OPTIMUM CONTROLLED FRACTIONAL ORDER DAMPED OSCILLATORY SYSTEM

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Precisely desired output and accurate performance are substantial requirements in systems and models design. Generally, in the dynamic systems design, these requirements can be fulfilled by applying controllers and optimizers to the systems' models. A forced variable order fractional damped vibrating system is modelled. The dissipation term in the model is represented by Caputo Fabrizio fractional derivative. The system is subject to a unit step external force. Numerical and time discretization techniques are applied to obtain the open loop system responses. The discrete-time PID controller is applied to the system model to generate closed loop system responses. The Particle Swarm Optimization along with the discrete-time PID controller are utilized to achieve predetermined desired responses. In this paper, our won Matlab script is utilized to numerically generate the response of the open-loop, closed-loop and optimum closed-loop control systems. The effectiveness of the applied techniques, controller, and optimizer are investigated by comparing the classical integer model responses with the fractional model responses. The results demonstrate the feasibility of the applied numerical method, and the effectiveness of the introduced controller and optimizer. The results show relatively low errors between the desired responses' metrics and the actual responses' metrics of the integer and the fractional models.

Keywords: Fractional damped vibrating system, Caputo Fabrizio fractional derivative, Discrete-time PID controller, Particle Swarm Optimization.

1. **Introduction**

The optimization techniques are widely implemented different sciences' researches. The optimization methods are applied to the dynamic systems to achieve desired outputs. Such as The Grey Wolf Optimizer (GWO) ([1]).

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An Ant Colony Optimization, Artificial Bee Colony algorithms ([2]), and Genetic Algorithm (GA) ([3]). The Particle Swarm Optimization (PSO) which is implemented in this work had been applied in various sectors of science ([4], [5], [6], [7]). These techniques are used to find the optimum solutions of different problems based on certain criteria and constraints. Recent researches in different science disciplines demonstrate the fractional calculus as an extraordinary modeling tool. The fractional calculus can precisely model and describe the fractional systems in pure and applied sciences ([8], [9], [10], [11], [12]). A fractional-order chaotic random number generator (RNG) is used as the target RNG to demonstrate a cryptanalysis technique ([15]). A different stability behaviors have been obtained for the coupled fractional Boussinesq-Burgers system model, when the model was upgraded to include time-fractional derivatives ([16]).

The Proportional Integral Derivative (PID) controller is widely applied to control and regulate system responses. The continuous form of PID controller need to be discretized to a form that can be used in computer implementations ([13]). Moreover, the PID controller is applied as a fractional controller ($PI^{\lambda}D^{\mu}$) ([14]).

In active suspension control, a fractional damping single DOF system is studied. A relationship is derived between the system critical damping coefficient and the fractional derivative order.([17]).

The PID controller along with PSO are implemented to obtain optimum results for different models. The PID-PSO controller is applied to automatic voltage regulator to prompt the output voltage convergence ([18]). A DC motor optimally controlled by PID controller. The optimized PID gains are obtained for the DC motor model using the firefly algorithm (FA)([19]).

An improved fractional sub-equation method is applied to fractional biological population equation. ([44]). The Einstein tensor and Einstein field equations are obtained by involving the local fractional derivatives ([33]). The simulations of systems as fractional models, in certain applications, show a proximated simulations compared to integer representations ([30]). Due to the mechanical behavior of some elements of dynamic systems, it is accurated to be modeled as fractional systems. These elements may include viscoelastic materials such as some types of dampers ([35], [36]). Specific diffusion processes and systems whose elemnts are exposed to a time accumulated damage are more precisely modeled as variable order fractional systems ([46]).

In some applications of the fractional representation a variable damping force is used to suppress dynamic systems. For instance, we may recall the Magneto-Rheological (MR) damper ([37]) and the brush disk sliding friction ([38]). In those systems, due to time varying parameters, the damping ratio increases.

The motivations behind this work are the generation of optimum fractional dynamic system responses, and investigation of the effects of the fractional derivative order on the optimization process. The novelties of the study are the modeling of a forced variable order damped fractional vibrating system by using the non-singular Caputo Fabrizio (C-F) fractional derivative, and the obtaining of optimum responses of the system. Which is accomplished by applying the PSO along with discrete-time PID controller to optimally control the fractional system responses.

