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AN INERTIAL ALGORITHM FOR SOLVING SPLIT VARIATIONAL INCLUSION PROBLEM

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This paper aims to investigate a new inertial algorithm for solving a split variational inclusion problem in real Hilbert spaces. Under very mild conditions, we prove a strong convergence theorem for the proposed algorithm by using self-adaptive stepsizes and demiclosedness principle. Furthermore, an application is given to illustrate the effectiveness of the algorithm. The results improve and extend the corresponding ones announced by some others in the earlier and recent literature.

Keywords: Inertial algorithm, Self-adaptive stepsizes, Split variational inclusion, Hilbert spaces.

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1. Introduction

Throughout this paper, let H be a real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. $F(T)$ is denoted as the set of fixed points of a nonlinear mapping T. We use $x_n \to x$ and $x_n \rightharpoonup x$ to indicate the strong convergence and the weak convergence of the sequence $\{x_n\}$ to x, respectively.

First, we recall some notations which are needed in sequel. A mapping $T : H \to H$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H.$$

A mapping $T : H \to H$ is called firmly nonexpansive if

$$\langle Sx - Sy, x - y \rangle \geq \|Sx - Sy\|^2, \ \forall x, y \in H.$$

A multi-valued mapping $B : D(B) \subseteq H \to 2^H$ is called monotone if, for all $x, y \in D(B), u \in Bx$ and $v \in By$ such that

$$\langle x - y, u - v \rangle \geq 0.$$

A monotone mapping B is maximal if the graph $G(B)$ is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping B is maximal if and only if for $(x, u) \in D(B) \times H, \langle x - y, u - v \rangle \geq 0$ for every $(y, v) \in G(B)$ implies that $u \in Bx$.

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Let $B : D(B) \subseteq H \to 2^H$ be a multi-valued maximal monotone mapping. The resolvent operator $J_\lambda^B : H_1 \to D(B)$ associated with B is defined by

$$J_\lambda^B x := (I + \lambda B)^{-1}(x), \ \forall x \in H,$$

for some $\lambda > 0$, where I stands for the identity operator on H_1. Observe that for all $\lambda > 0$, the resolvent operator J_λ^B is single-valued, nonexpansive and firmly nonexpansive.

Split monotone variational inclusion problem has already been used in practice as a model in intensity-modulated radiation therapy treatment planning; see e.g., [1, 2, 3]. This formalism is also at the core of the modeling of many inverse problems arising for phase retrieval and other real-world problems, further, in sensor networks in computerized tomography and data compression; see e.g., [4, 5] and references therein.

In 2011, Moudafi [6] introduced the following split monotone variational inclusion problem (in short, SMVIP): find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1 x^* + B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in f_2 y^* + B_2 y^*, \end{cases} \tag{1}$$

where $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two given single-valued mappings, $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two multi-valued maximal monotone mappings.

If $f_1 \equiv 0$ and $f_2 \equiv 0$, then the problem (1) reduces to the following split variational inclusion problem (in short, SVIP): find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in B_2 y^*. \end{cases} \tag{2}$$

Subsequently, Byrne et al. [7] proved a weak and strong convergence of the following iterative method for problem (2): for given $x_0 \in H_1$ and $\lambda > 0$, compute the following iterative sequence:

$$x_{n+1} = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n].$$

Very recently, Sumalai et al. [8] studied a new split monotone variational inclusion problem:

$$\begin{cases} 0 \in Kx^*, \\ L_\alpha x^* \in H_\alpha : 0 \in K_\alpha(L_\alpha x^*), \end{cases}$$

where $L_\alpha : H \to H_\alpha$ is bounded linear operator for every $\alpha = 1, 2, \cdots, N$, $K : H \to 2^H$ and $K_\alpha : H_\alpha \to 2^{H_\alpha}$ are multi-valued maximal monotone mappings. Moreover, they introduced the following scheme:

$$z_{n+1} = J_{\mu_{n,\alpha}}^K[\vartheta_n v + (1 - \vartheta_n) \sum_{\alpha=1}^{N} \varrho_{n,\alpha}(z_n - \eta_{n,\alpha} L_\alpha^*(I - J_{\mu_{n,\alpha}}^{K_\alpha})L_\alpha z_n)].$$

As a result, they proved a strong convergence of the algorithm above under appropriate assumptions on the parameters.

