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STOCHASTIC CONNECTIVITY ON A ALMOST-RIEMANNIAN STRUCTURE INDUCED BY SYMMETRIC POLYNOMIALS

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In this paper we introduce an almost-Riemannian structure on the real plane, generated by the exact differential $1-$forms $\omega_1 = dx + dy$, $\omega_2 = ydx + xdy$, and prove its stochastic connectivity property. More specifically, we show that it is possible to steer any admissible stochastic process, starting at a point P, to an arbitrarily small disk centered at a point Q, almost surely.

Keywords: Stochastic connectivity, almost-Riemannian structure, Wiener process.

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1. Introduction

Given a smooth manifold M, a sub-Riemannian structure on M is specified by a given distribution $\mathcal{D}$, i.e., a sub-bundle $\mathcal{D} \subseteq TM$, together with a metric g defined on $\mathcal{D} \times \mathcal{D}$. Most often, the distribution is specified by a family of vector fields and is non-integrable. For an introduction into the subject, we refer the reader to [3, 4, 12].

Any given distribution splits the tangent space, *locally*, into "forbidden" and, respectively, "allowed" directions (*horizontal vectors*). The natural curves on sub-Riemannian manifolds are called *horizontal (admissible) curves*, which are tangent to horizontal vectors. Thus, a classical problem, in the context of sub-Riemannian geometry, is to join two arbitrary points by admissible curves. A sufficient condition for this to be possible is that the vector fields, together with their iterated Lie brackets span the entire tangent space at each point $p \in M$ (known as bracket generating condition, Hörmander's condition [11], etc.). This fact is established by the famous Chow-Rashevskii Theorem [10, 14]. As a result, the so-called Carnot-Carathéodory distance can be defined, thus inducing a metric space structure on M.

Very often, especially in the context of Control Theory, one encounters rank-varying sub-Riemannian structures, called in the literature *almost-Riemannian* structures [1], as well as *Grushin manifolds* [4]. For examples the reader might consult [3, 4, 6, 7, 8, 9, 15, 16], An

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almost-Riemannian structure which is defined globally by a family of vector fields, is called *trivializable* (by analogy with parallelizable manifolds).

In this paper we are interested in the stochastic analogue of connectivity problems on almost-Riemannian manifolds. In other words, we look for stochastic analogues of the Chow-Rashevskii Theorem. To our knowledge, this problem has been raised, for an arbitrary sub-Riemannian manifold, in [6, 7]. The problem has been solved for the Grushin plane and its generalizations in [6, 7, 15, 16].

It is worth mentioning that, when passing to a stochastic setting, some important adjustments need to be made. Firstly, the admissible curves are replaced by *admissible stochastic processes* (defined below). Secondly, we have to replace the deterministic boundary conditions as well, since the probability of an admissible stochastic process, starting at a state P, to reach a fixed state Q, is zero. Thus, we look for some controls which steer the stochastic process into an arbitrarily small neighborhood of the state Q, in finite time, with probability arbitrarily close to one. It is appropriate to mention here that, if the bracket generating conditions is satisfied, then the base manifold topology and the induced metric space topology are equivalent (see for instance Corollary 2.6 in [3]). This allows us, in certain contexts, to chose the most convenient topology.

2. **An almost-Riemannian structure induced by elementary symmetric polynomials**

Consider the exact differential 1−forms $\omega_1 = dx + dy,\ \omega_2 = ydx + xdy$, obtained by taking the differential of the elementary symmetric polynomials, defined on the real plane $\mathbb{R}^2$. Clearly, the given 1−forms are functionally dependent on the set $\{x = y\}$ and functionally independent elsewhere. The kernels of the 1−forms ω_1 and ω_2 are generated by the vector fields

$$X_1 = \partial_x - \partial_y \text{ and } X_2 = x\partial_x - y\partial_y,$$

respectively, which span the distribution $\mathcal{D} = span\{X_1, X_2\}$, inducing an almost-Riemannian structure on $\mathbb{R}^2$. The *singular locus*, i.e., the set of points at which the vector fields X_1 and X_2 lose their linear independence, is the set $S = \{(x, y) \in \mathbb{R}^2 |\ x = y\}$. The sub-Riemannian metric g, which turns the pair $\{X_1, X_2\}$ into an orthonormal basis at each point $p \in \mathbb{R}^2 \backslash S$, is given by

