U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 1, 2021 ISSN 1223-7027

ITERATED F-CONTRACTIONS IN b-METRIC SPACES

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The aim of this paper is to introduce the classes of iterated F-contraction and fundamentally F-contractive type mappings in a b-metric space and obtain some fixed point existence results for these mappings, with some examples in support of the results.

Keywords: b-metric space, iterated contraction, iterated F-contraction, fundamentally F-contractive type mappings.

MSC2020: 47H10, 54H25.

1. Introduction

The study of fixed point theory has led to numerous generalizations of the underlying space with a view to generalize or extend existing fixed point results (one may refer to [2], [5], [7], [17] and the references therein). One such generalization of the metric space in the Banach fixed point theory is a b-metric space. The notion of b-metric was first introduced by Bakhtin [2] in 1989 and formally defined by Czerwik [5] in 1993. Since then, numerous research have been done on b-metric fixed point theory (one may refer to [1] [4], [8], [9], [10], [11], [16] and the references therein).

The notion of *iterated contraction* was first introduced in 1970 by Ortega and Rheinboldt [14], where the same is defined for self mappings on $\mathbb{R}^n$. As mentioned in their book, an iterated contraction need not be continuous nor its fixed point be unique. Nevertheless, iterated contractions turn out to be quite useful in considering certain iterative processes. Recently, Ghoncheh and Razani [7] introduced the class of *fundamentally nonexpansive* mappings in a metric space.

In 2012, Wardowski [17] introduced the class of F-contraction mappings (which is a generalization of contraction mappings) and derived some fixed point results with applications. Based on the definition of F-contraction mappings, in [8], the authors defined F-contrctive type mappings and Kannan F-contrctive type mappings in a b-metric space and derived some fixed point results.

Iterated F-contraction mappings and fundamental F-contractive type mappings are defined on b-metric spaces. Some of their properties are discussed and fixed point existence results are also obtained in this paper.

2. Preliminaries

In this section, some basic definitions and results required for our subsequent discussion are briefly stated, starting with the definition of a b-metric space.

Definition 2.1. [5] *Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0, \infty)$ is called b-metric if it satisfies the following properties.*

(1) $d(x, y) = 0$ *if and only if* $x = y$;

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(2) $d(x,y)=d(y,x)$; *and*
(3) $d(x,z)\le s[d(x,y)+d(y,z)]$, *for all* $x,y,z\in X$.
The triplet (X,d,s) *is called a b-metric space with coefficient* s.

A b-metric can be discontinuous unlike a metric. One may refer to Example 2.6 of [13]. Many examples of b-metric spaces are found in the literature. One may refer to [3] and [15] for some classical examples. Nonetheless, the following is an example of a b-metric space.

Example 2.1. Let $X=\{x_1,x_2,x_3,x_4\}$ and define a function $d:X\times X\longrightarrow[0,\infty)$ as follows.

$$d(x_i,x_i)=0 \qquad \text{for } i=1,2,3,4;$$
$$d(x_1,x_2)=d(x_2,x_1)=2; \qquad d(x_2,x_3)=d(x_3,x_2)=\frac{1}{4};$$
$$d(x_1,x_3)=d(x_3,x_1)=\frac{3}{2}; \qquad d(x_1,x_4)=d(x_4,x_1)=2;$$
$$d(x_2,x_4)=d(x_4,x_2)=d(x_3,x_4)=d(x_4,x_3)=\frac{5}{2}.$$

Then d is a b-metric on X with coefficient $s=\frac{3}{2}$ and (X,d,s) is a b-metric space.

The following lemma is useful in overcoming some of the problems due to the possible discontinuity of the b-metric d in showing the existence of fixed points.

Lemma 2.1. [12] *Let* (X,d) *be a b-metric space with coefficient* $s\ge 1$ *and* $\{x_n\}$ *be a convergent sequence in* X *with* $\lim_{n\to\infty}x_n=x$. *Then for all* $y\in X$

$$s^{-1}d(x,y)\le\lim_{n\to\infty}\inf d(x_n,y)\le\lim_{n\to\infty}\sup d(x_n,y)\le sd(x,y).$$

From the above lemma, it follows that if $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty}x_n=z$ for some $z\in X$, then

$$\lim_{n\to\infty}d(x_n,z)=0.$$

The following is the definition of iterated contraction in a metric space setting.

