SEVERAL SELF-ADAPTIVE ALGORITHMS FOR SOLVING SPLIT COMMON FIXED POINT PROBLEMS WITH MULTIPLE OUTPUT SETS

Wenlong Sun[1], Yuanfeng Jin[2], Tzu-Chien Yin[3]

In this article, we study split common fixed point problems with multiple output sets in real Hilbert spaces. In order to solve this problem, we present three new self-adaptive algorithms. We establish weak and strong convergence theorems for them. Using our methods, we can remove the assumptions imposed on the norms of the transfer operators.

Keywords: split common fixed point, multiple output sets, self-adaptive algorithm, demicontractive operator.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint A^*. Let $T : H_1 \longrightarrow H_1$ and $S : H_2 \longrightarrow H_2$ be two nonlinear operators. We denote by $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$ the sets of fixed points of T and S, respectively.

First, let us recall the split common fixed point problem:

$$\text{find } u \text{ such that } u \in \mathrm{Fix}(T) \text{ and } Az \in \mathrm{Fix}(S), \tag{1}$$

which can be regarded as a generalization of the following split feasibility problem:

$$\text{find } u \text{ such that } u \in C \text{ and } \mathcal{A}u \in Q. \tag{2}$$

Problem (2) introduced by Censor and Elfving [4] in order to model certain inverse problems plays an important role in medical image reconstruction and signal processing (see [2, 3, 6, 8, 12, 14, 16]). In [5], Censor and Segal introduced Problem (1) which can be regarded as a generalization of Problem (2). Since then, several iterative algorithms for solving split problems have been studied extensively (see [7, 9, 10, 13, 15, 17–25]).

Very recently, in 2022, Reich et al. [11] presented and studied the split common fixed point problem with multiple output sets in Hilbert spaces:

$$\text{find } u^\dagger \text{ such that } u^\dagger \in \bigcap_{i=1}^{N} \mathrm{Fix}\, T_i \text{ and } A_k u^\dagger \in \bigcap_{j=1}^{L_k} \mathrm{Fix}\, S_j^k,\ k = 1, 2, \ldots, M. \tag{3}$$

[1]Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110870, China; Department of Mathematics, Yanbian University, Yanji 133001, China; Liaoning Provincial Key Laboratory of Composite Metal Nanomaterials and Magnetic Technology, Shenyang University of Technology, Shenyang 110870, China, email: `sun_math@sut.edu.cn`

[2]Corresponding author. Department of Mathematics, Yanbian University, Yanji 133001, China, email: `yfkim@ybu.edu.cn`

[3]Corresponding author. Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, e-mail: `yintzuchien@hotmail.com`

After that, in 2023, Sun et al. [13] studied the Problem (3) and presented their algorithms in which they handled more general quasinonexpansive operators.

Inspired by these works in the literature, the main purpose of this paper is to extend Sun's results from the quasinonexpansive operators to the demicontractive operators. Subsequently, we construct three self-adaptive algorithms for solving the split common fixed point problem with multiple output sets (3). Weak and strong convergence theorems are given under some mild assumptions.

2. **Preliminaries**

In this section, we collect some definitions and lemmas which will be used to derive our main results in the next section.

Definition 2.1. *An operator $T : C \longrightarrow C$ is said to be*

(i) *nonexpansive if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in C$.*

(ii) *quasinonexpansive if $\|Tu - u^*\| \leq \|u - u^*\|$ for all $u \in C$ and $u^* \in \mathrm{Fix}(T)$, or equivalently,*

$$\langle u - Tu, u - u^* \rangle \geq \frac{1}{2}\|u - Tu\|^2$$

for all $u \in C$ and $u^* \in \mathrm{Fix}(T)$.

(iii) *ϱ-demicontractive if there exists a constant $\varrho \in [0, 1)$ such that*

$$\|Tu - u^*\|^2 \leq \|u - u^*\|^2 + \varrho\|Tu - u\|^2,$$

or equivalently,

$$\langle u - Tu, u - u^* \rangle \geq \frac{1 - \varrho}{2}\|u - Tu\|^2, \tag{4}$$

for all $u \in C$ and $u^* \in \mathrm{Fix}(T)$.