$$g = (g_{ij}) = \frac{1}{(x-y)^2} \begin{pmatrix} 1+y^2 & 1+xy \\ 1+xy & 1+x^2 \end{pmatrix}, \tag{1}$$

and clearly is singular on S. It is easily verified that the bracket generating condition is satisfied:

$$[X_1, X_2] = \partial_x + \partial_y,$$

which has a twofold implication. Namely, *a*) the resulting manifold has the global connectivity property by admissible curves (Chow-Rashevskii Theorem [10, 14]), i.e., the Carnot-Carathéodory distance d_C between any two points is finite; *b*) the topology induced by the metric d_C is equivalent to the Euclidean topology (Corollary 2.6 [3]).

3. The stochastic connectivity property

In the present section we shall state and prove the main result of this paper, but first one needs to define the stochastic admissible processes corresponding to the distribution $\mathcal{D}$.

Any admissible curve $c : [0, T] \to \mathbb{R}^2$, between two fixed points P and Q is described by the boundary value problem

$$\begin{cases} \dot{c}\,(t) = u_1(t)X_1(c(t)) + u_2X_2\,(c(t)) \\ c(0) = P,\ c(T) = Q, \end{cases} \tag{2}$$

for some control functions $u_1, u_2 \in L^1\left([0, T], \mathbb{R}\right)$, which rewrites as

$$\begin{cases} dx(t) = \left[u_1(t) + u_2(t)x(t)\right] dt \\ dy(t) = -\left[u_1(t) + u_2(t)y(t)\right] dt \\ (x(0), y(0)) = (x_P, y_P)\,,\ (x(T), y(T)) = (x_Q, y_Q)\,. \end{cases} \tag{3}$$

Using a pair of independent Wiener processes $\left(W_t^1, W_t^2\right)$, together with a pair of nonnegative constants (σ_1, σ_2), the ODE system (3) is stochastically perturbed, yielding the SDE system

$$\begin{cases} dx(t) = \left[u_1(t) + u_2(t)x(t)\right] dt + \sigma_1 dW_t^1 \\ dy(t) = -\left[u_1(t) + u_2(t)y(t)\right] dt + \sigma_2 dW_t^2. \end{cases} \tag{4}$$

Here the controls $u_i(s) = u_i(s, \omega),\ i = 1, 2$ are stochastic processes measurable with respect to the $\sigma-$algebra generated by $\{W_{s\wedge t}, t \geq 0\}$, taking values in a Borel set at any instant. The controls which do not depend on ω are called *deterministic* or *open loop controls*. Controls of the form $u(s, \omega) = u_0\left(t, c_t\left(\omega\right)\right)$, for some function u_0, are called *Markov controls*. Denote the set of deterministic controls by $\mathcal{U}_1$ and, respectively, the set of Markov controls by $\mathcal{U}_2$.

Definition 3.1. *A stochastic process $c_t = (x(t), y(t))$ solving the SDE system (4) is called admissible stochastic process.*

Theorem 3.1. *Let $P(x_P, y_P)$ and $Q(x_Q, y_Q)$ be two given points on $\mathbb{R}^2$ and denote by $D_C(Q, r)$ the Carnot-Carathéodory disk of radius r centered at Q. Then, for any $\varepsilon \in (0, 1)$ and $r > 0$, there exist an admissible stochastic process $c_t = (x(t), y(t))$, and a hitting time $T < \infty$, such that*

$$\mathbb{P}\left[c_T \in D_C(Q, r)\right] \geq 1 - \varepsilon, \tag{5}$$

and $x(0) = x_P,\ y(0) = y_P,\ \mathbb{E}\left[y(T)\right] = y_Q,\ \mathbb{E}\left[x(T)\right] = x_Q$.

Proof. According to the above made remarks, it is enough to prove the assertion of the theorem for Euclidean disks.