Definition 2.2. [6] *Let* (X,d) *be a b-metric space. A self mapping* $T:X\longrightarrow X$ *is said to an* iterated contraction *if there exists* $\alpha\in[0,1)$ *such that for every* $x\in X$,

$$d(T^2x,Tx)\le\alpha d(Tx,x).$$

This definition naturally extends to a b-metric space (X,d,s) with the relation

$$sd(T^2x,Tx)\le\alpha d(Tx,x).$$

An iterated contraction need not have a fixed point, as shown in the following example.

Example 2.2. Consider the b-metric space (X,d,s) with $X=[0,1]$ and the b-metric d given by $d(x,y)=|x-y|^2$ for all $x,y\in X$. Let $T:X\longrightarrow X$ be defined by

$$Tx=\begin{cases}1, & x=0\\ \frac{1}{2}x^2, & x\in(0,1]\end{cases}$$

Then

$$d(Tx,T^2x)=\left|\frac{x^2}{2}-\frac{x^4}{8}\right|^2=\frac{1}{4}\left|x^2-\left(\frac{x^2}{2}\right)^2\right|^2=\alpha\left|x-\frac{x^2}{2}\right|^2\le\alpha d(x,Tx)$$

since $\alpha=\frac{1}{4}\left(x+x^2/2\right)^2<1$ and thus T is an iterated contraction with discontinuity at $x=0$, which is fixed point free.

The b-metric space in the above example is compact. However, for continuous iterated contractions, with the help of Lemma 2.1, the existence of a fixed point in a complete b-metric space can be established following the usual proofs of Banach Contraction principle.

Theorem 2.1. *Let (X, d, s) be a complete b-metric space and $T : X \longrightarrow X$ be a continuous mapping such that for all $x, y \in X$,*

$$sd(T^2x, Tx) \leq \alpha d(Tx, x)$$

for some $\alpha \in [0, 1)$. Then the sequence $\{T^n x\}$ converge to a fixed point of T for each $x \in X$.

The following lemma is useful in the proof of the existence of fixed point for iterated F-contraction mappings in a complete b-metric space.

Lemma 2.2. [10] *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a self mapping. If $\{x_n\}$ is a sequence in X with $x_{n+1} = Tx_n$, satisfying*

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$$

for all $n = 0, 1, 2, \dots$ with $k \in [0, 1)$. Then $\{x_n\}$ is a Cauchy sequence.

3. Iterated F-contraction mappings

The following is a fixed point existence result for iterated contraction mappings which generalizes a result (Theorem 5.1) given in [6]. One may also refer to Corollary 3.2 below for a generalization of the theorem to a b-metric space.

Theorem 3.1. *Let X be a compact b-metric space with the b-metric being continuous and $T : X \longrightarrow X$ be a continuous mapping such that there exist a positive real number $\alpha < 1$ satisfying*

$$f\big(d(T^2x, Tx)\big) \leq \alpha f\big(d(Tx, x)\big) \tag{1}$$

for every $x \in X$ and for any continuous function $f : (0, \infty) \longrightarrow (0, \infty)$. Then T has a fixed point.

Proof. Because of the continuity of T, the real valued function $N : X \longrightarrow (0, \infty)$ defined by $N(x) = f\big(d(x, Tx)\big)$ for all $x \in X$, is continuous. Since X is compact, N attains its minimum at, say, $x_0 \in X$. Then $Tx_0 = x_0$.

For, if $x_0 \neq Tx_0$ then (1) implies

$$N(Tx_0) = f\big(d(Tx_0, T^2x_0)\big) < f\big(d(x_0, Tx_0)\big) = N(x_0),$$

which contradicts the fact that x_0 is a minimum point of N. □

The above result have interesting consequences. For instance, if $f(x) = \frac{1}{x}$, then we have the following corollary.

Corollary 3.1. *Let X be a compact b-metric space (the b-metric being continuous) and $T : X \longrightarrow X$ be a continuous mapping such that there exist a positive real number $\beta > 1$ satisfying*

$$\beta d(Tx, x) \leq d(T^2x, Tx) \tag{2}$$

for every $x \in X$. Then T has a fixed point.

Trivial examples of functions satisfying the relation (2) are the constant functions and the function $T : X \longrightarrow X$ defined by $Tx = x$ for all $x \in X$.

