ON GENERALIZED NOTIONS OF APPROXIMATE AMENABILITY AND BIFLATNESS ON BANACH ALGEBRAS

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In this paper, the notions of approximate (σ,τ)-amenability and approximate (σ,τ)-biflatness for a Banach algebra $\mathcal{A}$ are introduced, where σ and τ are bounded homomorphisms on $\mathcal{A}$. Moreover, some known hereditary properties concerning the (approximate) amenability of Banach algebras are studied. Some examples regarding the main results are given as well. Furthermore, an example shows that the class of approximately (σ,τ)-amenable Banach algebras is large than the class of approximately amenable Banach algebras.

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1. Introduction

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-module is inner, that is $H^1(\mathcal{A},X^*)=\{0\}$ for every Banach $\mathcal{A}$-module X, where $H^1(\mathcal{A},X^*)$ is the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X^*. This concept was introduced by B. E. Johnson in [12]. In addition, $\mathcal{A}$ is called *contractible* if $H^1(\mathcal{A},X)=\{0\}$ for every Banach $\mathcal{A}$-bimodule X. He showed that a Banach algebra is amenable if and only if it has a virtual diagonal if and only if it has approximate diagonal; see also [6, 13]. Later, Ghahramani, Loy and Zang introduced and studied the approximate amenable Banach algebras [7, 8]. They proved that for any locally compact group G, the correspondence group algebra $L^1(G)$ is approximate amenable if and only if G is amenable; for module case of derivations into iterated duals of Banach algebras which is a generalization of the classical case, we refer to [3].

Helemskii studied the structure of Banach algebras through the concepts of biprojectivity and biflatness [10, 11]. One of his main results is that, a Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is biflat and has a bounded approximate identity. Recall that a Banach algebra $\mathcal{A}$ is *biflat*, if there is a bimodule homomorphism $\rho:\mathcal{A}\longrightarrow(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\Delta^{**}\circ\rho$ is identity on $\mathcal{A}$, where $\Delta:\mathcal{A}\hat{\otimes}\mathcal{A}\longrightarrow\mathcal{A}$; $a\otimes b\mapsto ab$ is the canonical morphism and Δ^{**} is the second adjoint of Δ. Biflatness and biprojectivity is important notions in the category of commutative Banach algebras. For instance, each commutative Banach algebra has a discrete character space if it is biprojective, and the converse holds for all commutative C^*-algebras [15]. Next, Ghorbani and Bami [9] investigated the φ-approximate biflatness and φ-amenability of Banach algebras, whenever φ is an bounded homomorphism on $\mathcal{A}$. For another extensions of amenability and biflaness of Banach algebras, we refer to [1, 2, 5, 14, 16, 17, 18] and references therein.

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Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Throughout this paper, we denote the set of all bounded homomorphism form $\mathcal{A}$ to $\mathcal{B}$ by $\mathrm{Hom}(\mathcal{A},\mathcal{B})$, and denote $\mathrm{Hom}(\mathcal{A},\mathcal{A})$ by $\mathrm{Hom}(\mathcal{A})$. Let $\mathcal{A}$ be a Banach algebra and $\sigma,\tau \in \mathrm{Hom}(\mathcal{A})$. Suppose that X is a Banach $\mathcal{A}$-bimodule. A bounded linear map $D:\mathcal{A}\longrightarrow X$ is called a (σ,τ)-derivation if

$$D(ab)=\sigma(a)\cdot D(b)+D(a)\cdot\tau(b), \quad (a,b\in\mathcal{A}).$$

For each $x\in X$, define

$$ad_x^{(\sigma,\tau)}(a)=\sigma(a)\cdot x-x\cdot\tau(a), \quad (a\in\mathcal{A}),$$

then $ad_x^{(\sigma,\tau)}$ is a (σ,τ)-derivation which is called a (σ,τ)-inner derivation [14]; for more details we also refer to [4].