Noticing that the square of the Euclidean distance $d^2(c_t, Q)$ is a nonnegative random variable, according to Markov's inequality, one has

$$\mathbb{P}\left(d^2(c_T, Q) \geq r^2\right) \leq \frac{1}{r^2}\mathbb{E}\left[d^2(c_T, Q)\right],$$

which is the same as

$$\mathbb{P}\left(d^2(c_T, Q) \leq r^2\right) \geq 1 - \frac{1}{r^2}\mathbb{E}\left[d^2(c_T, Q)\right]. \tag{6}$$

Case I: Suppose at first that the controls are *deterministic*. Each equation of the SDE system (4) can be solved independently with respect to the unspecified deterministic controls u_1 and u_2 respectively. Four our purposes, since we do not have any optimality criteria, it is sufficient to take the controls to be some nonnegative constant functions determined below. First, we prove the existence of a finite time T satisfying the conditions of the problem.

Solving the SDE system (4), we obtain the solution

$$\begin{aligned} x(t) &= e^{u_1 t}x_P - \frac{u_2}{u_1}\left(1 - e^{u_1 t}\right) + \sigma_1\int_0^t e^{u_1(t-s)}dW_s^1 \\ y(t) &= e^{-u_1 t}y_P - \frac{u_2}{u_1}\left(1 - e^{-u_1 t}\right) + \sigma_2\int_0^t e^{-u_1(t-s)}dW_s^2 \end{aligned} \tag{7}$$

(for more details see, for instance, [2], Example 2.4.1). Imposing the boundary conditions $\mathbb{E}[x(T)] = x_Q$, $\mathbb{E}[y(T)] = y_Q$, yields

$$e^{u_1 T}x_P - \frac{u_2}{u_1}\left(1 - e^{u_1 T}\right) = x_Q,\ e^{-u_1 T}y_P - \frac{u_2}{u_1}\left(1 - e^{-u_1 T}\right) = y_Q. \tag{8}$$

Using the last equalities, together with the properties of stochastic integral, we compute

$$\begin{aligned} \mathbb{E}\left[d^2(c_T, Q)\right] &= \mathbb{E}\left[(x(T) - x_Q)^2\right] + \mathbb{E}\left[(y(T) - y_Q)^2\right] \\ &+ \mathbb{E}\left[\left(\sigma_1\int_0^T e^{u_1(T-s)}dW_s^1\right)^2\right] + \mathbb{E}\left[\left(\sigma_2\int_0^T e^{-u_1(T-s)}dW_s^2\right)^2\right] \\ &= \sigma_1^2\int_0^T e^{2u_1(T-s)}ds + \sigma_2^2\int_0^T e^{-2u_1(T-s)}ds \\ &= \frac{1}{2u_1}\left[\sigma_1^2\left(e^{2u_1 T} - 1\right) + \sigma_2^2\left(1 - e^{-2u_1 T}\right)\right]. \end{aligned}$$

Take $u_1 = \dfrac{1}{2}$ and consider the function $\varphi(T) = \sigma_1^2\left(e^T - 1\right) + \sigma_2^2\left(1 - e^{-T}\right)$ which is unbounded and $\varphi(0) = 0$. Noticing that the derivative $\varphi'(T) = \sigma_1^2 e^T + \sigma_2^2 e^{-T}$ is strictly positive, it follows that the function $\varphi(T)$ is strictly increasing. Hence the equation $\varphi(T) = \varepsilon r^2$ has a unique solution $T < \infty$, i.e., $\mathbb{E}\left[d^2(c_T, Q)\right] = \varepsilon r^2$, which means that

$$\mathbb{P}\left(d^2(c_T, Q) \leq r^2\right) \geq 1 - \varepsilon.$$

The control u_2 is determined from the boundary conditions (8):

$$u_2 = \frac{e^{\frac{1}{2}T} x_P - x_Q}{2\left(1 - e^{\frac{1}{2}T}\right)}.$$

Case II: For the case of Markov controls, it is enough to take $u_1(t, c_t) = \dfrac{a+b}{x-y}$ and $u_2(t, c_t) = \dfrac{ay+bx}{y-x}$, for a and b some constants determined below. In this case the SDE (4) system becomes