In this work, the PSO method is applied to forced variable order damped fractional vibrating system. A discrete-time PID controller is implemented to the system before applying the optimizer. The application of the PID controller to the system generates a controlled system responses. These controlled responses are yielded from arbitrary controller gains. So that these responses don't match the desired responses. In order to reach the desired system responses, the PSO is applied to the controlled system. The optimizer generated optimum controller gains, by which the desired response can be achieved.

In this study, the optimum controller gains are obtained based on the desired response characteristics. The objective function of the optimizer is obtained, at each iteration, by comparing the desired response characteristics with the actual response characteristics. In section 2 the procedure and algorithm of PSO method are introduced. Section 3 illustrates the PID controller both in continuous and discrete forms. A forced variable order damped fractional vibrating system is modelled in section 4. The damping force in the system model is represented by non-singular C-F fractional derivative. The open and closed loop system responses are generated in section 5. The responses are obtained by using a time discretization numerical technique. The resulted open and closed loop system responses verify the accuracy of the applied numerical technique. In section 6, the PSO is implemented to the closed loop system to achieve desired system responses. The optimum system responses are generated for different desired system responses and fractional derivative orders.

The errors between desired system responses' characteristics and corresponding optimum actual responses' characteristics are calculated for all cases. The results indicated very slight errors that verify the effectiveness of the applied techniques.

2. Particle swarm optimization

The particle swarm optimization method was discovered through studying of behaviours of the bird flocking, and the fish schooling. The simulation of these behaviours generates a strongly effective optimization technique ([20]). This technique is known as the particle swarm optimization (PSO). The swarm which represents the particles population is randomly initialized. Each particle in the swarm represents a potential solution and has its position X_i and velocity V_i in a hyperspace. The positions and the velocities of the population are, respectively, defined as

$$X_i = (x_{i,1}, x_{i,2}, ..., x_{i,d}), \tag{1a}$$

$$V_i = (v_{i,1}, v_{i,2}, ..., v_{i,d}), \quad i = 1, 2, 3, ..., P, \tag{1b}$$

where d represents the dimension of the particle, and P is the population size. The fitnesses of the solutions are compared to determine the best solution. The value of the current best solution is stored. The stored solutions represent global solutions. Each particle adjusts its position and velocity based on the best solution (*Pbest*) and the best global (*gbest*) solution as following:

$$\begin{aligned} v_{i,m}^{t+1} = \omega * v_{i,m}^{t} + c_1 * rand() * \left(Pbest_{i,m} - x_{i,m}^{t}\right) \\ + c_2 * rand() * \left(gbest_{i,m} - x_{i,m}^{t}\right), \end{aligned} \tag{2a}$$

$$x_{i,m}^{t+1} = x_{i,m}^{t} + v_{i,m}^{t+1}, \qquad m = 1, 2, 3, ..., d. \tag{2b}$$

The velocity right here is used as a memory of a certain particle previous direction, which means s movement in the immediate past. This memory term can be considered as a momentum, which leads the particle towards the current direction. This component is also referred to as the inertia component ([21]). The inertia component represents the prior velocity, which provides the particles with suitable momentum to move over the search space([22]). Therefore the velocity can be considered as just a factor used to push the particle to change its position (location).

The optimization process is running upto predetermined number of iterations N. The following pseudo code illustrates the particle swarm optimization procedure:

(1) **Initialize the population randomly**
(2) ***while*** No. iterations < N
(3) ***for*** i=1:population size
Compute the objective functions
if Objective function value is the best
{
The associated particle is the best solution
}
end
(4) **Save the best solution**
(5) **Assign the global solution**
(6) ***for*** i=1:population size
Based on the best and global solutions
Calculate a new velocity of particle(i)
Update the new position of particle(i)
end
end

3. **Discrete time PID controller**

The PID controller is composed of three coefficients or gains, namely, proportional, integral, and derivative. These gains can be tuned to achieve a desired controlled system responses. The general formulas of outputs of the proportional (K_p), integral (K_i), and derivative K_d gains are given, respectively, as following:

$$f_p(t) = K_p e(t), \tag{3a}$$

$$f_i(t) = K_i \int_0^t e(\tau) d\tau, \tag{3b}$$

$$f_d(t) = K_d \frac{de(t)}{dt}, \tag{3c}$$

where $f_p(t),\ f_i(t),\ and\ f_d(t)$ are the outputs of the controller and their summation is the input of the controlled system (Fig. 1), $e(t)$ is the continuous error between the desired output $y_d(t)$ and the actual output $y(t)$.