Definition 2.2. *An operator T is said to be demiclosed at v if, for any sequence $\{u_n\}$ which weakly converges to u, and if $Tu_n \to v$, then $Tu = v$.*

Definition 2.3. *A sequence $\{u_n\}$ is called* Fejér*-monotone with respect to a given nonempty set Ω if for every $u \in \Omega$, the inequality $\|u_{n+1} - u\| \leq \|u_n - u\|$ holds for all $n \geq 0$.*

In this paper, we denote by Proj_C the projection from H onto C, and by $\omega_w(u_n)$ the set of cluster points in the weak topology, that is, $\omega_w(u_n) = \{u : \exists u_{n_j} \rightharpoonup u\}$.

Lemma 2.1 ([19])**.** *Assume that $\{\varpi_n\}$ is a sequence of nonnegative real numbers such that*

$$\varpi_{n+1} \leq (1 - \alpha_n)\varpi_n + \alpha_n \delta_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\limsup_{n\to\infty} \delta_n \geq 0$.

Lemma 2.2 ([1])**.** *Let C be a nonempty closed convex subset in H. If the sequence $\{u_n\}$ is* Fejér*-monotone with respect to Ω, then we have the following conclusions:*

(i) *$u_n \rightharpoonup u \in \Omega$ iff $\omega_w(u_n) \subset \Omega$;*

(ii) *the sequence $\{\mathrm{Proj}_\Omega u_n\}$ converges strongly;*

(iii) *if $u_n \rightharpoonup u \in \Omega$, then $u = \lim_{n\to\infty} \mathrm{Proj}_\Omega u_n$.*

Lemma 2.3 ([20])**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \in N,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) *$\sum_{n=1}^{\infty} \gamma_n = \infty$;*

(ii) *$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.*

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

Let $H,\ H_k,\ k=1,2,\dots,M$, be real Hilbert spaces. Let $T_i : H \longrightarrow H,\ i=1,\dots,N$, $S_j^k : H_k \longrightarrow H_k,\ k=1,2,\dots,M,\ j=1,2,\dots,L_k$, be ϱ-demicontractive operators. Let $A_k : H \longrightarrow H_k,\ k=1,\dots,M$, be bounded linear operators with adjoints A_k^*. Let $I-T_i,\ i=1,\dots,N,\ I-S_j^k,\ k=1,2,\dots,M,\ j=1,2,\dots,L_k$, be demiclosed at zero. We denote by $\Omega := \{u \in \bigcap_{i=1}^N \text{Fix } T_i$ and $A_k u \in \bigcap_{j=1}^{L_k} \text{Fix } S_j^k,\ k=1,2,\dots,M$ the solution set of the problem (3).

Next, we propose several iterative algorithms for solving the problem (3).

Algorithm 3.1. *Let $u_1 \in H$ and assume the current iterate $\{u_n\}$ is known.*

1: *Compute* $\phi_n = \max\{\|u_n - T_i u_n\|,\ i=1,2,\dots,N\}$,
2: $\Phi_n = \{i \in \{1,2,\dots,N\} :\ \|x_n - T_i u_n\| = \phi_n\}$,
3: $\psi_n = \max\{\|A_k u_n - S_j^k A_k u_n\| : k=1,2,\dots,M,\ j=1,2,\dots,L_k\}$,
4: *and* $\Psi_n = \{(k,j) \in \{1,2,\dots,M\} \times \{1,2,\dots,L_k\} :\ \|A_k u_n - S_{jn}^k A_k u_n\| = \psi_n\}$.
5: *Compute* $\Gamma_n = \max\{\phi_n, \psi_n\}$. *If* $\Gamma_n = 0$, *then stop else if* $\phi_n = \Gamma_n$, *choose* $i_n \in \Phi_n$,
6: *and compute* $u_{n+1} = u_n - \theta_n(u_n - T_{i_n} u_n)$ *else if* $\psi_n = \Gamma_n$, *choose* $(k_n, j_n) \in \Psi_n$,
7: *set* $\tau_n = \frac{\theta_n \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2}{\|A_{k_n}^*(A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n)\|^2}$, $\theta_n \in [c,d] \subset (0, 1-\varrho)$,
8: *and compute* $u_{n+1} = u_n - \tau_n A_{k_n}^*(A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n)$.
9: *Set* $n := n+1$ *and go back to 1.*

Remark 3.1. *Obviously, when $\Gamma_n = 0$, x_n is the solution of the problem* (3).