$$\begin{cases} dx(t) = adt + \sigma_1 dW_t^1 \\ dy(t) = bdt + \sigma_2 dW_t^2, \end{cases}$$

with the solutions $x(t) = x_P + at + \sigma_1 W_t^1$ and $y(t) = x_P + bt + \sigma_2 W_t^2$. From the boundary conditions it follows that $x_P + aT = x_Q$ and $y_P + bT = y_Q$. Thus

$$\mathbb{E}\left[d^2\left(c_T, Q\right)\right] = \sigma_1^2 T + \sigma_2^2 T$$

and for

$$T = \frac{\varepsilon r^2}{\sigma_2^1 + \sigma_2^2},$$

$\mathbb{E}\left[d^2\left(c_T, Q\right)\right] = \varepsilon r^2$. The constants a and b respectively are determined by the boundary conditions:

$$a = \frac{\left(x_Q - x_P\right)\left(\sigma_2^1 + \sigma_2^2\right)}{\varepsilon r^2},\ b = \frac{\left(y_Q - y_P\right)\left(\sigma_2^1 + \sigma_2^2\right)}{\varepsilon r^2}.$$

□

REFERENCES

[1] *A. A. Agrachev, U. Boscain, M. Sigalotti,* A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds, Discrete Contin. Dyn. Syst., 20,4 (2008), 801-822.

[2] *A. G. Ladde, G. S. Ladde*, An Introduction to Differential Equations II: Stochastic Modeling, Methods and Analysis, World Scientific, Singapore, 2013.

[3] *A. Bellaïche, J. J. Risler* (eds.), Sub-Riemannian Geometry, Progress in Mathematics 144, Birkhäuser, Basel, 1996.

[4] *O. Călin, D. C. Chang*, Sub-Riemannian Geometry: General Theory and Exemples, EMIA 126, Cambridge University Press, Cambridge, 2009.

[5] *O. Călin, C. Udrişte*, Geometric Modeling in Probability and Statistics, Springer, 2014.

[6] *O. Călin, C. Udrişte, I. Ţevy*, A stochastic variant of Chow-Rashevski Theorem on the Grushin distribution, Balkan J. Geom. Appl., 19, 1 (2014), 1-12.

[7] *O. Călin, C. Udrişte, I. Ţevy*, Stochastic Sub-Riemannian geodesics on Grushin distribution, Balkan J. Geom. Appl., 19, 2 (2014), 37-49.

[8] *C. H. Chang, D. C. Chang, B. Gaveau, P. Greiner, H. P. Lee*, Geometric analysis on a step 2 Grushin operator, Bull. Inst. Math. Acad. Sinica (New Series) 4, 2 (2009), 119-188.

[9] *D. C. Chang, Y. Li*, SubRiemannian geodesics in the Grushin plane, J. Geom. Anal. 22 (2012), 800-826.

[10] *W. L. Chow*, Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, Math. Ann. 177 (1939), 98-105.

[11] *L. Hörmander*, Hypoelliptic second order differential operators, Acta. Math. 119 (1967), 147-171.

[12] *R. Montgomery*, A Tour of Subriemannian Geometries, Their Geodesics and Applications, Math Surveys and Monographs 91, AMS, Providence, RI, 2002.

[13] *B. Øksendal*, Stochastic Differential Equations, 6-th ed., Springer, 2003.

[14] *P. K. Rashevskii*, About connecting two points of complete nonholonomic space by admissible curve (in Russian), Uch. Zapiski Ped. Instit. K. Liebknexta, 2 (1938), 83-94.

[15] *T. Ţurcanu, C. Udrişte*, Stochastic perturbation and connectivity based on Grushin distribution, U. Politeh. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 79, 1 (2017), 3-10.

[16] *T. Ţurcanu, C. Udrişte*, Stochastic accessibility on Grushin-type manifolds, Statist. Probab. Lett., 125 (2017), 196-201.