,m,p,q}\} < \delta$ for arbitrary $\delta > 0$ (the solution is not unique: a big cell can be written as a joint of small cells). This remark justify the approximations from above, so that the solution can be approximated by means of the cell - decomposition of some known open sets depending on $\widetilde{h}$.

3. A moment problem not involving polynomials

We consider an application of Theorem 2.2 to an arbitrary $X = L^1_\nu(M)$ space, where M is a measure space. Let Y be an order complete Banach lattice with solid norm: $|y_1| \le |y_2| \Rightarrow \|y_1\| \le \|y_2\|$. Let $F_2 \in B(X,Y)$ be a linear positive bounded operator from X into Y, and let $\{\varphi_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ be given arbitrary families.

Theorem 3.1. *Consider the following statements:*

(a) *there exists* $F \in B(X,Y)$ *such that*

$$F(\varphi_j) = y_j, \quad j \in J, \quad 0 \le F(\psi) \le F_2(\psi), \quad \psi \in X_+, \|F\| \le \|F_2\|;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j;\, j \in J_0\} \subset R$, *we have*

$$\sum_{j\in J_0} \lambda_j y_j \le \sum_{j\in J_0} \lambda_j F_2(\varphi_j).$$

Then $(b) \Rightarrow (a)$.

Proof. We verify the implication from (b), Theorem 2.2:

$$\sum_{j\in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1, \psi_j \in X_+,\ j = 1,2 \Rightarrow$$

$$\sum_{j\in J_0} \lambda_j y_j \le F_2\left(\sum_{j\in J_0} \lambda_j \varphi_j\right) \le F_2(\psi_2) = F_2(\psi_2) - F_1(\psi_1), F_1 := 0.$$

The implication $(b) \Rightarrow (a)$ of Theorem 2.2 leads to the existence of a linear operator F from X into Y such that

$$0 \le F(\psi) \le F_2(\psi), \forall \psi \in X_+, F(\varphi_j) = y_j, \quad j \in J.$$

If $\varphi \in X$ is arbitrary, we have

$$|F(\varphi)| \le F(\varphi^+) + F(\varphi^-) \le F_2(|\varphi|).$$

Since the norm on Y is solid, we derive that

$$\|F(\varphi)\| \le \|F_2(|\varphi|)\| \le \|F_2\| \cdot \|\varphi\|_1, \forall \varphi \in X.$$

Consequently, one obtains $\|F\| \le \|F_2\|$. This concludes the proof. □

Here is the "scalar" version of the above theorem.

Corollary 3.1. *Let* $Y = R, \nu$ *a* $\sigma -$ *finite measure on* M. *Consider the following statements:*

(a) *there exists* $h \in L_{\nu}^{\infty}(M), 0 \le h(t) \le 1$ *a.e. in* M, *such that*

$$\int_M \varphi_j \cdot h d\nu = y_j,\ j \in J;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j;\ j \in J_0\} \subset R$, *we have*

$$\sum_{j \in J_0} \lambda_j y_j \le \sum_{j \in J_0} \lambda_j \int_M \varphi_j d\nu$$

Then $(b) \Rightarrow (a)$.

Remark 3.1. In the above statements of this Section, M can be a closed unbounded non semi-algebraic subset of $R^n, n \ge 2$.

4. Conclusions

In section 1 one recalls some results on polynomial approximation on unbounded subsets. On the other hand, one formulates the aims and the content of this work, as well as some related results in the literature. The main new results of section 2 concern solving (approximately) systems of nonlinear systems of infinitely many equations with infinitely many unknowns, as inverse problems solved starting from the moments (see also [12]). A related generalization of a result of M. G. Krein is also stated (Theorem 2.2). Finally, in section 3 an application of Theorem 2.2 to a sufficient condition for the existence of the solution of a moment problem on the space of absolutely integrable functions is discussed. The scalar version is stated as well.

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