Motivated and inspired by the results above, we introduce an inertial algorithm with adaptive stepsize that does not depend on the norms of the bounded linear operators. Under some suitable assumptions, a strong convergence of the proposed algorithm is proved for solving a split variational inclusion problem. Finally, we apply the main results to solve a split feasibility problem.

2. **Preliminaries**

In this section, we first recall some lemmas which are needed in sequel.

Lemma 2.1. *([9]) Let a multivalued mapping $K : D(K) \subset H \to 2^H$ be monotone, then the following statements are satisfied:*

(i) *For positive numbers $\rho \leq \mu$ and for any $z \in R(I + \mu K) \cap R(I + \rho K)$, we get*

$$\|z - J_\rho^K z\| \leq 2\|z - J_\mu^K z\|,$$

where $R(I + \mu K)$ and $R(I + \rho K)$ denote the range of the operators $I + \mu K$ and $I + \rho K$, respectively.

(ii) *For all $z, \overline{z} \in R(I + \mu K)$ with $\mu > 0$, we have*

$$\|J_\mu^K z - J_\mu^K \overline{z}\|^2 \leq \langle z - \overline{z}, J_\mu^K z - J_\mu^K \overline{z}\rangle;$$

$$\|(I - J_\mu^K)z - (I - J_\mu^K)\overline{z}\|^2 \leq \langle (I - J_\mu^K)z - (I - J_\mu^K)\overline{z}, z - \overline{z}\rangle;$$

$$\|J_\mu^K z - s\|^2 \leq \|z - s\|^2 - \|z - J_\mu^K z\|^2,$$

where $s \in \Gamma = K^{-1}(0) \neq \emptyset$.

Lemma 2.2. *([10]) Let K be a nonexpansive mapping on a closed convex subset C of a real Hilbert space H. The mapping $I - K$ is said to be demiclosed on C, if for any sequence z_n in C, such that $z_n \rightharpoonup s \in C$ and $(I - K)(z_n) \to s^*$, we have $(I - K)(s) = s^*$.*

Lemma 2.3. *([11]) Let $\{r_n\}$ be a sequence of nonnegative numbers, $\{\vartheta_n\}$ be a sequence in $(0, 1)$, and $\{q_n\}$ be a sequence of real numbers. Let the following conditions be satisfied:*

$$r_{n+1} \leq (1 - \vartheta_n) r_n + \vartheta_n q_n,$$

$\sum_{n=0}^{\infty} \vartheta_n = +\infty$ and $\limsup_{n\to\infty} q_n \leq 0$. Then, $\lim_{n\to\infty} r_n = 0$.

Lemma 2.4. *([12]) Let $\{r_n\}$ be a sequence of real numbers which does not decrease at infinity; that is, there is a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ such that, for some $j_0 \in \mathbb{N}$,*

$$r_{n_j} \leq r_{n_j+1} \text{ for all } j \geq j_0.$$

For some α_0 large enough, define a sequence of integers $\{\sigma(n)\}$ by

$$\sigma(n) := \max\{\alpha_0 \leq j \leq n : r_j \leq r_{j+1}\}.$$

Then, $\lim_{n\to\infty} \sigma(n) = \infty$ and for all $n > \alpha_0$, $\max\{r_{\sigma(n)}, r_n\} \leq r_{\sigma(n)+1}$.

3. **Main results**

In this section, we introduce a new inertial algorithm to approximating a solution of the following split variational inclusion problem:

$$\begin{cases} 0 \in Bx^*, \\ A_\alpha x^* \in H_\alpha : 0 \in B_\alpha(A_\alpha x^*). \end{cases} \tag{3}$$

Let Γ denote the solution set of problem (3). Subsequently, we give the main results about our algorithm.