Theorem 3.1. *If Algorithm 3.1 does not stop in a finite number of iterations, and $\Omega \neq \emptyset$, then the sequence $\{u_n\}$ generated by Algorithm 1 3.1 converges weakly to a solution $z^\dagger (= \lim_{n\to\infty} \text{Proj}_\Omega u_n)$ of the problem (3).*

Proof. First, let $z^* \in \Omega$. If $\phi_n = \Gamma_n$, then owing to (4) and the ϱ-demicontractivity of T, we obtain

$$\begin{aligned} \|u_{n+1} - z^*\|^2 &= \|u_n - z^*\|^2 + \theta_n^2 \|u_n - T_{i_n} u_n\|^2 - 2\theta_n \langle u_n - z^*, u_n - T_{i_n} u_n\rangle \\ &\le \|u_n - z^*\|^2 + \theta_n^2 \|u_n - T_{i_n} u_n\|^2 - \theta_n(1-\varrho)\|u_n - T_{i_n} u_n\|^2 \\ &= \|u_n - z^*\|^2 - \theta_n(1-\varrho-\theta_n)\|(u_n - T_{i_n} u_n)\|^2 \end{aligned} \tag{5}$$

and therefore, according to Algorithm 3.1, we get

$$\|u_{n+1} - z^*\|^2 \le \|u_n - z^*\|^2 - \theta_n(1-\varrho-\theta_n)\phi_n^2 = \|u_n - z^*\|^2 - \theta_n(1-\varrho-\theta_n)\Gamma_n^2.$$

Then, we have

$$\Gamma_n^2 \le \frac{1}{\theta_n(1-\varrho-\theta_n)}(\|u_n - z^*\|^2 - \|u_{n+1} - z^*\|^2). \tag{6}$$

Otherwise, if $\psi_n = \Gamma_n$, then we choose $(k_n, j_n) \in \Psi_n$. Similar to (5), we deduce

$$\begin{aligned} \|u_{n+1} - z^*\|^2 &= \|u_n - z^*\|^2 + \tau_n^2 \|A_{k_n}^*(A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n)\|^2 \\ &\quad - 2\tau_n \langle A_{k_n} u_n - A_{k_n} z^*, A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\rangle \\ &\le \|u_n - z^*\|^2 + \tau_n^2 \|A_{k_n}^*(A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n)\|^2 \\ &\quad - \tau_n(1-\varrho)\|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2 \\ &= \|u_n - z^*\|^2 - \theta_n(1-\varrho-\theta_n)\frac{\|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^4}{\|A_{k_n}^*(A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n)\|^2}. \end{aligned} \tag{7}$$

It follows from (5) and (7) that the sequence $\{u_n\}$ is Fejér-monotone with respect to Ω and hence it is bounded. Let $L := \sup_{n,k}\{\|A_k^*(A_k u_n - S_j^k A_k u_n)\|\}$, and we have $L < +\infty$. We see from (7) that

$$\begin{aligned}
\|u_{n+1} - z^*\| &\leq \|u_n - z^*\|^2 - \theta_n(1-\varrho-\theta_n)\frac{\|A_{k_n}u_n - S_{j_n}^{k_n}A_{k_n}u_n\|^4}{\|A_{k_n}^*(A_{k_n}u_n - S_{j_n}^{k_n}A_{k_n}u_n)\|^2} \\
&\leq \|u_n - z^*\|^2 - \frac{\theta_n(1-\varrho-\theta_n)}{L^2}\|A_{k_n}u_n - S_{j_n}^{k_n}A_{k_n}u_n\|^4 \\
&= \|u_n - z^*\|^2 - \frac{\theta_n(1-\varrho-\theta_n)}{L^2}\psi_n^4 \\
&= \|u_n - z^*\|^2 - \frac{\theta_n(1-\varrho-\theta_n)}{L^2}\Gamma_n^4.
\end{aligned}$$