Example 3.1. Consider the b-metric space (X, d), where $X = \{1, 2, 3, 4\}$ and $d : X \longrightarrow [0, \infty)$ is defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ x + y, & \text{if } x \neq y \end{cases}$$

Then (X, d) is a compact b-metric space, and the metric d is also continuous. Consider the function $T : X \longrightarrow X$ defined by $Tn = n + 1$ for all $n = 1, 2, 3$ and $T4 = 4$. Then one can easily check that for $\beta = \frac{8}{7}$,

$$\beta d(Tn, n) \leq d(T^2 n, Tn)$$

for all $n \in X$ and hence by Corollary 3.1, T has a fixed point which by inspection is $n = 4$.

Based on the definition of F-contraction, we define *iterated F-contraction mappings* and prove a fixed point result in a complete b-metric space.

Definition 3.1. *Let $F : (0, \infty) \longrightarrow \mathbb{R}$ be a mapping satisfying the following conditions.*

(F1) *F is strictly increasing;*

(F2) *For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*

(F3) *There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

(F4) *Let $s \geq 1$ be a real number. For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$ for all $n \in \mathbb{N}$ and some $\tau > 0$, then*

$$\tau + F(s^n \alpha_n) \leq F\left(s^{n-1}\alpha_{n-1}\right) \qquad \text{for all } n \in \mathbb{N}$$

For a b-metric space (X, d, s), a mapping $T : X \longrightarrow X$ is said to be an iterated F-contraction *if there exists $\tau > 0$ such that $d(Tx, x) > 0$ implies*

$$\tau + F\left(sd(T^2x, Tx)\right) \leq F(d(Tx, x)) \qquad \text{for all } x \in X.$$

Some examples of functions satisfying conditions $F1 - F4$ are $F_1(x) = \log x$, $F_2(x) = x + \log x$, $F_3 = \log(x + x^2)$, etc (one may refer to [4, 12]).

We note here that F-contraction mapping is an iterated F-contraction mapping. Moreover, an iterated contraction is also an iterated F-contraction. This can be seen by putting $F(x) = \log x$ in the definition of iterated F-contraction. However, since every F-contraction is a continuous mapping (refer to [17]) and an iterated F-contraction need not be continuous, it follows that an iterated F-contraction need not be an F-contraction.

Further, it may be noted that F-contractive type mappings are iterated F-contraction. This can be easily seen by replacing x by Tx and y by x in the definition of F-contractive type mappings ([8], Definition 1.6). It is also easily seen that an iterated contraction is necessarily an iterated F-contraction.

Theorem 3.2. *Let X be a complete b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a continuous mapping such that there exist a positive real number τ satisfying*

$$\tau + F\big(sd(T^2x, Tx)\big) \leq F\big(d(Tx, x)\big) \tag{3}$$

for every $x \in X$. Then T has a fixed point.

Proof. For an arbitrary (but fixed) point $x_0 \in X$, one can construct a sequence $\{x_n\}$ in X with $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ Let $\mu_n = d(x_n, x_{n+1})$, $n = 0, 1, 2, \ldots$

Without loss of generality, assume that $\mu_n > 0$ for all $n = 0, 1, 2, \ldots$ By (3), the following holds for all $n \in \mathbb{N}$:

$$F(s\mu_n) \leq F(\mu_{n-1}) - \tau,$$

and by condition (F4), we have

$$F(s^n\mu_n) \leq F(s^{n-1}\mu_{n-1}) - \tau,$$

which by induction yields

$$F(s^n\mu_n) \leq F(s^{n-1}\mu_{n-1}) - \tau \leq F(s^{n-2}\mu_{n-2}) - 2\tau \leq \cdots \leq F(\mu_0) - n\tau. \tag{4}$$

In the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} F(s^n \mu_n) = -\infty \qquad \text{so that} \qquad \lim_{n\to\infty} s^n \mu_n = 0.$$

By condition (F3), there exists $k \in (0,1)$ such that $\lim_{n\to\infty}(s^n\mu_n)^k F(s^n\mu_n) = 0$. Multiplying (4) by $(s^n\mu_n)^k$, we get

$$0 \leq (s^n\mu_n)^k F(s^n\mu_n) + n(s^n\mu_n)^k\tau \leq (s^n\mu_n)^k F(\mu_0).$$

Taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} n(s^n\mu_n)^k = 0$.