Definition 1.1. *Let $\mathcal{A}$ be a Banach algebra and $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$. Then*

(i) *$\mathcal{A}$ is called approximate (σ,τ)-amenable if for any Banach $\mathcal{A}$-bimodule X, all (σ,τ)-derivation $D:\mathcal{A}\longrightarrow X^*$ is approximate (σ,τ)-inner, i.e., $D=\lim_\alpha ad_{f_\alpha}^{(\sigma,\tau)}$ for some net $\{f_\alpha\}_\alpha$ in X^*.*

(ii) *$\mathcal{A}$ is said to be approximate (σ,τ)-contractible if for each (σ,τ)-derivation from $\mathcal{A}$ to every Banach $\mathcal{A}$-bimodule X is approximate (σ,τ)-inner.*

Note that if σ and τ are identity maps on $\mathcal{A}$, then approximate (σ,τ)-amenability (resp. (σ,τ)-contractibility) coincide with approximate amenability (resp. contractibility).

In this paper, we study the structure of approximate (σ,τ)-amenable Banach algebras, whenever σ and τ are two bounded homomorphisms on $\mathcal{A}$. We present the notions of (σ,τ)-pseudo (virtual) diagonals and find some sufficient conditions for the approximate (σ,τ)-amenability of $\mathcal{A}$ which are equivalent to the existence of a (σ,τ)-pseudo virtual diagonal. Moreover, we introduce and investigate the notions of approximate (σ,τ)-biflatness and (σ,τ)-biprojectivity, and generalize the well-known results due to Johnson and Helemskii.

2. Approximate (σ,τ)-amenability

In this section, we extend the concept of approximate amenability of Banach algebras, and study the hereditary properties of approximate (σ,τ)-amenable Banach algebras. From now on, we consider $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$ unless otherwise stated explicitly.

Definition 2.1. *A Banach $\mathcal{A}$-bimodule X is called (σ,τ)-unital if,*

$$X=\{\tau(a)\cdot x\cdot\sigma(b):\ a,b\in\mathcal{A}, x\in X\}.$$

Lemma 2.1. *Let $D:\mathcal{A}\longrightarrow X^*$ be a (σ,τ)-derivation, where X is a Banach $\mathcal{A}$-bimodule. Then, D is (σ,τ)-inner, if one of the following statements holds.*

(i) *$\mathcal{A}$ has a right bounded approximate identity and $\tau(\mathcal{A})\cdot X=0$.*

(ii) *$\mathcal{A}$ has a left bounded approximate identity and $X\cdot\sigma(\mathcal{A})=0$.*

Proof. Suppose that (i) is valid. Then, $X^*\cdot\tau(\mathcal{A})=0$ and hence

$$D(ab)=\sigma(a)\cdot D(b), \quad (a,b\in\mathcal{A}).$$

Now, by letting $\phi\in X^*$ as in the proof of Proposition 2.1.3 of [15], we get $D=ad_\phi^{(\sigma,\tau)}$. This means that D is (σ,τ)-inner.

If (ii) holds, then $\sigma(\mathcal{A})\cdot X^*=0$ and so for all $a,b\in\mathcal{A}$,

$$D(ab)=D(a)\cdot\tau(b).$$

Again by a similar argument as in the proof of of [15, Proposition 2.1.3], we conclude that D is (σ,τ)-inner. □

Proposition 2.1. *Suppose that $\mathcal{A}$ has a bounded approximate identity. Then, $\mathcal{A}$ is approximate (σ,τ)-amenable if and only if each (σ,τ)-derivation from $\mathcal{A}$ into any (σ,τ)-unital Banach $\mathcal{A}$-bimodule is approximate (σ,τ)-inner.*

Proof. Note that each (σ,τ)-derivation from $\mathcal{A}$ into any (σ,τ)-unital Banach $\mathcal{A}$-bimodule is approximate (σ,τ)-inner. Let X be a Banach $\mathcal{A}$-bimodule. Consider the following closed submodules of X,

$$X_1=\{\tau(a)\cdot x\cdot\sigma(b):\ a,b\in\mathcal{A},x\in X\}\,,\quad X_2=\{x\cdot\sigma(b):\ a,b\in\mathcal{A},x\in X\}\,.$$