The forms of the continuous PID controller that are given by (3) can be reintroduced in a discrete forms ([13]) as following:

$$f_p(k) = K_p e(k), \tag{4a}$$

$$f_i(k) \approx f_i(k-1) + K_i \frac{h[e(k) + e(k-1)]}{2}, \tag{4b}$$

$$f_d(k) \approx K_d \frac{[e(k) - e(k-1)]}{h}, \tag{4c}$$

where k is a pointer of the discrete signals sample points, and h is a discrete time increment.

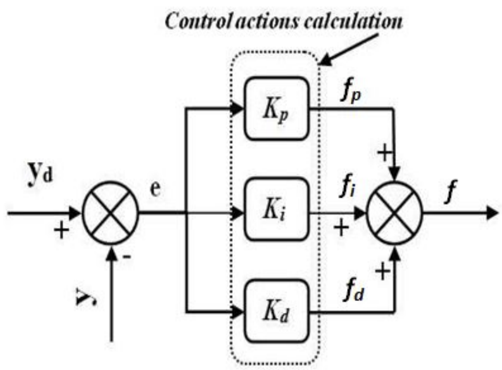

FIGURE 1. Block diagram of PID controller.

In order to achieve the desired fractional system response, the discrete time PID controller along with particle swarm optimizer are implemented to the introduced fractional dynamic system.

4. **Fractional oscillatory system modelling**

The Kelvin–Voigt system ([23]) which is composed of dashpot and spring elements connected in parallel is modeled as a simple fractional oscillatory system. In which the dashpot element generates a dissipation force and is modelled as fractional term in the system model. The dissipation force is proportional to the fractional derivative of the displacement. The Kelvin–Voigt model is connected to a mass M to generate a single degree of freedom fractional system as shown in (Fig. 2). The system is subjected to external force $u(t)$, The response of the system is represented by the reciprocating motion $y(t)$ of the mass M.

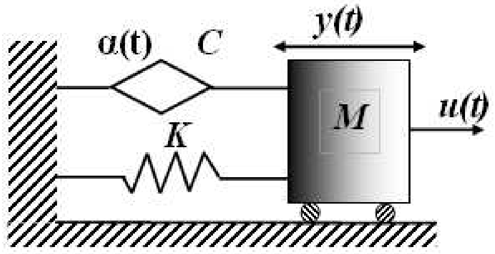

FIGURE 2. A single degree of freedom fractional oscillatory system.

The model of the system that is shown in (Fig. 2) is given as following:

$$M\ddot{y}(t) + C\ {}_{0}^{CF}D_{t}^{\alpha(t)}y(t) + Ky(t) = f(t), \qquad \alpha(t) \in \Re, \tag{5}$$

where C is the damping coefficient of the dashpot, $\alpha(t)$ is a real variable derivative order of the fractional dissipation force, K represents the spring stiffness.

The damping term in the model (5) is defined by the non-singular variable order C-F fractional derivative ([24, 25]) as following:

$${}_{0_+}^{CF}D_{t}^{\alpha(t)}y\,(t) = \frac{Q[\alpha(t)]}{[1-\alpha(t)]}\int_{0^+}^{t}\frac{d}{d\eta}y\,(\eta)\,exp\left[-\frac{\alpha(t)(t-\eta)}{1-\alpha(t)}\right]d\eta,\ \ 0<\alpha\,(t)<1, \tag{6}$$

where $Q(\alpha(t))$ is a normalized function in the interval of $\alpha(t) \in [0,1]$ such that $Q(0) = Q(1) = 1$ ([24]).