Following the proof of Theorem 3.2 in [12], it is seen that $\{x_n\}$ is a Cauchy sequence. The completeness of X then ensures the existence of a limit point of $\{x_n\}$ in X, that is, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Finally, Lemma 2.1 and the continuity of T implies

$$d(z, Tz) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

and the proof is complete. □

As a corollary to Theorem 3.2, we get a fixed point result for iterated contractions in a complete b-metric space (X, d, s) with the b-metric d not necessarily continuous.

Corollary 3.2. *Let (X, d, s) be a complete b-metric space and $T : X \longrightarrow X$ be a continuous mapping such that there exist a positive real number $\alpha < 1$ satisfying*

$$sd(T^2x, Tx) \leq \alpha d(Tx, x) \tag{5}$$

for every $x \in X$ and for any continuous function $f : (0, \infty) \longrightarrow (0, \infty)$. Then T has a fixed point.

Proof. The proof follows from Theorem 3.2 by noting that for $F(x) = \log x$ the relation (3) reduces to

$$\frac{sd(T^2x, Tx)}{d(Tx, x)} \leq e^{-\tau} \qquad \text{or,} \qquad d(T^2x, Tx) \leq \alpha d(Tx, x)$$

for some $\alpha < 1$. □

4. Fundamentally F-contractive type mappings

Recently, Ghoncheh and Razani [7] introduced a class of mappings called *fundamentally nonexpansive* mappings in a metric space which generalizes Suzuki mappings, where a mapping $T : X \longrightarrow X$, X a metric space, is said to be *fundamentally nonexpansive* if

$$d(T^2x, Ty) \leq d(Tx, y)$$

for all $x, y \in X$.

In a similar manner, *fundamental contraction* mappings may be defined and extended to b-metric spaces.

Definition 4.1. *Let (X, d, s) be a b-metric space. A mapping $T : X \longrightarrow X$ is said to be a* fundamental contraction *if there exists $\alpha < 1$ such that*

$$sd(T^2x, Ty) \leq \alpha d(Tx, y)$$

for all $x, y \in X$.

Similarly, a mapping $T : X \longrightarrow X$ is said to be a fundamental F-contraction *if there exists a positive number τ such that $d(Tx, y) \neq 0$ implies*

$$\tau + F\big(sd(T^2x, Ty)\big) \leq F\big(d(Tx, y)\big) \tag{6}$$

for all $x, y \in X$.

We note here that a fundamental contraction is a fundamental F-contraction. Also, a fundamental F-contraction is an iterated F-contraction. This follows from putting $y = x$ in (6). However, an iterated F-contraction need not be a fundamental F-contraction as shown in the following example.

Example 4.1. Consider the function T as defined in Example 2.2 which is an iterated F-contraction. Now, for $x = 0$ and $y = \frac{1}{2}$, we have

$$d\left(T^2 0, T\frac{1}{2}\right) = \left|\frac{1}{2} - \frac{1}{8}\right|^2 = \frac{3^2}{8^2} \qquad \text{and} \qquad d\left(T0, \frac{1}{2}\right) = \left|\frac{1}{2} - \frac{1}{8}\right|^2 = 0$$

so that $\tau + F\big(sd(T^2x, Ty)\big) \not\leq F\big(d(Tx, y)\big)$ for all $x, y \in X$, and hence T is not a fundamental F-contraction.

As stated above, a fundamental F-contraction is necessarily an iterated F-contraction and therefore, the above fixed point results on iterated F- contractions holds true for this class of mappings too.

Following the definition of F-contractive type mappings [8], we define the following class of mappings.

Definition 4.2. *For a b-metric space (X, d, s), a mapping $T : X \longrightarrow X$ is said to be fundamentally F-contractive type if there exists $\tau > 0$ such that $d(x, Tx)d(y, Ty) \neq 0$ implies*

$$\tau + F\left(sd(T^2x, Ty)\right) \leq \frac{1}{2}\Big\{F\left(d(Tx, y)\right) + F\left(d(y, Ty)\right)\Big\} \tag{7}$$

and $d(x, Tx)d(y, Ty) = 0$ implies

$$\tau + F\left(sd(T^2x, Ty)\right) \leq \frac{1}{2}\Big\{F\left(d(x, Ty)\right) + F\left(d(Tx, y)\right)\Big\} \tag{8}$$

for all $x, y \in X$.