Consequently,

$$\Gamma_n^4 \leq \frac{L^2}{\theta_n(1-\varrho-\theta_n)}(\|u_n - z^*\|^2 - \|u_{n+1} - z^*\|^2). \tag{8}$$

By the Fejér-monotonicity of the sequence $\{u_n\}$, we get

$$\|u_n - z^*\|^2 - \|u_{n+1} - z^*\|^2 \to 0, \text{ as } n \to \infty$$

and thereby, $\Gamma_n \to 0$, due to (6) and (8). By the definition of Γ_n, we get that $\lim_{n\to\infty}\|u_n - T_iu_n\| = 0,\ i = 1, 2, \ldots, N$, $\lim_{n\to\infty}\|A_ku_n - S_j^kA_ku_n\| = 0,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k$.

Thanks to the hypothesis of the demiclosedness, we have $\omega_w(u_n) \subset \Omega$. In the end, applying Lemma 2.2, we obtain that $u_n \rightharpoonup z^\dagger = \lim_{n\to\infty}\mathrm{Proj}_\Omega u_n$. The proof is completed. □

Algorithm 3.2. *Let $u_1 \in H$ and the current iterate u_n be known.*

1: *Compute* $y_{jn}^k = A_k^*(A_ku_n - S_j^kA_ku_n)$,
2: $\psi_n = \max\{\|u_n - T_iu_n + y_{jn}^k\| : \ i = 1, 2, \ldots, N, k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k\}$,
3: $\Psi_n = \{(i, k, j) \in \{1, 2, \ldots, N\} \times \{1, 2, \ldots, M\} \times \{1, 2, \ldots, L_k\} : \|u_n - T_iu_n + y_{jn}^k\| = \psi_n\}$.
4: *If $\psi_n = 0$, then stop; else choose $(i_n, k_n, j_n) \in \Psi_n$,*
5: *and let* $\tau_n = \theta_n\frac{\|u_n - T_{i_n}u_n\|^2 + \|A_{k_n}u_n - S_{j_n}^{k_n}A_{k_n}u_n\|^2}{\|u_n - T_{i_n}u_n + y_{j_nn}^{k_n}\|^2}$, $\theta_n \in [c, d] \subset (0, 1-\varrho)$.
6: *Compute* $u_{n+1} = u_n - \tau_n(u_n - T_{i_n}u_n + y_{j_nn}^{k_n})$.
7: *Set $n := n + 1$ and go back to 1.*

Remark 3.2. *In Algorithm 3.2, the equality $\psi_n = 0$ holds if and only if $u_n \in \Omega$.*

It is obvious that if $u_n \in \Omega$, then $\psi_n = 0$ holds. In the sequel, we show that $u_n \in \Omega$ if $\psi_n = 0$. Owing to the ϱ-demicontractivity of $T_i : H \longrightarrow H,\ i = 1, \ldots, N$, $S_j^k : H_k \longrightarrow H_k,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k$, for any $z^* \in \Omega$, we obtain

$$\begin{aligned}
0 &= \langle u_n - T_iu_n + y_{jn}^k, u_n - z^*\rangle \\
&= \langle u_n - T_iu_n + A_k^*(A_ku_n - S_j^kA_ku_n), u_n - z^*\rangle \\
&= \langle u_n - T_iu_n, u_n - z^*\rangle + \langle A_k^*(A_ku_n - S_j^kA_ku_n), u_n - z^*\rangle \\
&= \langle u_n - T_iu_n, u_n - z^*\rangle + \langle A_ku_n - S_j^kA_ku_n, A_ku_n - A_kz^*\rangle \\
&\geq \frac{1-\varrho}{2}(\|u_n - T_iu_n\|^2 + \|A_ku_n - S_j^kA_ku_n\|^2)
\end{aligned} \tag{9}$$

for all $T_i : H \longrightarrow H,\ i = 1, \ldots, N$, $S_j^k : H_k \longrightarrow H_k,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k$, which implies that

$$u_n \in \bigcap_{i=1}^N Fix\ T_i$$

and

$$A_k u_n \in \bigcap_{j=1}^{L_k} Fix\ S_j^k,\ \ k = 1, 2, \ldots, M.$$

So, $u_n \in \Omega$.