Substituting (6) into the system model in (5) yields,

$$M\ddot{y}(t) + C\,\frac{Q[\alpha(t)]}{[1-\alpha(t)]}\int_{0^+}^{t}\frac{d}{d\eta}y\,(\eta)\,exp\left[-\frac{\alpha(t)(t-\eta)}{1-\alpha(t)}\right]d\eta + Ky(t) = f(t). \tag{7}$$

The response $y(t)$ of the fractional system model in (7)can be obtained numerically. The obtained response represents the open loop system response.

The discrete time PID controller that is introduced in section 3 can be applied to the fractional system model to generate a closed loop system responses.

5. **Open and closed system responses**

A numerical method and discretization technique are utilized to approximate the open loop response of the introduced model (7). The second derivative in the inertia term ($M\ddot{y}(t)$) in the model is approximated by using the finite differences method as follows

$$\ddot{y}(t_k) = \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + O\left(h^2\right). \tag{8}$$

The damping term in the model, which is represented by C-F fractional derivative is approximated numerically as following:

$$\begin{aligned} &C\left[\frac{Q[\alpha(t)]}{[1-\alpha(t)]}\right]\int_{0^+}^{t}\frac{d}{d\eta}y\left(\eta\right)exp\left[-\frac{\alpha(t)(t-\zeta)}{1-\alpha(t)}\right]d\eta = \\ &C\left[\frac{Q[\alpha(t)]}{[1-\alpha(t)]}\right]\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\frac{d}{d\eta}y\left(\eta\right)exp\left[-\frac{\alpha(t_k)(t_k-\eta)}{1-\alpha(t_k)}\right]d\eta. \end{aligned} \tag{9}$$

The first order derivative $\frac{dy(\eta)}{d\eta}$ is approximated via the forword finite difference method at the point k as following:

$$\frac{dy_k\left(\eta\right)}{d\eta} = \frac{y_{k+1} - y_k}{h} + O\left(h\right)\ , \tag{10}$$

Substituting (10) into (9) yields

$$\begin{aligned} &C\left[\frac{Q[\alpha(t)]}{[1-\alpha(t)]}\right]\int_{0^+}^{t}\frac{d}{d\eta}y\left(\eta\right)exp\left[-\frac{\alpha(t)(t-\eta)}{1-\alpha(t)}\right]d\eta = \\ &\qquad C\left[\frac{Q[\alpha(t)]}{[1-\alpha(t)]}\right]\sum_{j=0}^{k-1}\frac{y_{k+1}-y_k}{h}\int_{t_j}^{t_{j+1}}exp\left[-\frac{\alpha(t_k)(t_k-\eta)}{1-\alpha(t_k)}\right]d\eta. \end{aligned} \tag{11}$$

Substitute (8) and (11) into (7) to generate the approximated form of the system model as

$$\begin{aligned} &M\frac{y_{k+1}-2y_k+y_{k-1}}{h^2} + C\left[\frac{Q[\alpha(t)]}{[1-\alpha(t)]}\right]\sum_{j=0}^{k-1}\frac{y_{k+1}-y_k}{h} \\ &\int_{t_j}^{t_{j+1}}exp\left[-\frac{\alpha(t_k)(t_k-\eta)}{1-\alpha(t_k)}\right]d\eta + Ky(t) = f(t). \end{aligned} \tag{12}$$

The open loop system response can be obtained by solving the integral in (12) and applying the initial conditions, and the external force, for more details refer to ([10]).

The external force is given as a unit step as shown in (Fig. 3). The system actual response that is obtained based on the initial conditions, and the external force reaches the desired position at a certain time interval. This time interval depends on the system parameters and the controller gains. Each actual response has its own characteristics which are rise time, maximum overshoot, peak time, and settling time, see (Fig. 3).

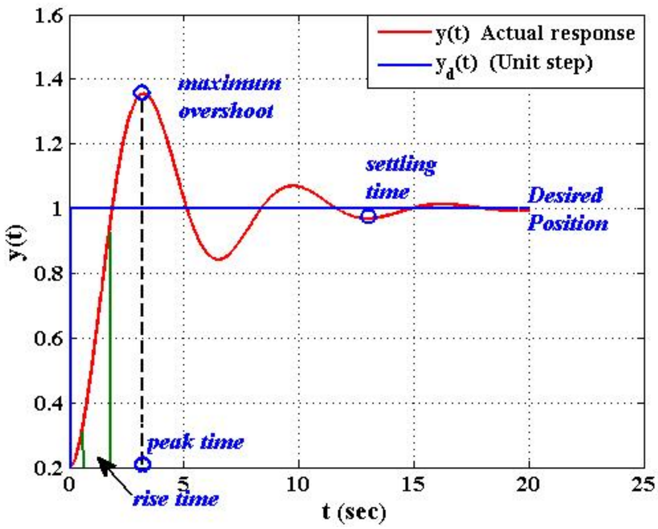

FIGURE 3. Unit step and actual response of oscillatory system.