We note here that on putting $y = x$ in the above definition, we get the definition of iterated F-contraction and thus the existence of a fixed point is ensured by Theorem 3.2. The following is a unique fixed point existence result for a fundamentally F-contractive type mapping in a complete b-metric space without assuming the continuity of the mapping.

Theorem 4.1. *Let (X, d, s) be a complete b-metric space and $T : X \longrightarrow X$ be a fundamentally F-contractive type mapping. Then T have a unique fixed point.*

Proof. For an arbitrary (but fixed) point $x_0 \in X$, one can construct a sequence $\{x_n\}$ in X with $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ Let $\mu_n = d(x_n, x_{n+1})$, $n = 0, 1, 2, \ldots$

Without loss of generality, assume that $\mu_n > 0$ for all $n = 0, 1, 2, \ldots$ By (7), putting $x = x_n$ and $y = x_{n+2}$ the following holds for all $n \in \mathbb{N}$:

$$F(s\mu_{n+3}) \leq F(\mu_{n+2}) - 2\tau,$$

and by condition (F4), we have

$$F(s^n\mu_n) \leq F(s^{n-1}\mu_{n-1}) - 2\tau,$$

which by induction yields

$$F(s^n\mu_n) \leq F(s^{n-1}\mu_{n-1}) - 2\tau \leq F(s^{n-2}\mu_{n-2}) - 4\tau \leq \cdots \leq F(\mu_0) - 2n\tau.$$

As in the proof of Theorem 3.2, we see that $\{x_n\}$ is a Cauchy sequence. Since (X, d, s) is a complete b-metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Now, from (7) the following holds for all $n \in \mathbb{N}$.

$$F\big(d(x_{n+2}, Tz)\big) = F\big(d(T^2 x_n, Tz)\big) \le \frac{1}{2}\Big\{F\big(d(Tx_n, z)\big) + F\big(d(z, Tz)\big)\Big\}$$
$$= \frac{1}{2}\Big\{F\big(d(x_{n+1}, z)\big) + F\big(d(z, Tz)\big)\Big\}.$$

Now, by Lemma 2.1, $\lim_{n\to\infty} d(x_{n+1}, z) = 0$ and $\lim_{n\to\infty} d(x_{n+2}, Tz) = d(z, Tz)$. Hence taking the limit as $n \to \infty$ in the above relation, we get

$$F\big(d(z, Tz)\big) = -\infty \qquad \text{or,} \qquad d(z, Tz) = 0,$$

showing that z is a fixed point of T.

Finally, if possible let $w \neq z$ be another fixed point of T, then by (8) we get

$$\tau + F\big(d(w, z)\big) \le F\big(d(w, z)\big)$$

implying $\tau \le 0$, a contradiction and hence the fixed point is unique. This completes the proof. □

Example 4.2. Consider the complete metric space (X, d) where $X = [0, 1] \cup [2, \infty)$ and $d(x, y) = \min\{x + y, 2\}$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, and the discontinuous function T defined by

$$Tx = \begin{cases} \frac{1}{2}, & 0 \le x < 1 \\ 0, & x = 1 \\ \frac{1}{2} - \frac{1}{x}, & x \ge 2 \end{cases}$$

as in Example 2.4 [8]. First we note that for the function $F(x) = \log x$, conditions (7) and (8) reduces respectively to

$$sd(T^2x, Ty) \le e^{-2\tau} d(Tx, y) d(y, Ty) \tag{9}$$

and

$$sd(T^2x, Ty) \le e^{-\tau} H(x, y) \tag{10}$$

where $H(x, y) = \max\Big\{d(x, y), d(x, Ty), d(Tx, y)\Big\}$.

Now, for $x, y \ge 2$ and $x \neq y$, $d(x, Tx)d(y, Ty) \neq 0$ and we have

$$d(T^2x, Ty) = d\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{y}\right) = \min\left\{1 - \frac{1}{y}, 2\right\} = 1 - \frac{1}{y} \le 1$$

and

$$d(Tx, y)d(y, Ty) = d\left(\frac{1}{2} - \frac{1}{x}, y\right) d\left(y, \frac{1}{2} - \frac{1}{y}\right) = 4$$

so that (9) is satisfied for $e^{-2\tau} = \frac{1}{4}$ or $\tau = \log 2$.