Lemma 3.1. *Let $H, H_\alpha, \alpha = 1, 2, \cdots, N$ be real Hilbert spaces. Assume that $B : H \to 2^H, B_\alpha : H_\alpha \to 2^{H_\alpha}$ are maximal monotone operators, and $A_\alpha : H \to H_\alpha$ is a bounded linear operator with adjoint operator A^*_α. Define the following algorithm :*

Algorithm 1 *Let $u \in H$ be a fixed point and choose two arbitrary initial guesses $x_0, x_1 \in H$. For $n \in \mathbb{N}$, let $\{x_n\}$ be a sequence of H generated by:*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J^B_{\mu_{n,\alpha}}[\beta_n u + (1 - \beta_n) \sum\limits_{\alpha=1}^{N} \rho_{n,\alpha}(y_n - \eta_{n,\alpha} A^*_\alpha (I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n)], \end{cases} \tag{4}$$

where

$$\theta_n := \begin{cases} \min\{\frac{\xi_n}{\|x_n - x_{n-1}\|}, \theta\}, & \|x_n - x_{n-1}\| \neq 0, \\ \theta, & \text{otherwise}; \end{cases}$$

$$\eta_{n,\alpha} := \zeta_{n,\alpha} \frac{\|(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^2}{\|A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}}$$

and the parameters satisfy the following conditions:
(i) $0 < \xi_n < 1, \theta > 0$;
(ii) $\{\rho_{n,\alpha}\} \subset [a, b] \subset (0, 1), \sum\limits_{\alpha=1}^{N} \rho_{n,\alpha} = 1$;
(iii) $\{\beta_n\}, \{\mu_{n,\alpha}\} \subset (0, 1), \min\limits_\alpha\{\liminf\limits_n \mu_{n,\alpha}\} = \mu > 0, \{\zeta_{n,\alpha}\} \subset [c, d] \subset (0, 2)$;
(iv) $\varepsilon_{n,\alpha} > 0, \max\limits_\alpha\{\limsup\limits_n \varepsilon_{n,\alpha}\} = N_1 < \infty$.
Then, for any $p \in \Gamma$, the following inequality holds:

$$\begin{aligned} \|l_n - p\|^2 \le{}& \|x_n - p\|^2 + \theta_n(2\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|^2) \\ &- \sum_{\alpha=1}^{N} \rho_{n,\alpha}\zeta_{n,\alpha}(2 - \zeta_{n,\alpha}) \frac{\|(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^4}{\|A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}}, \end{aligned}$$

where $l_n = \sum\limits_{\alpha=1}^{N} \rho_{n,\alpha}(y_n - \eta_{n,\alpha}A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n)$.

Proof. Take $p = P_\Gamma u$. From the conclusion studied in [6], we have $x \in \Gamma \Leftrightarrow x \in F(J^B_{\mu_{n,\alpha}})$ and $A_\alpha x \in F(J^{B_\alpha}_{\mu_{n,\alpha}})$, which implies that $J^B_{\mu_{n,\alpha}} p = p$ and $(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha p = 0$. By (4), we get

$$\begin{aligned} \|y_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{5}$$

Setting $l_n = \sum\limits_{\alpha=1}^{N} \rho_{n,\alpha}(y_n - \eta_{n,\alpha} A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n)$. For each $\alpha = 1, 2, \cdots, N$, by the convexity of $\|\cdot\|^2$, we obtain

$$\begin{aligned}\|l_n - p\|^2 &= \|\sum_{\alpha=1}^{N} \rho_{n,\alpha}(y_n - \eta_{n,\alpha} A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n) - p\|^2 \\ &\le \sum_{\alpha=1}^{N} \rho_{n,\alpha}\|y_n - \eta_{n,\alpha} A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n - p\|^2. \end{aligned} \tag{6}$$