For same system parameters and same initial conditions, the open loop fractional system response at $\alpha(t) = 0.99$ is compared the classical integer system response in (Fig. 4).

The introduced discrete-time PID controller is applied to the open loop systems. Closed loop responses of both integer and fractional systems are generated. (Fig. 5) shows the integer controlled system response and fractional controlled system response at $\alpha(t) = 0.99$.

It's deduced from (Fig. 4) and (Fig. 5) that the integer and fractional responses are identical in uncontrolled and controlled cases, respectively. The fractional model responses in these cases are obtained at close to one $\alpha(t)$ values. These identical responses verify the accuracy of the applied procedure and the introduced techniques. The closed loop system responses at $\alpha(t) = 0.5$, and $\alpha(t) = 0.8$ are shown in (Fig. 6-a) and (Fig. 6-b), respectively. Both fractional models are subject to the same PID controller gains. (Fig. 6) demonstrates that the fractional system response at $\alpha(t) = 0.8$ reaches the steady state faster than the fractional system response at $\alpha(t) = 0.5$.

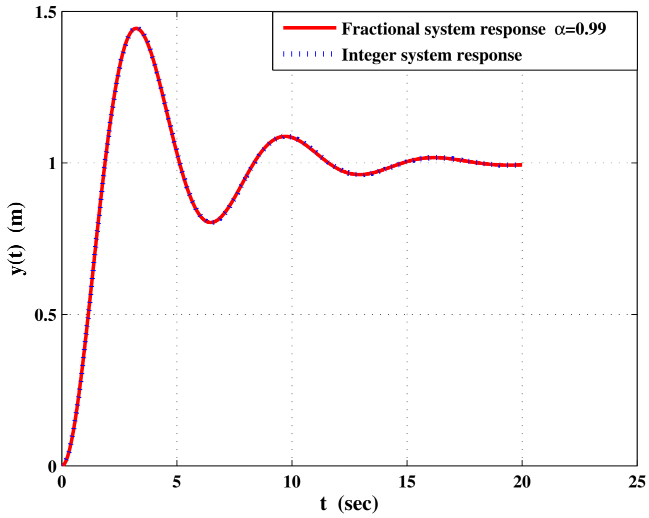

FIGURE 4. Open loop integer system and fractional (at $\alpha(t) = 0.99$) system responses.

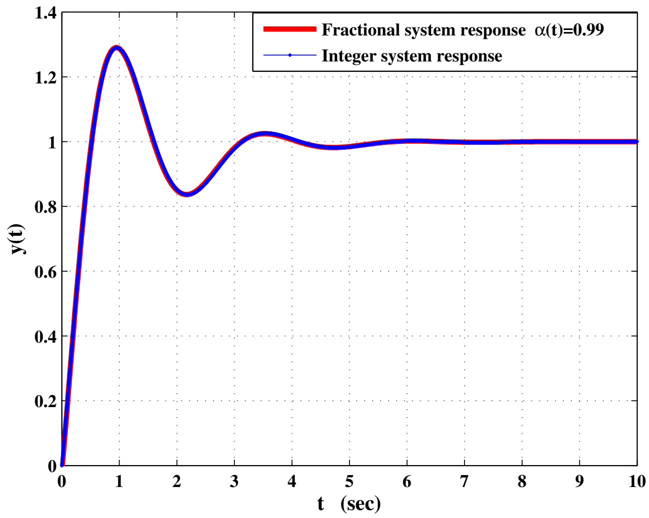

FIGURE 5. Closed loop integer system and fractional (at $\alpha(t) = 0.99$) system responses.