And, if $\frac{1}{2} \neq x \in [0, 1)$ and $y \ge 2$, $d(x, Tx)d(y, Ty) \neq 0$ and we have

$$d(T^2x, Ty) = d\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{y}\right) = \min\left\{1 - \frac{1}{y}, 2\right\} = 1 - \frac{1}{y} \le 1$$

and

$$d(Tx, y)d(y, Ty) = d\left(\frac{1}{2}, y\right) d\left(y, \frac{1}{2} - \frac{1}{y}\right) = 4$$

so that (9) is satisfied for $\tau = \log 2$.

Again, if $\frac{1}{2} \neq x \in [0, 1)$

$$d(T^2x, T1) = d\left(\frac{1}{2}, 0\right) = \min\left\{\frac{1}{2} + 0, 2\right\} = \frac{1}{2}$$

and

$$d(Tx,1)d(1,T1) = d\left(\frac{1}{2},1\right)d\left(1,0\right) = \frac{3}{2}$$

so that (9) is satisfied for $e^{-2\tau} = \frac{1}{3}$ or $\tau = \log\sqrt{3}$.

Also, for $x \geq 2$

$$d(T^2x,T1) = d\left(\frac{1}{2}-\frac{1}{x},0\right) = \min\left\{\frac{1}{2}-\frac{1}{x},2\right\} \leq \frac{1}{2}$$

and

$$d(Tx,1)d(1,T1) = d\left(\frac{1}{2}-\frac{1}{x},1\right)d\left(1,0\right) = \min\left\{\frac{3}{2}-\frac{1}{x},2\right\} \geq 1$$

so that (9) is satisfied for $e^{-2\tau} = \frac{1}{2}$ or $\tau = \log\sqrt{2}$.

Further, for $0 \leq x \leq 1$

$$d\left(T^2x,T\frac{1}{2}\right) = d\left(\frac{1}{2},\frac{1}{2}\right) = 0$$

and finally, for $x \geq 2$

$$d\left(T^2x,T\frac{1}{2}\right) = d\left(\frac{1}{2},\frac{1}{2}\right) = 0$$

so that (10) is trivially satisfied for all $x, y \in X$ with $d(Tx,y)d(y,Ty) = 0$. Thus for $\tau = \log\sqrt{2}$, (9) is satisfied for all $x, y \in X$ with $d(Tx,y)d(y,Ty) \neq 0$ and (10) is satisfied for all $x, y \in X$ with $d(Tx,y)d(y,Ty) = 0$.

Hence all the conditions of Theorem 4.1 are satisfied and therefore T has a unique fixed point, which is $x = \frac{1}{2}$.

Definition 4.2 can be obtained from the definition of *F-contractive type mappings* ([8], Definition 1.6) as F-contractive type mappings are iterated F-contractions and the fact that, for any positive numbers a and b, either

$$\frac{1}{3}(2a+b) \leq \frac{1}{2}(a+b) \qquad \text{or} \qquad \frac{1}{3}(a+2b) \leq \frac{1}{2}(a+b). \tag{11}$$

For, replacing x by Tx in condition (1) of Definition 1.6 [8], we get,

$$\tau + F\big(d(T^2x,Ty)\big) \leq \frac{1}{3}\Big\{F\big(d(Tx,y)\big) + F\big(d(T^2x,Tx)\big) + F\big(d(y,Ty)\big)\Big\}$$

If $F\big(d(T^2x,Tx)\big) \leq F\big(d(Tx,y)\big)$ or $F\big(d(T^2x,Tx)\big) \leq F\big(d(y,Ty)\big)$, we get (7). If $F\big(d(T^2x,Tx)\big) > F\big(d(Tx,y)\big)$ and $F\big(d(T^2x,Tx)\big) > F\big(d(y,Ty)\big)$, we get a contradiction to the fact that F-contractive type mappings are iterated F-contractions.

Now, if x is a fixed point of T, condition (2) of Definition 1.6 [8] gives

$$\tau + F\big(d(T^2x,Ty)\big) \leq \frac{1}{3}\Big\{2F\big(d(Tx,y)\big) + F\big(d(x,Ty)\big)\Big\}$$

which by (11) implies (8). Similar result is obtained if y is a fixed point. We have thus proved the following proposition.