Theorem 3.2. *Assume that Algorithm 3.2 does not stop in a finite number of iterations, and* $\Omega \neq \emptyset$. *Then the sequence* $\{u_n\}$ *generated by Algorithm 3.2 converges weakly to a solution* $z^\dagger (= \lim_{n\to\infty} \mathrm{Proj}_\Omega u_n)$ *of the problem (3).*

Proof. Let $z^* \in \Omega$. In the light of (9), we derive

$$\begin{aligned}
\|u_{n+1} - z^*\|^2 &= \|u_n - \tau_n(u_n - T_{i_n} u_n + y_{j_n n}^{k_n}) - z^*\|^2 \\
&= \|u_n - z^*\|^2 + \tau_n^2 \|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\|^2 \\
&\quad - 2\tau_n \langle u_n - z^*, u_n - T_{i_n} u_n + y_{j_n n}^{k_n} \rangle \\
&= \|u_n - z^*\|^2 - 2\tau_n \langle A_{k_n}^* u_n - A_{k_n}^* z^*, A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n \rangle \\
&\quad + \tau_n^2 \|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\|^2 - 2\tau_n \langle u_n - z^*, u_n - T_{i_n} u_n \rangle \\
&\leq \|u_n - z^*\|^2 + \tau_n^2 \|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\|^2 \\
&\quad - \tau_n (1 - \varrho)(\|u_n - T_{i_n} u_n\|^2 + \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2) \\
&= \|u_n - p\|^2 - \theta_n (1 - \varrho - \theta_n) \frac{(\|u_n - T_{i_n} u_n\|^2 + \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2)^2}{\|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\|^2}.
\end{aligned} \tag{10}$$

It follows from (10) that the sequence $\{u_n\}$ is Fejér-monotone with respect to Ω. We also see from (10) that

$$\begin{aligned}
&\|u_n - T_{i_n} u_n\|^2 + \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2 \\
&\quad \leq \frac{\|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\|}{\sqrt{\theta_n (1 - \varrho - \theta_n)}} \sqrt{\|u_n - z^*\|^2 - \|u_{n+1} - z^*\|^2}.
\end{aligned} \tag{11}$$

Setting

$$\Gamma := \sup \Big\{ \frac{\|u_n - T_i u_n + y_{jn}^k\|}{\sqrt{\theta_n (1 - \varrho - \theta_n)}} :\ i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k,\ n \in \mathbb{N}^+ \Big\},$$

by (11), we have

$$\|u_n - T_{i_n} u_n\|^2 + \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2 \leq \Gamma \sqrt{\|u_n - z^*\|^2 - \|x_{n+1} - z^*\|^2}.$$

Therefore,

$$\lim_{n\to\infty} \|u_n - T_{i_n} u_n\|^2 + \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|^2 = 0$$

which yields that

$$\lim_{n\to\infty} \|u_n - T_{i_n} u_n\| = 0 \text{ and } \lim_{n\to\infty} \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\| = 0. \tag{12}$$

Observe that

$$\|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\| \leq \|u_n - T_{i_n} u_n\| + \|y_{j_n n}^{k_n}\| \leq \|u_n - T_{i_n} u_n\| + \|A_{k_n}\| \times \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|.$$

Defining $\Upsilon = \max\{\|A_k\| :\ k = 1, 2, \ldots, M\}$, it follows from the inequality above that

$$\psi_n = \|u_n - T_{i_n} u_n + y_{j_n n}^{k_n}\| \leq \|u_n - T_{i_n} u_n\| + \Upsilon \|A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n\|.$$

Hence, by (12), we get that $\psi_n \to 0$ as $n \to \infty$. In virtue of the definition of ψ_n, this implies that

$$\lim_{n\to\infty} \|u_n - T_i u_n + y_{jn}^k\| = 0,\ i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k. \tag{13}$$