The controlled fractional system responses that are shown in (Fig. 6) are generated from applying arbitrary PID controller gains. In order to reach desired fractional system responses, the PID controller gains need to be optimized. The introduced PSO technique is applied to the considered fractional oscillatory system to achieve desired responses.

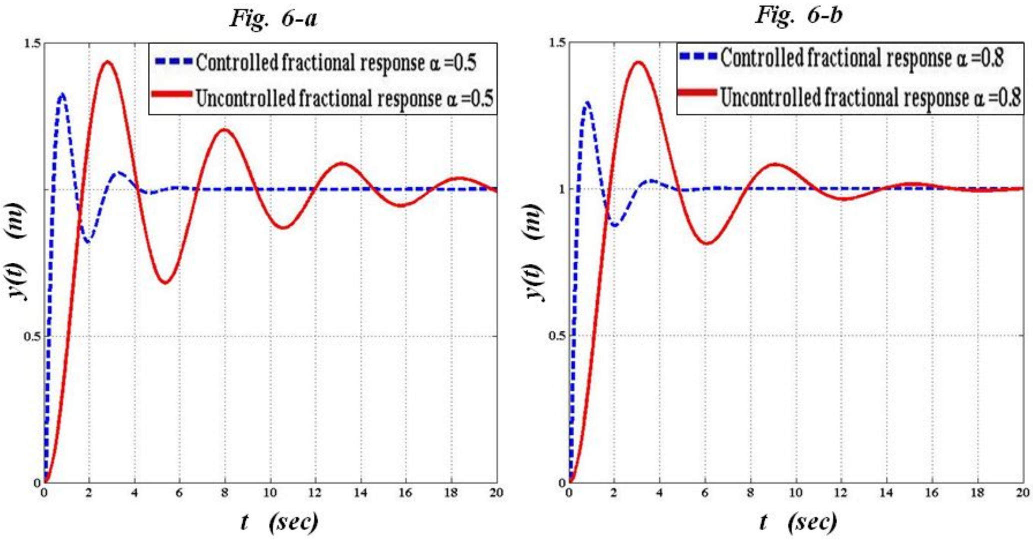

FIGURE 6. Controlled vs. uncontrolled fractional system responses, a- $\alpha(t) = 0.5$, b- $\alpha(t) = 0.8$.

6. **Optimal responses**

High efficiencies and performances of dynamic systems can be achieved when their desired outputs are verified. By applying arbitrary-gains controllers these desired outputs can not be achieved. The applied controllers gains need to be tuned to reach the desired system output. An optimization technique is applied along with the controller to tune the controller parameters. The introduced PSO method is applied to the considered controlled damped fractional vibrating system. The discrete-time PID controller gains are tuned by the PSO. The controller gains are introduced as a population of the optimizer. Each particle of the population is composed of a set of the controller gains. In order to decrease the computation cost, the population size of swarm is chosen to be four. The PSO procedure that is introduced in section 2 is applied to controlled system model that given by (12). A bunch of desired responses are obtained from the integer model and different orders fractional models. (Fig. 7) shows a comparison between the responses of optimal-controlled, controlled, and uncontrolled damped fractional vibrating system. The comparison is done for $\alpha(t) = 0.5$ at (Fig. 7-a) and $\alpha(t) = 0.8$ at (Fig. 7-b). Both fractional systems have same parameters (mass, stiffness, and damping coefficient). The desired response characteristics in the optimal cases in (Fig. 7) are percent overshoot=18, settling time=3, and peak time=0.6. The actual optimum response of the fractional system at $\alpha(t) = 0.5$ has settling time =2.9728, percent over shoot =17.9309, and peak time = 0.5. Whereas at $\alpha(t) = 0.8$ the actual optimum response has settling time =2.9288, percent over shoot =18.0252, and peak time = 0.5.