Proposition 4.1. *Every F-contractive type mapping is a fundamentally F-contractive type mapping.*

Finally, we consider the following example to show that a fundamentally F-contractive type mapping need not be an F-contractive type mapping.

Example 4.3. Consider the complete metric space (X, d) where $X = [0, 1] \cup [2, \infty)$ with the metric d on X defined by

$$d(x, y) = \begin{cases} \min\{x + y, 1.75\}, & x \neq y \\ 0, & x = y \end{cases}$$

Consider the function $T : X \longrightarrow X$ defined by

$$Tx = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{1}{x}, & x \geq 2 \end{cases}$$

For $x, y \in [0, 1]$ with $x \neq 1$ and $y \neq 1$, we have $F\big(d(T^2x, Ty)\big) = F\big(d(1, 1)\big) = -\infty$, and (7) is trivially satisfied.

For $x, y \geq 2$, we have

$$F\big(d(T^2x, Ty)\big) = F\left(d\left(1, \frac{1}{y}\right)\right) = F\left(1 + \frac{1}{y}\right) \leq F\big(1.5\big)$$

and

$$\frac{1}{2}\left\{F\big(d(Tx, y)\big) + F\big(d(y, Ty)\big)\right\} = \frac{1}{2}\left\{F\left(\frac{1}{x} + y\right) + F\left(y + \frac{1}{y}\right)\right\} = F\big(1.75\big)$$

and since F is strictly increasing, we can find $\tau > 0$ such that (7) holds.

For $1 \neq x \in [0, 1]$ and $y \geq 2$ or vice versa, we have

$$F\big(d(T^2x, Ty)\big) = F\left(d\left(1, \frac{1}{y}\right)\right) = F\left(1 + \frac{1}{y}\right) \leq F\big(1.5\big)$$

and

$$\frac{1}{2}\left\{F\big(d(Tx, y)\big) + F\big(d(y, Ty)\big)\right\} = \frac{1}{2}\left\{F\left(1 + y\right) + F\left(y + \frac{1}{y}\right)\right\} = F\big(1.75\big)$$

and hence (7) holds for some $\tau > 0$.

Also, for $y \in [0, 1]$, $F\big(d(T^21, Ty)\big) = F\left(d\left(1, 1\right)\right) = -\infty$, and (8) is trivially satisfied.

Finally, for $y \geq 2$, we have

$$F\big(d(T^21, Ty)\big) = F\left(d\left(1, \frac{1}{y}\right)\right) = F\left(1 + \frac{1}{y}\right) \leq F\big(1.5\big)$$

and

$$\frac{1}{2}\left\{F\big(d(1, Ty)\big) + F\big(d(T1, y)\big)\right\} = \frac{1}{2}\left\{F\left(1 + \frac{1}{y}\right) + F\left(1 + y\right)\right\} > F\big(1.5\big).$$

Since F is strictly increasing, we can find some $\tau > 0$ for which (8) holds true. Thus, T is a fundamentally F-contractive type mapping.

We shall show that T is not an F-contractive type mapping. As in Example 1.7 [8], for $F = \log x$ the F-contractive conditions are:
$d(x, Tx)d(y, Ty) \neq 0$ implies

$$s^3 d(Tx, Ty)^3 \leq e^{-3\tau} d(x, y)d(x, Tx)d(y, Ty) \tag{12}$$

and $d(x, Tx)d(y, Ty) = 0$ implies

$$s^3 d(Tx, Ty)^3 \leq e^{-3\tau} d(x, y)d(x, Ty)d(y, Tx). \tag{13}$$

Now, for $x = 0$ and $y = 2$, we have

$$d(Tx, Ty)^3 = d\left(1, \frac{1}{2}\right)^3 = \left(\frac{3}{2}\right)^3 = 3.375$$

and

$$e^{-3\tau}d(x,y)d(x,Tx)d(y,Ty) = e^{-3\tau}d(0,2)d(0,1)d\left(2,\frac{1}{2}\right)$$
$$= e^{-3\tau} \times 1.75 \times 1 \times 1.75 = e^{-3\tau} \times 3.0625.$$

Thus equation (12) is not satisfied for any $\tau > 0$ and hence T is not an F-contractive type mapping.

5. **Conclusion**

We introduced new classes of mappings called iterated F-contraction and fundamentally F-contractive type mappings in a b-metric space and obtained some fixed point existence results for such classes of mappings.

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