According to (9) and the boundedness of u_n, we deduce

$$\begin{aligned} D\|u_n - T_i u_n + y_{jn}^k\| &\geq \langle u_n - T_i u_n + y_{jn}^k, u_n - z^*\rangle \\ &\geq \frac{1-\varrho}{2}(\|u_n - T_i u_n\|^2 + \|A_k u_n - S_j^k A_k u_n\|^2), \end{aligned}$$

where $D = \sup\{\|u_n - z^*\| : n \in \mathbb{N}^+\}$. Therefore, we get from (13) that

$$\lim_{n\to\infty} \|u_n - T_i u_n\| = 0 \text{ and } \lim_{n\to\infty} \|A_k u_n - S_j^k A_k u_n\| = 0$$

for all $i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k$. Thanks to the hypothesis of the demiclosedness, we have $\omega_w(u_n) \subset \Omega$. In the end, applying Lemma 2.2, we obtain that $u_n \rightharpoonup z^\dagger = \lim_{n\to\infty} \mathrm{Proj}_\Omega u_n$. The proof is completed. □

Algorithm 3.3. *Let $u \in H$ and $u_1 \in H$. Let the current iterate $\{u_n\}$ be given.*

1: *Compute* $\phi_n = \max\{\|u_n - T_i u_n\|,\ i = 1, 2, \ldots, N\}$,
2: $\Phi_n = \{i \in \{1, 2, \ldots, N\} :\ \|x_n - T_i u_n\| = \phi_n\}$,
3: $\psi_n = \max\{\|A_k u_n - S_{jn}^k A_k u_n\| : k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k\}$,
4: *and* $\Psi_n = \{(k, j) \in \{1, 2, \ldots, M\} \times \{1, 2, \ldots, L_k\} : \|A_k u_n - S_{jn}^k A_k u_n\| = \psi_n\}$.
5: *Compute* $\Gamma_n = \max\{\phi_n, \psi_n\}$. *If (Case 1)* $\Gamma_n = 0$, *then* $v_n = u_n$ *else if (Case 2)* $\phi_n = \Gamma_n$, *choose* $i_n \in \Phi_n$,
6: *and compute* $v_n = u_n - \theta_n(u_n - T_{i_n} u_n)$ *else if (Case 3)* $\psi_n = \Gamma_n$, *choose* $(k_n, j_n) \in \Psi_n$,
7: *and set* $\tau_n = \frac{\theta_n \|A_{k_n} u_n - S_j^{k_n} A_{k_n} u_n\|^2}{\|A_{k_n}^*(A_{k_n} u_n - S_j^{k_n} A_{k_n} u_n)\|^2}$, $\theta_n \in [c, d] \subset (0, 1-\varrho)$,
8: *and compute* $v_n = u_n - \tau_n A_{k_n}^*(A_{k_n} u_n - S_{j_n}^{k_n} A_{k_n} u_n)$.
9: *Compute* $u_{n+1} = \alpha_n u + (1-\alpha_n) v_n$ *where* $\{\alpha_n\} \subset (0, 1)$.
10: *Set* $n := n + 1$ *and go back to 1.*

Theorem 3.3. *If* $\lim_{n\to+\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = +\infty$, *then the sequence* $\{u_n\}$ *generated by Algorithm 3.3 converges strongly to the solution* $z^\dagger (= \mathrm{Proj}_\Omega u)$ *of the problem (3).*

Proof. Set $z^\dagger = \mathrm{Proj}_\Omega u$. In the light of (5) and (7), we can see

$$\|v_n - z^\dagger\| \leq \|u_n - z^\dagger\|.$$

Therefore, we obtain

$$\begin{aligned} \|u_{n+1} - z^\dagger\| &= \|\alpha_n(u - z^\dagger) + (1-\alpha_n)(v_n - z^\dagger)\| \\ &\leq \alpha_n\|u - z^\dagger\| + (1-\alpha_n)\|v_n - z^\dagger\| \\ &\leq \alpha_n\|u - z^\dagger\| + (1-\alpha_n)\|u_n - z^\dagger\| \\ &\leq \max\{\|u - z^\dagger\|, \|u_n - z^\dagger\|\}. \end{aligned}$$