Assume that $D:\mathcal{A}\longrightarrow X_2^*$ is a (σ,τ)-derivation. Then, $\pi_1 D:\mathcal{A}\longrightarrow X_1^*$ is a (σ,τ)-derivation, where $\pi_1 : X_2^*\longrightarrow X_1^*$ is the restriction map. Since X_1 is (σ,τ)-unital, there is a net $\{f_\alpha\}$ in X_1^* such that $\pi_1 D=\lim_\alpha ad_{f_\alpha}^{(\sigma,\tau)}$. For fix f_α, consider $g_\alpha\in X_2^*$ as an extension of f_α. Clearly,

$$\tilde{D}:=D-\lim_\alpha ad_{g_\alpha}^{(\sigma,\tau)}:\mathcal{A}\to X_2^*\cap X_1^\perp\cong(X_2/X_1)^*,$$

is a (σ,τ)-derivation. As $\tau(\mathcal{A})\cdot(X_2/X_1)$ is zero, it follows from Lemma 2.1 that there is $h\in X_2^*\cap X_1^\perp$ such that $\tilde{D}=ad_h^{(\sigma,\tau)}$ and therefore $D=\lim_\alpha ad_{g_\alpha+h}^{(\sigma,\tau)}$. Repeating the above argument, we conclude that every (σ,τ)-derivation from $\mathcal{A}$ to X^* is approximate (σ,τ)-inner. □

The proof of the following lemma is similar to the proof of Proposition 2.4 from [1], and so omitted. It should be pointed out that the map $\sigma:\mathcal{A}\longrightarrow\mathcal{A}$ is called dense range if $\sigma(\mathcal{A})$ is norm dense in $\mathcal{A}$.

Lemma 2.2. *If $\mathcal{A}$ is approximate amenable, then it is approximate (σ,τ)-amenable. The converse is true when $\sigma=\tau$ has dense range.*

Lemma 2.3. *For two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, let $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$ and $\varphi,\psi\in\mathrm{Hom}(\mathcal{B})$. If $\theta\in\mathrm{Hom}(\mathcal{A},\mathcal{B})$ is surjective, such that $\theta\sigma=\varphi\theta$ and $\theta\tau=\psi\theta$. Then, the approximate (φ,ψ)-amenability of $\mathcal{B}$ follows from the approximate (σ,τ)-amenability of $\mathcal{A}$.*

Proof. Assume that X is a Banach $\mathcal{B}$-bimodule and $D:\mathcal{B}\longrightarrow X^*$ is a (φ,ψ)-derivation. We turn X into a Banach $\mathcal{A}$-bimodule via θ. It is immediate that $D\circ\theta:\mathcal{A}\longrightarrow X^*$ is a (σ,τ)-derivation. The approximate (σ,τ)-amenability of $\mathcal{A}$ implies that there exists a net $\{f_\alpha\}_\alpha$ in X^* such that for $a\in\mathcal{A}$,

$$\begin{aligned}D(\theta(a))&=\lim_\alpha\big(\sigma(a)\cdot f_\alpha-f_\alpha\cdot\tau(a)\big),\\&=\lim_\alpha\big(\theta(\sigma(a))\cdot f_\alpha-f_\alpha\cdot\theta(\tau(a))\big),\\&=\lim_\alpha\big(\varphi(\theta(a))\cdot f_\alpha-f_\alpha\cdot\psi(\theta(a))\big).\end{aligned}$$

Consequently, it follows from the surjectively of θ that $D=\lim_\alpha ad_{f_\alpha}^{(\sigma,\tau)}$. □

Let $\mathcal{A}$ be a Banach algebra. Then, $\mathcal{A}^\sharp=\mathcal{A}\oplus\mathbb{C}$, the unitization of $\mathcal{A}$, with the multiplication

$$(a,\lambda)(b,\gamma)=(ab+\lambda b+\gamma a,\lambda\gamma),$$

and norm $\|(a,\lambda)\|=\|a\|+|\lambda|$ is a unital Banach algebra. For each bounded homomorphism $\sigma:\mathcal{A}\longrightarrow\mathcal{A}$, define $\sigma^\sharp:\mathcal{A}^\sharp\longrightarrow\mathcal{A}^\sharp$ by

$$\sigma^\sharp(a+\lambda e)=\sigma(a)+\lambda e,\quad(a\in\mathcal{A},\lambda\in\mathbb{C}).$$

It is obvious that $\sigma^\sharp\in\mathrm{Hom}(\mathcal{A}^\sharp)$ and is an extension of σ.