From (ii) of Lemma 2.1, condition (iv), the definition of $\eta_{n,\alpha}$ and (5), we estimate

$$\begin{aligned}
&\|y_n - \eta_{n,\alpha} A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n - p\|^2 \\
&= \|y_n - p\|^2 - 2\eta_{n,\alpha}\langle A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n, y_n - p\rangle + \eta_{n,\alpha}^2\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 \\
&= \|y_n - p\|^2 - 2\eta_{n,\alpha}\langle (I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n, A_\alpha y_n - A_\alpha p\rangle + \eta_{n,\alpha}^2\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 \\
&= \|y_n - p\|^2 - 2\eta_{n,\alpha}\langle (I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n - (I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha p, A_\alpha y_n - A_\alpha p\rangle \\
&+\eta_{n,\alpha}^2\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 \\
&= \|y_n - p\|^2 - 2\eta_{n,\alpha}\langle (I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n - (I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha p, A_\alpha y_n - A_\alpha p\rangle \\
&+\eta_{n,\alpha}^2\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 \\
&\le \|y_n - p\|^2 - 2\eta_{n,\alpha}\|(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \eta_{n,\alpha}^2(\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}) \\
&\le \|y_n - p\|^2 - 2\zeta_{n,\alpha}\frac{\|(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^4}{\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}} + \zeta_{n,\alpha}^2\frac{\|(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^4}{\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}} \\
&\le \|y_n - p\|^2 - \zeta_{n,\alpha}(2 - \zeta_{n,\alpha})\frac{\|(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^4}{\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}} \\
&\le \|x_n - p\|^2 + \theta_n(2\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|^2) \\
&-\zeta_{n,\alpha}(2 - \zeta_{n,\alpha})\frac{\|(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^4}{\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}}.
\end{aligned} \tag{7}$$

Combining (6) with (7) yields that

$$\begin{aligned}\|l_n - p\|^2 \le\ & \|x_n - p\|^2 + \theta_n(2\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|^2) \\ & -\sum_{\alpha=1}^{N} \rho_{n,\alpha}\zeta_{n,\alpha}(2 - \zeta_{n,\alpha})\frac{\|(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^4}{\|A_\alpha^*(I - J_{\mu_{n,\alpha}}^{B_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}}.\end{aligned} \tag{8}$$

□

Theorem 3.1. *Let $H, H_\alpha, \alpha = 1, 2, \cdots, N$ be real Hilbert spaces. Assume that $B : H \to 2^H, B_\alpha : H_\alpha \to 2^{H_\alpha}$ are two maximal monotone operators. Let $A_\alpha : H \to H_\alpha$ be bounded linear operators with adjoint operators A_α^*. Assume that $\Gamma \neq \emptyset$, conditions (i)-(iv) hold and $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0, \sum_{n=1}^{\infty} \beta_n = +\infty$. Then the sequence $\{x_n\}$ generated by algorithm 1 converges strongly to the point $p = P_\Gamma u$.*

Proof. First, we claim that the sequence $\{x_n\}$ generated by (4) is bounded. Indeed, take $p = P_\Gamma u$, from (4) and (5), we deduce

$$\begin{aligned}
\|x_{n+1} - p\| &= \|J^B_{\mu_{n,\alpha}}[\beta_n u + (1-\beta_n)\sum_{\alpha=1}^{N}\rho_{n,\alpha}(y_n - \eta_{n,\alpha}A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n)] - p\| \\
&\le \|\beta_n u + (1-\beta_n)l_n - p\| \\
&\le \beta_n\|u-p\| + (1-\beta_n)\|l_n - p\| \\
&\le \beta_n\|u-p\| + (1-\beta_n)\|x_n - p\| + (1-\beta_n)\theta_n\|x_n - x_{n-1}\| \\
&\le \beta_n\|u-p\| + (1-\beta_n)\|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
&\le \beta_n\|u-p\| + (1-\beta_n)\|x_n - p\| + \xi_n \\
&\le \max\{\|u-p\|, \|x_n - p\|\} + 1 \\
&\;\vdots \\
&\le \max\{\|u-p\|, \|x_0 - p\|\} + 1,
\end{aligned}$$