By induction, we derive that $\|u_{n+1} - z^\dagger\| \leq \max\{\|u - z^\dagger\|, \|u_0 - z^\dagger\|\}$ and thereby, the sequence $\{u_n\}$ is bounded. It follows that from (5) and (7) that (Case 1)

$$\begin{aligned} \|u_{n+1} - z^\dagger\|^2 =& \|\alpha_n(u - z^\dagger) + (1-\alpha_n)(u_n - z^\dagger)\|^2 \\ \leq& (1-\alpha_n)\|u_n - z^\dagger\|^2 + 2\alpha_n\langle u - z^\dagger, u_{n+1} - z^\dagger\rangle, \end{aligned} \tag{14}$$

and (Case 2)

$$\begin{aligned} \|u_{n+1} - z^\dagger\|^2 =& \|\alpha_n(u - z^\dagger) + (1-\alpha_n)(v_n - z^\dagger)\|^2 \\ \leq& (1-\alpha_n)\|v_n - z^\dagger\|^2 + 2\alpha_n\langle u - z^\dagger, u_{n+1} - z^\dagger\rangle \\ \leq& (1-\alpha_n)\|u_n - z^\dagger\|^2 + \alpha_n[2\langle u - z^\dagger, u_{n+1} - z^\dagger\rangle \\ &- \frac{\theta_n(1-\alpha_n)(1-\varrho-\theta_n)}{\alpha_n}\|(u_n - T_{i_n} u_n)\|^2], \end{aligned} \tag{15}$$

and (Case 3)

$$\begin{aligned}\|u_{n+1}-z^\dagger\|^2 =&\|\alpha_n(u-z^\dagger)+(1-\alpha_n)(v_n-z^\dagger)\|^2\\ \leq&(1-\alpha_n)\|v_n-z^\dagger\|^2+2\alpha_n\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle\\ \leq&(1-\alpha_n)\|u_n-z^\dagger\|^2+\alpha_n[2\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle\\ &-\frac{\theta_n(1-\alpha_n)(1-\varrho-\theta_n)}{\alpha_n}\frac{\|A_{k_n}u_n-S_{j_n}^{k_n}A_{k_n}u_n\|^4}{\|A_{k_n}^*(A_{k_n}u_n-S_{j_n}^{k_n}A_{k_n}u_n)\|^2}].\end{aligned} \tag{16}$$

Next, we rewrite our results in (14), (15) and (16):

$$\begin{aligned}\|u_{n+1}-z^\dagger\|^2 \leq&(1-\alpha_n)\|u_n-z^\dagger\|^2+\alpha_n[2\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle\\ &-\frac{\theta_n(1-\alpha_n)(1-\varrho-\theta_n)}{\alpha_n}D_n],\end{aligned} \tag{17}$$

where

$$D_n:=\begin{cases}0, & \text{Case 1},\\ \|(u_n-T_{i_n}u_n)\|^2, & \text{Case 2},\\ \dfrac{\|A_{k_n}u_n-S_{j_n}^{k_n}A_{k_n}u_n\|^4}{\|A_{k_n}^*(A_{k_n}u_n-S_{j_n}^{k_n}A_{k_n}u_n)\|^2}, & \text{Case 3},\end{cases}$$

Let

$$M:=\sup\Big\{\|A_k^*(A_ku_n-S_j^kA_ku_n)\|^2:\ k=1,2,\ldots,M,\ j=1,2,\ldots,L_k,\ n\in\mathbb{N}^+\Big\},$$

and

$$J=\min\{1,\frac{1}{M}\}.$$

Set

$$\bar{D}_n:=\begin{cases}0, & \text{Case 1},\\ \|(u_n-T_{i_n}u_n)\|^2, & \text{Case 2},\\ \|A_{k_n}u_n-S_{j_n}^{k_n}A_{k_n}u_n\|^4, & \text{Case 3}.\end{cases}$$

In fact, we can see

$$\bar{D}_n=\begin{cases}\Gamma_n, & \text{Case 1},\\ \Gamma_n^2, & \text{Case 2},\\ \Gamma_n^4, & \text{Case 3},\end{cases}$$