Corollary 2.1. *A Banach algebra $\mathcal{A}$ is approximate (σ,τ)-amenable if and only if $\mathcal{A}^\sharp$ is approximate $(\sigma^\sharp,\tau^\sharp)$-amenable.*

Proof. Let $\mathcal{A}$ be an approximate (σ,τ)-amenable, X be a $(\sigma^\sharp,\tau^\sharp)$-unital Banach $\mathcal{A}^\sharp$-bimodule and $D:\mathcal{A}^\sharp\longrightarrow X^*$ be a $(\sigma^\sharp,\tau^\sharp)$-derivation. Put e as the identity of $\mathcal{A}^\sharp$. Then, $\sigma^\sharp(e)=\tau^\sharp(e)=e$ and hence $D(e)=0$. On the other hand, $D|_{\mathcal{A}}$ is (σ,τ)-approximate inner, by the approximate (σ,τ)-amenability of $\mathcal{A}$. Therefore, D is $(\sigma^\sharp,\tau^\sharp)$-approximate inner. It now follows from Proposition 2.1 that $\mathcal{A}^\sharp$ is approximate $(\sigma^\sharp,\tau^\sharp)$-amenable.

For the converse, let $\mathcal{A}^\sharp$ be approximate $(\sigma^\sharp,\tau^\sharp)$-amenable and $\theta\in \text{Hom}(\mathcal{A}^\sharp,\mathcal{A})$ be the quotient map. By Lemma 2.3, $\mathcal{A}$ is approximate (σ,τ)-amenable. □

Let $\sigma\in\text{Hom}(\mathcal{A})$ and I be a closed ideal of $\mathcal{A}$ such that $\sigma(I)\subseteq I$. We denote the coset of $a\in\mathcal{A}$ in $\mathcal{A}/I$ by $\bar{a}$. We may define $\tilde{\sigma}:\mathcal{A}/I\longrightarrow\mathcal{A}/I$ by $\tilde{\sigma}(\bar{a})=\overline{\sigma(a)}$.

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra and I be a closed ideal of $\mathcal{A}$ such that $\sigma(I)I=I=I\tau(I)$. If I is $(\sigma|_I,\tau|_I)$-amenable and $\mathcal{A}/I$ is approximate $(\tilde{\sigma},\tilde{\tau})$-amenable, then $\mathcal{A}$ is approximate (σ,τ)-amenable.*

Proof. Let X be a Banach $\mathcal{A}$-bimodule. Take

$$E=\{f\in X^*:\ \sigma(a)\cdot f=f\cdot\tau(b)=0,\ \forall a,b\in I\},$$

and let F be a closed linear span of

$$\{\sigma(a)\cdot x+y\cdot\tau(b)\ :\ a,b\in I,\ x,y\in X\}.$$

Since $\sigma(I)I=I=I\tau(I)$, F is a closed I-bimodule of X and thus $E\cong(X/F)^*$ is a dual Banach $\mathcal{A}/I$-bimodule. Now, let $D:\mathcal{A}\to X^*$ be a (σ,τ)-derivation. Consider f in X^* such that $D|_I=ad_f^{(\sigma,\tau)}$ and $\tilde{D}:=D-ad_f^{(\sigma,\tau)}$. Then, $\tilde{D}$ vanishes on I and therefore it induces a (σ,τ)-derivation from $\mathcal{A}/I$ to X^*, that we denote likewise by $\tilde{D}$. Moreover, for each $a\in\mathcal{A}$ and $b\in I$ we obtain that $\sigma(b)\cdot\tilde{D}(a)=\tilde{D}(a)\cdot\tau(b)=0$. Hence, $\tilde{D}(\mathcal{A}/I)\subseteq E$. Consequently, the approximate $(\tilde{\sigma},\tilde{\tau})$-amenability of $\mathcal{A}/I$ implies that $\tilde{D}$ is approximate (σ,τ)-inner and so is D. Therefore, $\mathcal{A}$ is approximate (σ,τ)-amenable. □

Proposition 2.3. *Let $\mathcal{A}$ be a Banach algebra and I be a closed ideal of $\mathcal{A}$ with bounded approximate identity. Let also X be a $(\sigma|_I,\tau|_I)$-unital Banach I-bimodule. If $D:I\longrightarrow X^*$ is a $(\sigma|_I,\tau|_I)$-derivation, then X is a Banach $\mathcal{A}$-bimodule in a canonical fashion. Furthermore, there is a unique (σ,τ)-derivation $\tilde{D}:\mathcal{A}\longrightarrow X^*$ that*

(i) $\tilde{D}|_I=D$.
(ii) *$\tilde{D}$ is continuous with respect to the strict topology on $\mathcal{A}$ and the w^*-topology on X^*.*