which implies that $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{l_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_n - p\| = 0$, where $p = P_\Gamma u$. Indeed, setting $z_n = \beta_n u + (1-\beta_n)l_n$, we have

$$\|z_n - l_n\| = \beta_n\|u - l_n\| \to 0 \text{ as } n \to \infty, \tag{9}$$

it follows that $\{z_n\}$ is bounded. From the definition of z_n and (8), we estimate

$$\begin{aligned}
&\|z_n - p\|^2 \\
&= \|\beta_n u + (1-\beta_n)l_n - p\|^2 \\
&= \|\beta_n(u-p) + (1-\beta_n)(l_n - p)\|^2 \\
&\le (1-\beta_n)\|l_n - p\|^2 + 2\beta_n\langle u-p, z_n - p\rangle \\
&\le (1-\beta_n)\|x_n - p\|^2 + \theta_n(2\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|^2) \\
&\quad +2\beta_n\langle u-p, z_n - p\rangle - (1-\beta_n)\sum_{\alpha=1}^{N}\rho_{n,\alpha}\zeta_{n,\alpha}(2-\zeta_{n,\alpha})\frac{\|(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^4}{\|A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}}.
\end{aligned} \tag{10}$$

Due to (ii) of Lemma 2.1 and (10), one has

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|J^B_{\mu_{n,\alpha}}[\beta_n u + (1-\beta_n)\sum_{\alpha=1}^{N}\rho_{n,\alpha}(y_n - \eta_{n,\alpha}A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n)] - p\|^2 \\
&= \|J^B_{\mu_{n,\alpha}}(z_n) - J^B_{\mu_{n,\alpha}}p\|^2 \\
&\le \|z_n - p\|^2 - \|(I - J^B_{\mu_{n,\alpha}})z_n\|^2 \\
&\le (1-\beta_n)\|x_n - p\|^2 + 2\beta_n\langle u-p, z_n - p\rangle + \theta_n(2\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|^2) \\
&\quad -(1-\beta_n)\sum_{\alpha=1}^{N}\rho_{n,\alpha}\zeta_{n,\alpha}(2-\zeta_{n,\alpha})\frac{\|(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^4}{\|A^*_\alpha(I - J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}} - \|(I - J^B_{\mu_{n,\alpha}})z_n\|^2,
\end{aligned} \tag{11}$$

and consequently

$$r_{n+1} \le (1-\beta_n)r_n + \beta_n q_n, \tag{12}$$

where $r_n = \|x_n - p\|^2$ and $q_n = 2\langle u-p, z_n - p\rangle + \frac{\theta_n}{\beta_n}(2\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\|^2)$.

In addition, we note that

$$q_n = 2\langle u-p, z_n-p\rangle + \frac{\theta_n}{\beta_n}(2\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|^2)$$
$$\leq 2\|u-p\|\|z_n-p\| + \frac{\theta_n}{\beta_n}(2\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|^2),$$

by $\lim_{n\to\infty}\frac{\theta_n}{\beta_n}=0$ and the boundedness of $\{x_n\}$ and $\{z_n\}$, it follows that q_n is bounded.

We divide the rest of the proof into two cases.