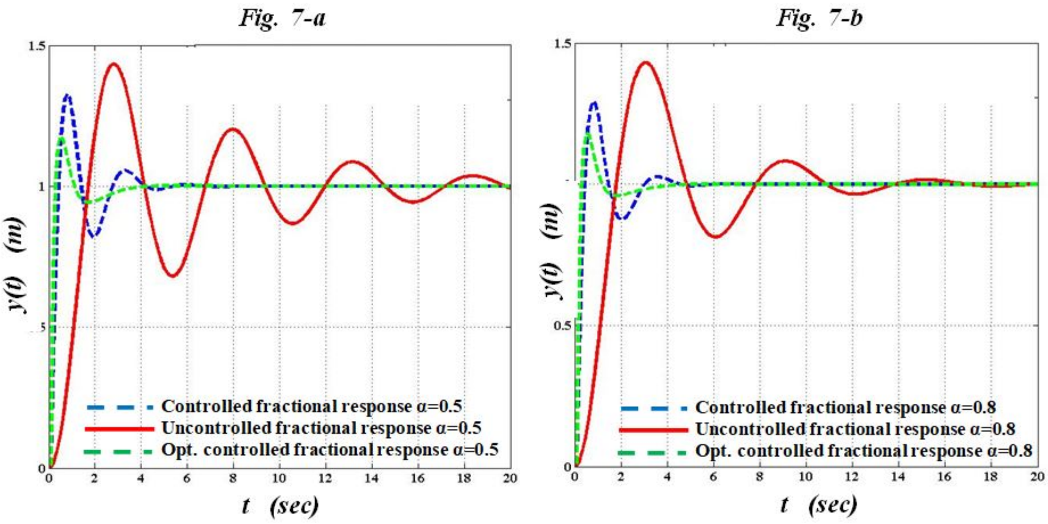

FIGURE 7. Optimal-controlled vs. Controlled vs. uncontrolled fractional system responses, a- $\alpha(t) = 0.5$, b- $\alpha(t) = 0.8$.

TABLE 1. Desired vs actual responses characteristics (Integer case)

Desired Res. Characteristics			Actual Res. Characteristics			Opt. Controller Gains		
P-OS	S-T	P-T	P-OS	S-T	P-T	K_p	K_i	K_d
15	5	0.6	15.0296	4.8156	0.65	11.3283	2.8891	4.2658

The errors of the systems' actual optimum responses is calculated as following:

$$E = \frac{|\sum DRCs - \sum ORCs|}{\sum DRCs} * 100, \tag{13}$$

where $DRCs$ represents the desired response characteristics and $ORCs$ is the optimum response characteristics. The actual optimum responses that are shown in (Fig. 7-a) and (Fig. 7-b) yields absolute errors 0.9088% and 0.6759% compared to the desired responses, respectively.

The characteristics of desired and actual responses of the unit step forced damped integer oscillatory system are demonstrated in Table 1.

The optimum response of the integer system in Table 1 yields an absolute error 0.5087% compared to the desired response. The introduced PSO method is applied to the unit step forced damped fractional oscillatory systems. The optimum responses' characteristics of these system are obtained for different fractional derivative orders and different desired responses as shown in Table 2 through Table 5.

Table 2. Desired vs actual res. characteristics (Fractional $\alpha = 0.99$)

Desired Res. Characteristics			Actual Res. Characteristics			Opt. Controller Gains		
P-OS	S-T	P-T	P-OS	S-T	P-T	K_p	K_i	K_d
12	3	0.2	11.8759	2.8471	0.2	6.6152	8.1040	10.3410
15	5	0.6	15.4820	4.8768	0.6	10.5246	3.6877	4.5687
10	3	0.3	10.1094	2.9801	0.4	9.3417	19.1008	8.132
8	3	0.3	8.014	3.0264	0.4	8.3406	17.2389	8.4848
17	3	0.5	16.9941	2.8701	0.5	11.4388	9.7034	5.3731
20	4	0.6	20.0317	3.7259	0.5	9.5822	30.6348	6.6175

Table 3. Desired vs actual res. characteristics (Fractional $\alpha = 0.8$)

Desired Res. Characteristics			Actual Res. Characteristics			Opt. Controller Gains		
P-OS	S-T	P-T	P-OS	S-T	P-T	K_p	K_i	K_d
12	3	0.2	12.1503	2.9964	0.2	7.1425	32.5041	10.0680
15	5	0.6	15.1789	4.8761	0.63	11.0885	3.5053	4.2308
10	3	0.3	9.9910	3.0286	0.4	8.6798	8.2458	7.5368
8	3	0.3	7.9593	2.9467	0.3	7.9438	8.3051	8.6869
17	3	0.5	17.057	2.8634	0.5	10.1365	9.2018	5.5625
20	4	0.6	19.9476	3.7897	0.6	11.3617	5.0314	4.5203