Proof. For $x\in X$, let $a,b\in I$ and $y,z\in X$ be such that $x=\tau(a)\cdot y$ and $x=z\cdot\sigma(a)$. For $c\in\mathcal{A}$, define

$$c\cdot x:=c\tau(a)\cdot y,\quad x\cdot c:=z\cdot\sigma(a)c.$$

We claim that $c\cdot x$ is well defined, i.e., it is independent of the choices of a and y. Let y' and a' be such that $x=\tau(a')\cdot y'$. If $\{e_\alpha\}$ is a bounded approximate identity of I, then for each $c\in\mathcal{A}$, we have

$$c\tau(a)\cdot y=\lim_\alpha ce_\alpha\tau(a)\cdot y=\lim_\alpha ce_\alpha\tau(a')\cdot y'=c\tau(a')\cdot y'.$$

Similarly, $x\cdot c$ is well-defined. It is routinely checked that X is a Banach $\mathcal{A}$-bimodule with the above actions. Consider

$$\tilde{D}:\mathcal{A}\longrightarrow X^*,\quad \tilde{D}(a)=w^*-\lim_\alpha D(ae_\alpha),\quad (a\in\mathcal{A}).$$

Clearly, $\tilde{D}$ extends D. Consider $x\in X$, $a\in I$ and $y\in X$ such that $x=y\cdot\sigma(a)$. Then

$$\begin{aligned}\langle x,D(be_\alpha)\rangle&=\langle y,\sigma(a)\cdot D(be_\alpha)\rangle\\&=\langle y,D(abe_\alpha)-D(a)\cdot\tau(be_\alpha)\rangle\longrightarrow\langle y,D(ab)\rangle-\langle\tau(b)\cdot y,D(a)\rangle.\end{aligned}$$

for all $b \in \mathcal{A}$. Therefore, $\tilde{D}$ is well-defined. Moreover,

$$\begin{aligned}\tilde{D}(b) \cdot \tau(a) &= w^* - \lim_{\alpha} D(be_\alpha) \cdot \tau(a) \\ &= w^* - \lim_{\alpha} \big(D(be_\alpha a) - \sigma(be_\alpha) \cdot D(a)\big) \\ &= D(ba) - \sigma(b) \cdot D(a), \qquad (a \in I, \quad b \in \mathcal{A}).\end{aligned}$$

Now, if $b_\alpha \longrightarrow b$ in strict topology on $\mathcal{A}$ and $a \in I$, then $ab_\alpha \longrightarrow ab$ and $b_\alpha a \longrightarrow ba$ in norm topology. Hence, for $a \in I$ and $y \in X$,

$$\begin{aligned}\langle \tau(a) \cdot y, \tilde{D}(b_\alpha)\rangle &= \langle y, D(b_\alpha a) - \sigma(b_\alpha) \cdot D(a)\rangle \\ &\longrightarrow \langle y, D(ba) - \sigma(b) \cdot D(a)\rangle \\ &= \langle \tau(a) \cdot y, \tilde{D}(b)\rangle.\end{aligned}$$

Thus, $\tilde{D}$ is continuous with respect to the strict topology on $\mathcal{A}$ and the w^*-topology on X^*. Therefore, the continuity of $\tilde{D}$ implies that it is a (σ, τ)-derivation. □

By Propositions 2.1 and 2.3, we reach to the next result.

Corollary 2.2. *Let I be a closed ideal of $\mathcal{A}$ with bounded approximate identity. If $\mathcal{A}$ is approximate (σ, τ)-amenable, then I is approximate $(\sigma|_I, \tau|_I)$-amenable.*

Proposition 2.4. *Let $\mathcal{A}$ be (σ, τ)-amenable and $\mathcal{B}$ be approximate (φ, ψ)-amenable. If $\mathcal{A}$, $\mathcal{B}$ are unital and both $\varphi(e_2)$ and $\psi(e_2)$ contained in the center of $\mathcal{B}$, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma \otimes \varphi, \tau \otimes \psi)$-amenable, where e_2 is an identity of $\mathcal{B}$.*

Proof. Let X be a Banach $\mathcal{A}\hat{\otimes}\mathcal{B}$-bimodule and $D : \mathcal{A}\hat{\otimes}\mathcal{B} \longrightarrow X^*$ be a $(\sigma \otimes \varphi, \tau \otimes \psi)$-derivation. From the (σ, τ)-amenability of $\mathcal{A}$, there is f in X^* such that

$$D(a \otimes e_2) = (\sigma(a) \otimes \varphi(e_2)) \cdot f - f \cdot (\tau(a) \otimes \psi(e_2)), \qquad (a \in \mathcal{A}).$$