Case 1. If the sequence $\{r_n\}$ is decreasing, i.e., for an integer number n_0, the sequence $\{r_n\}$ is decreasing for all $n \geq n_0$. Thus, the sequence $\{r_n\}$ eventually must converges. Moreover, from (11), we have

$$\begin{aligned}
&(1-\beta_n)\sum_{\alpha=1}^{N}\rho_{n,\alpha}\zeta_{n,\alpha}(2-\zeta_{n,\alpha})\frac{\|(I-J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^4}{\|A^*_\alpha(I-J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|^2+\varepsilon_{n,\alpha}} + \|(I-J^{B}_{\mu_{n,\alpha}})z_n\|^2\\
&\leq (1-\beta_n)\|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\beta_n\langle u-p, z_n-p\rangle + \theta_n(2\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|^2)\\
&\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\beta_n\langle u-p, z_n-p\rangle + \theta_n(2\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|^2)\\
&\leq r_n - r_{n+1} + 2\beta_n\langle u-p, z_n-p\rangle + \theta_n(2\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n\|x_n-x_{n-1}\|^2). \qquad (13)
\end{aligned}$$

From conditions (ii), (iii), the boundedness of $\{x_n\},\{z_n\}$ and $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\frac{\theta_n}{\beta_n}=0$, we infer that

$$\lim_{n\to\infty}\|(I-J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\| = \lim_{n\to\infty}\|(I-J^{B}_{\mu_{n,\alpha}})z_n\| = 0, \forall\alpha=1,2,\cdots,N. \quad (14)$$

Furthermore, for every $\alpha=1,2,\cdots,N$, from (i) of Lemma 2.1 and condition (iii), we obtain

$$\|(I-J^{B_\alpha}_{\mu})A_\alpha y_n\| \leq 2\|(I-J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\| \to 0 \text{ as } n\to\infty \quad (15)$$

and

$$\|(I-J^{B}_{\mu})z_n\| \leq 2\|(I-J^{B}_{\mu_{n,\alpha}})y_n\| \to 0 \text{ as } n\to\infty.$$

By the definition of $\{l_n\}$ and (14), we deduce that

$$\begin{aligned}
\|l_n-y_n\| &= \|\sum_{\alpha=1}^{N}\rho_{n,\alpha}\eta_{n,\alpha}A^*_\alpha(I-J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\|\\
&\leq \sum_{\alpha=1}^{N}\rho_{n,\alpha}\|\eta_{n,\alpha}A^*_\alpha(I-J^{B_\alpha}_{\mu_{n,\alpha}})A_\alpha y_n\| \to 0 \text{ as } n\to\infty.
\end{aligned}$$

In view of (9) and (15), it turns out that

$$\|z_n-y_n\| \leq \|z_n-l_n\| + \|l_n-y_n\| \to 0 \text{ as } n\to\infty. \quad (16)$$

Meanwhile, since the sequence $\{z_n\}$ is bounded, there is a subsequence $\{z_{n_j}\}$ of $\{z_n\}$, such that $z_{n_j}\rightharpoonup p^*$ (without loss of generality, we may denote $z_n\rightharpoonup p^*$). It follows from (14) and Lemma 2.2 that $p^*\in F(J^{B}_{\mu_{n,\alpha}})$. On the other hand, since A_α is bounded linear operator, we obtain from (16) and $z_n\rightharpoonup p^*$ that $Ay_n\rightharpoonup Ap^*$. By (14) and Lemma 2.2, it follows that $A_\alpha p^*\in F(J^{B_\alpha}_{\mu_{n,\alpha}})$. Thus $p^*\in\Gamma$.

Now, since $p = P_\Gamma u$, utilizing the characterization of metric projection, we deduce

$$\limsup_{n\to\infty}\langle u-p, z_n-p\rangle = \langle u-p, p^*-p\rangle = \langle u-P_\Gamma u, p^*-P_\Gamma u\rangle \leq 0,$$

this together with $\lim_{n\to\infty}\frac{\theta_n}{\beta_n}=0$, implies that $\limsup_{n\to\infty} q_n \leq 0$. Applying Lemma 2.3 to (12), we obtain $\lim_{n\to\infty} r_n = 0$, i.e., $\lim_{n\to\infty}\|x_n-p\|=0$.