Table 4. Desired vs actual res. characteristics (Fractional $\alpha = 0.5$)

Desired Res. Characteristics			Actual Res. Characteristics			Opt. Controller Gains		
P-OS	S-T	P-T	P-OS	S-T	P-T	K_p	K_i	K_d
12	3	0.2	11.9772	3.0753	0.4	10.2617	8.3879	8.1653
15	5	0.6	15.0772	4.9354	0.5	9.9598	3.4099	5.7938
10	3	0.3	9.9918	3.0068	0.4	8.0560	8.9825	8.2438
8	3	0.3	8.0006	2.8466	0.4	7.9137	3.6576	8.3782
17	3	0.5	17.1737	3.1663	0.5	10.9202	5.7171	4.7161
20	4	0.6	20.0551	4.3255	0.6	10.7775	3.9560	6.6929

In Table. 2 through Table. 5, different optimum actual responses' characteristics and different optimum controller gains are generated based on desired responses' characteristics and fractional derivative orders. The absolute errors of all cases are shown in Table. 6.

Table 5. Desired vs actual res. characteristics (Fractional $\alpha = 0.3$)

Desired Res. Characteristics			Actual Res. Characteristics			Opt. Controller Gains		
P-OS	S-T	P-T	P-OS	S-T	P-T	K_p	K_i	K_d
12	3	0.2	11.8196	2.9801	0.2	5.6882	11.5260	10.3881
15	5	0.6	15.1545	5.1509	0.5	9.8644	3.2688	5.8159
10	3	0.3	10.0379	2.9863	0.3	8.6220	11.6948	9.8514
8	3	0.3	8.0451	3.1638	0.3	7.7770	7.3242	9.5057
17	3	0.5	17.0462	2.9938	0.5	9.5632	9.2852	6.1051
20	4	0.6	19.9449	4.0346	0.2	7.5947	40.9156	10.7268

Table 6. Optimal actual responses' errors of fractional systems

	Desired Res. Characteristics [12 3 0.2]	Desired Res. Characteristics [15 5 0.6]	Desired Res. Characteristics [10 3 0.3]	Desired Res. Characteristics [8 3 0.3]	Desired Res. Characteristics [17 3 0.5]	Desired Res. Characteristics [20 4 0.6]
α	Error %	Error %	Error %	Error %	Error %	Error %
0.99	1.8224	1.7417	1.4248	1.2425	0.6624	1.3919
0.8	0.9651	0.4126	0.8992	0.8319	0.3883	1.0679
0.5	1.6612	0.4243	0.7414	0.4673	1.6585	1.5472
0.3	1.3178	0.9971	0.1820	1.8487	0.1951	1.7093

7. Conclusion

The PSO method is applied to controlled forced damped fractional vibrating systems. The generated optimum responses are compared to the corresponding desired responses. The comparison is done based on the system response characteristics. An integer case and bunch of fractional cases are studied. The open loop responses and the closed loop responses that are shown in (Fig.4), and (Fig.5), respectively, verify the effectiveness of the introduced techniques; that is because the integer response and the fractional (at α close to one) response that are shown in the figures are identical. The optimal-controlled, controlled and uncontrolled responses of the fractional systems are compared in (Fig.7). The comparison illustrates the effectiveness of the applied optimizer, and shows the difference between the implementation of arbitrary controller gains and optimum controller gains. The actual optimum responses that are shown in (Fig. 7-a) and (Fig. 7-b) yields so slight absolute errors 0.9088% and 0.6759% compared to the desired responses, respectively.

The optimum responses' characteristics of fractional systems are obtained for different fractional derivative orders and different desired responses as shown in Table 2 through Table 5. The Optimal actual responses' errors of these systems are demonstrated in Table 6. The yielded errors between the optimum and desired responses characteristics are less than 2%. Such slight errors verify the high performance of the applied optimization method, the effectiveness of the implemented numerical technique, and the robustness of the introduced controller.

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