Let $\tilde{D} = D - ad_f^{(\sigma\otimes\varphi,\tau\otimes\psi)}$. Then, $\tilde{D}$ is zero on $\mathcal{A}\hat{\otimes}e_2$. Consider F be closed linear span of

$$\{(\tau(a) \otimes \psi(e_2) \cdot x - x \cdot (\sigma(a) \otimes \varphi(e_2)) \ : \ a \in \mathcal{A}, \ x \in X\}.$$

If e_1 be an identity of $\mathcal{A}$, then for $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in X$ we obtain

$$\begin{aligned}(e_1 \otimes b).[(\tau(a) \otimes \psi(e_2) \cdot x - x \cdot (\sigma(a) \otimes \varphi(e_2))] &= (\tau(a) \otimes b\psi(e_2)) \cdot x - z \cdot (\sigma(a) \otimes \varphi(e_2)) \\ &= (\tau(a) \otimes \psi(e_2)b) \cdot x - z \cdot (\sigma(a) \otimes \varphi(e_2)) \\ &= (\tau(a) \otimes \psi(e_2)) \cdot z - z \cdot (\sigma(a) \otimes \varphi(e_2)),\end{aligned}$$

where $z = (e_1 \otimes b) \cdot x \in X$. Therefore, F is a left $(e_1 \otimes \mathcal{B})$-submodule of X. Similarly, F is a right $(e_1 \otimes \mathcal{B})$-submodule of X and so it is a Banach $(e_1 \otimes \mathcal{B})$-bimodule. Obviously,

$$\tilde{D}(a \otimes b) = (\sigma(a) \otimes \varphi(e_2)) \cdot \tilde{D}(e_1 \otimes b) = \tilde{D}(e_1 \otimes b) \cdot (\tau(a) \otimes \psi(e_2)).$$

Thus,

$$\langle (\tau(a) \otimes \psi(e_2)) \cdot x - x \cdot (\sigma(a) \otimes \varphi(e_2)), \tilde{D}(e_1 \otimes b)\rangle =$$
$$\langle x, \tilde{D}(e_1 \otimes b) \cdot (\tau(a) \otimes \psi(e_2)) - (\sigma(a) \otimes \varphi(e_2)) \cdot \tilde{D}(e_1 \otimes b)\rangle = 0.$$

Hence, $\tilde{D} : (e_1 \otimes \mathcal{B}) \longrightarrow F^0 \cong (X/F)^*$. Consequently, the approximate (φ, ψ)-amenability of $\mathcal{B}$ implies that $\tilde{D}$ is approximate $(\sigma \otimes \varphi, \tau \otimes \psi)$-inner and so is D. This means that, $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma \otimes \varphi, \tau \otimes \psi)$-amenable. □

Corollary 2.3. *Let $\mathcal{A}$ be (σ, τ)-amenable and $\mathcal{B}$ be approximate (φ, ψ)-amenable. Let $\sigma, \tau \in$*Hom*$(\mathcal{A})$ and $\varphi, \psi \in$*Hom*$(\mathcal{B})$ such that one of the following is satisfied:*

(i) *$\mathcal{A}$ and $\mathcal{B}$ are unital, and both φ, ψ are surjective.*
(ii) *$\mathcal{B}$ has a bounded approximate identity, and σ, τ, φ and ψ are surjective.*

Then, $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma\otimes\varphi, \tau\otimes\psi)$-amenable.

Proof. (i) The result follows from Proposition 2.4. Suppose that (ii) holds. Corollary 2.1 necessitates that $\mathcal{A}^\sharp$ is $(\sigma^\sharp, \tau^\sharp)$-amenable and thus $\mathcal{B}^\sharp$ is approximate $(\varphi^\sharp, \psi^\sharp)$-amenable. It follows from (i) that $\mathcal{A}^\sharp\hat{\otimes}\mathcal{B}^\sharp$ is approximate $(\sigma^\sharp\otimes\varphi^\sharp, \tau^\sharp\otimes\psi^\sharp)$-amenable. Since σ and τ are surjective, $\mathcal{A}$ has a bounded approximate identity and therefore $\mathcal{A}\hat{\otimes}\mathcal{B}$ has a bounded approximate identity. Now, Corollary 2.2 implies that $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma\otimes\varphi, \tau\otimes\psi)$-amenable. □

3. Characterization of (σ, τ)-pseudo (virtual) diagonals

In this section, we extend the notions of classical diagonals and investigate the relations between these concepts and approximate (σ, τ)-amenability.