Case 2. If the sequence $\{r_n\}$ does not decrease at infinity. Then, using Lemma 2.4, for large enough $n \geq \alpha_0$, we defined a integer sequence $\{\sigma_n\}$ by

$$\sigma_n := \max\{\alpha_0 \leq j \leq n : r_j \leq r_{j+1}\}.$$

It is easily seen that $\{\sigma_n\}$ is increasing and $\lim_{n\to\infty}\sigma_n = +\infty$. Moreover, for all $n \geq \alpha_0$, $r_{\sigma_n} \leq r_{\sigma_n+1}$. From (12), we have

$$0 \leq r_{\sigma_n+1} - r_{\sigma_n} \leq \beta_{\sigma_n} q_{\sigma_n}.$$

We obtain from $\lim_{n\to\infty}\beta_{\sigma_n}=0$ and the boundedness of q_{σ_n} that

$$\lim_{n\to\infty}(r_{\sigma_n+1}-r_{\sigma_n}) = 0. \tag{17}$$

By using the similar arguments as case 1 above, we obtain

$$r_{\sigma_n+1} \leq (1-\beta_{\sigma_n})r_{\sigma_n} + \beta_{\sigma_n}q_{\sigma_n},$$

$$\|(I-J_\mu^{B_\alpha})A_\alpha y_{\sigma_n}\| \to 0 \text{ and } \|(I-J_\mu^{B})z_{\sigma_n}\| \to 0 \text{ as } n\to\infty,$$

and $\limsup_{n\to\infty} q_{\sigma_n} \leq 0$, this together with $r_{\sigma_n} \leq r_{\sigma_n+1}$ and $\lim_{n\to\infty}\beta_{\sigma_n}=0$, we infer that

$$r_{\sigma_n} \leq \beta_{\sigma_n}q_{\sigma_n} \to 0 \text{ as } n\to\infty. \tag{18}$$

It follows from (17), (18) and Lemma 2.4 that

$$0 \leq r_n \leq \max\{r_{\sigma_n}, r_n\} \leq r_{\sigma_n+1} \to 0 \text{ as } n\to\infty.$$

That is, $\lim_{n\to\infty}\|x_n-p\|=0$. This completes the proof.

□

Remark 3.1. *Compared with Theorem 3.2 of Sumalai et al.* [8]*, our Theorem 3.1 extends, improves and develops it in the following aspects:*
(i) Our iterative scheme is more general than it in [8]*. Especially, a self-adaptive inertial scheme is added to construct our iteration process, which is not applied in* [8]*.*
(ii) There is a gap in the proof of Theorem 3.2 in [8]*. That is, after (26) in* [8]*, from the boundedness of the sequences* $\{z_n\}$ *and* $\{l_n\}$*, we can not obtain that* $\|z_{n_j+1}-l_{n_j}\| = \vartheta_{n_j}\|v-l_{n_j}\|$*. So we modify the proof, which makes the results more applicable and valid.*

4. **Application**

In this section, by applying Lemma 3.1 and Theorem 3.1, we prove a strong convergence theorem for solving a split feasibility problem.

Let $H, H_\alpha, \alpha = 1, 2, \cdots, N$ be Hilbert spaces and C, Q_α be nonempty closed convex subsets of H and H_α respectively. Suppose that $A_\alpha : H \to H_\alpha$ is a bounded linear operator and A_α^* is the adjoint of A_α. The split feasibility problem (SFP) is the problem of finding a point with the property:

$$x \in C \text{ and } A_\alpha x \in Q_\alpha. \tag{19}$$

We denote the solution of SFP (19) by Γ.

Theorem 4.1. *Let P_C be the metric projection from H onto C and P_{Q_α} be the metric projection from H_α onto Q_α. Choose $u \in H$ and arbitrary initial guesses $x_0, x_1 \in H$. For $n \in \mathbb{N}$, let $\{x_n\}$ be a sequence of H generated by:*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = P_C[\beta_n u + (1-\beta_n)\sum\limits_{\alpha=1}^{N} \rho_{n,\alpha}(y_n - \eta_{n,\alpha}A_\alpha^*(I - P_{Q_\alpha})A_\alpha y_n)], \ n \in \mathbb{N}, \end{cases}$$