It is obvious that $D_n \geq J\bar{D}_n$. This together with (17) implies that

$$\begin{aligned}\|u_{n+1}-z^\dagger\|^2 \leq&(1-\alpha_n)\|u_n-z^\dagger\|^2+\alpha_n[2\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle\\ &-\frac{\theta_n(1-\alpha_n)(1-\varrho-\theta_n)}{\alpha_n}D_n]\\ \leq&(1-\alpha_n)\|u_n-z^\dagger\|^2+\alpha_n[2\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle\\ &-\frac{\theta_n(1-\alpha_n)(1-\varrho-\theta_n)}{\alpha_n}J\bar{D}_n]\\ =&(1-\alpha_n)\|u_n-z^\dagger\|^2+\alpha_n[2\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle-\frac{F_n}{\alpha_n}],\end{aligned} \tag{18}$$

where

$$F_n:=\theta_n(1-\alpha_n)(1-\varrho-\theta_n)\times J\bar{D}_n.$$

Set $\chi_n=\|u_n-z^\dagger\|^2$ and

$$\phi_n=2\langle u-z^\dagger,u_{n+1}-z^\dagger\rangle-\frac{F_n}{\alpha_n}. \tag{19}$$

Then, we can rewrite the above inequality (18) as

$$\chi_{n+1} \leq (1-\alpha_n)\chi_n + \alpha_n \phi_n.$$

In view of (19), we see

$$\phi_n \leq 2\langle u - z^\dagger, u_{n+1} - z^\dagger \rangle \leq 2\|u - z^\dagger\| \times \|u_{n+1} - z^\dagger\|.$$

Hence, $\limsup_{n\to\infty} \phi_n < +\infty$. Furthermore, from Lemma 2.1, we have that

$$\lim \sup_{n\to\infty} \phi_n \geq 0.$$

So, there exists a subsequence $\{n_s\}$ such that

$$\lim \sup_{n\to\infty} \phi_n = \lim_{s\to\infty} \phi_{n_s} = 2\langle u - z^\dagger, u_{n_s+1} - z^\dagger \rangle - \frac{F_{n_s}}{\alpha_{n_s}}.$$

Since $\langle u - z^\dagger, u_{n_s+1} - z^\dagger \rangle$ is bounded, without loss of generality, assume $\lim_{s\to\infty}\langle u - z^\dagger, u_{n_s+1} - z^\dagger \rangle$ exists. Consequently, $\lim_{s\to\infty} \frac{F_{n_s}}{\alpha_{n_s}}$ exists. Hence,

$$\lim_{s\to\infty} F_{n_s} = 0. \tag{20}$$

So, by the definition of F_{n_s}, we get that

$$\lim_{s\to\infty} \|u_{n_s} - T_i u_{n_s}\| = 0 \text{ and } \lim_{s\to\infty} \|A_k u_{n_s} - S_j^k A_k u_{n_s}\| = 0$$

for all $i = 1, 2, \ldots, N,\ k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, L_k$. Thanks to the hypothesis of the demiclosedness, we have $\omega_w(u_{n_s}) \subset \Omega$. Note that

$$\begin{aligned} \|u_{n+1} - u_n\| =& \|\alpha_n(u - u_n) + (1-\alpha_n)(v_n - u_n)\| \\ \leq& \alpha_n \|u - u_n\| + (1-\alpha_n)\|v_n - u_n\|. \end{aligned}$$

In virtue of (20), we can duduce easily that

$$\lim_{s\to\infty} \|u_{n_s+1} - u_{n_s}\| = 0.$$

This implies that the weak cluster point set $\omega_w(u_{n_s})$ also belong to the set Ω. Without loss of generality, we can assume that $\{u_{n_s}\}$ converges weakly to $z^* \subset \Omega$. Hence, in view of $z^\dagger = \mathrm{Proj}_\Omega u$, we get

$$\limsup_{n\to\infty} \phi_n \leq 2\langle u - z^\dagger, u_{n_s+1} - z^\dagger \rangle = 2\langle u - z^\dagger, z^* - z^\dagger \rangle \leq 0.$$

.

In the end, from Lemma 2.3, we obtain $\lim_{n\to\infty} x_n = z^\dagger$. The proof is completed. □

4. **Conclusion**

In this paper, we study split common fixed point problems with multiple output sets in real Hilbert spaces. In order to solve this problem, we present three new self-adaptive algorithms. Weak and strong convergence theorems are established under some mild assumptions.

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