where

$$\theta_n := \begin{cases} \min\{\frac{\xi_n}{\|x_n - x_{n-1}\|}, \theta\}, \ \|x_n - x_{n-1}\| \neq 0, \\ \theta, \ \text{otherwise}; \end{cases}$$

$$\eta_{n,\alpha} := \zeta_{n,\alpha}\frac{\|(I - P_{Q_\alpha})A_\alpha y_n\|^2}{\|A_\alpha^*(I - P_{Q_\alpha})A_\alpha y_n\|^2 + \varepsilon_{n,\alpha}}$$

and the following conditions hold:

(i) $0 < \xi_n < 1, \theta > 0, \{\zeta_{n,\alpha}\} \subset [c, d] \subset (0, 2)$;

(ii) $\{\rho_{n,\alpha}\} \subset [a, b] \subset (0, 1), \sum\limits_{\alpha=1}^{N} \rho_{n,\alpha} = 1$;

(iii) $\{\beta_n\} \subset (0, 1), \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0, \sum_{n=1}^{\infty} \beta_n = +\infty$;

(iv) $\varepsilon_{n,\alpha} > 0, \max\limits_{\alpha}\{\limsup\limits_{n} \varepsilon_{n,\alpha}\} = N_1 < \infty$.

Assume that $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ generated above converges strongly to the point $p = P_\Gamma u$.

Proof. Taking $J^B_{\mu_{n,\alpha}} = P_C$ and $J^{B_\alpha}_{\mu_{n,\alpha}} = P_{Q_\alpha}$ in (4), by using the same method of Lemma 3.1 and Theorem 3.1, we obtain the desired conclusion directly. □

5. **Conclusions**

We not only extend the iterative algorithm of [8] by adding a self-adaptive inertial scheme to construct our iteration process but also modify the proof of Theorem 3.2 in [8], which makes our results more applicable and valid. At last, we give an application of the modified algorithm.

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REFERENCES

[1] *Y. Censor, T. Elfving*, A multi projection algorithm using Bregman projections in a product space, Numer. Algorithms **8**(1994), 221-239 .

[2] *Y. Censor, T. Bortfeld, B. Martin, et al.*, A unified approach for inversion problems in intensity modulated radiation therapy, Phys. Med. Biol., **51**(2006), 2353-2365.

[3] *Y. Censor, A. Gibali, S. Reich*, Algorithms for the split variational inequality problem, Numer. Algorithms, **59**(2012), 301-323.

[4] *P.L. Combettes*, The convex feasibility problem in image recovery, Adv. Imag. Electron Phys., **95**(1996), 155-453.

[5] *C. Byrne*, Iterative oblique projection onto convex sets and the split feasibility problem, Inverse Prob. **8**(2002), 441-453.

[6] *A. Moudafi*, Split monotone variational inclusions, J. Optim. Theory Appl., **150**(2011), 275-283.

[7] *C. Byrne, Y. Censor, A. Gibali, et al.*, Weak and strong convergence of algorithms for the split common null point problem, J. Nonlinear Convex Anal., **13**(2012), 759-775.

[8] *P. Sumalai, J. Abubakar, P.Kumam, et al.*, A unified scheme for solving split inclusions with applications, Math. Meth. Appl. Sci., **46**(2023), 14622-14639.

[9] *S. Reich, T. M. Tuyen*, Iterative methods for solving the generalized split common null point problem in Hilbert spaces, Optimization, **69**(2020), 5, 1013-1038.

[10] *K. Goebel, W. A. Kirk*, Topics in metric fixed point theory, Cambridge University Press, 1990.

[11] *H.K. Xu*, Strong convergence of an iterative method for nonexpansive and accretive operators, J. Math. Anal. Appl., **314**(2006), 2, 631-643.

[12] *P.E. Maing, A. Moudafi*, Convergence of new inertial proximal methods for dc programming, SIAM J. Optim., **19**(2008), 1, 397-413.