Definition 3.1. *Let $\mathcal{A}$ be a Banach algebra and $\sigma, \tau \in Hom(\mathcal{A})$.*

(i) *A net $\{\boldsymbol{M}_\alpha\}_\alpha$ in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ is called a (σ, τ)-pseudo virtual diagonal for $\mathcal{A}$, if*

$$\sigma(a)\cdot\boldsymbol{M}_\alpha - \boldsymbol{M}_\alpha\cdot\tau(a)\to 0, \quad \Delta^{**}\boldsymbol{M}_\alpha\cdot\sigma(a) = \sigma(a), \quad \tau(a)\cdot\Delta^{**}\boldsymbol{M}_\alpha = \tau(a) \quad (a\in\mathcal{A}).$$

(ii) *A net $\{\boldsymbol{m}_\alpha\}_\alpha$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ is called a (σ, τ)-pseudo diagonal for $\mathcal{A}$, if*

$$\sigma(a)\cdot\boldsymbol{m}_\alpha - \boldsymbol{m}_\alpha\cdot\tau(a)\to 0, \quad \Delta\boldsymbol{m}_\alpha\cdot\sigma(a) = \sigma(a), \quad \tau(a)\cdot\Delta\boldsymbol{m}_\alpha = \tau(a) \quad (a\in\mathcal{A}).$$

Proposition 3.1. *Let σ and τ be surjective such that $\sigma^2 = \sigma\tau$ or $\tau^2 = \tau\sigma$. If $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \tau^\sharp)$-pseudo virtual diagonal, then $\mathcal{A}$ is approximate (σ, τ)-amenable.*

Proof. First, assume that $\sigma^2 = \sigma\tau$. Suppose that $D : \mathcal{A}^\sharp \longrightarrow X^*$ is a (σ, τ)-derivation, where X is a $(\sigma^\sharp, \tau^\sharp)$-unital Banach $\mathcal{A}^\sharp$-bimodule. Let $\{\mathbf{M}_\alpha\}_\alpha$ be a (σ, τ)-pseudo virtual diagonal for $\mathcal{A}^\sharp$. Fix $\mathbf{M}_\alpha$, then by Goldstine's theorem, there exists a bounded net $\{\mathbf{m}_{i,\alpha}\}_i$ in $\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp$ such that w^*-converges to $\mathbf{M}_\alpha$. Suppose that $\mathbf{m}_{i,\alpha} = \sum_{n=1}^\infty a_n^{(i,\alpha)}\otimes b_n^{(i,\alpha)}$ where $\left\{a_n^{(i,\alpha)}\right\}_n$ and $\left\{b_n^{(i,\alpha)}\right\}_n$ are bounded sequences in $\mathcal{A}^\sharp$. Then, the bounded net $\left\{\sum_{n=1}^\infty a_n^{(i,\alpha)} D(b_n^{(i,\alpha)})\right\}_i$ has a w^*-accumulation point in X^*. Without loss of generality, we may suppose that there is $f_\alpha \in X^*$ such that

$$f_\alpha = w^* - \lim_i \sum_{n=1}^\infty \sigma^\sharp(a_n^{(i,\alpha)})\cdot D(b_n^{(i,\alpha)}).$$

Since σ and τ are surjective, every $\Delta^{**}(\mathbf{M}_\alpha)$ is an identity for $\mathcal{A}^\sharp$, thus $\left\{\sigma^\sharp(\Delta\mathbf{m}_{i,\alpha})\right\}_i$ is a w^*-bounded approximate identity for $\mathcal{A}^\sharp$ and for $a\in\mathcal{A}^\sharp$,

$$w^* - \lim_i (\sum_{n=1}^\infty \sigma^\sharp(a) a_n^{(i,\alpha)}\otimes b_n^{(i,\alpha)} - \sum_{n=1}^\infty a_n^{(i,\alpha)}\otimes b_n^{(i,\alpha)}\tau^\sharp(a)) \xrightarrow{\alpha} 0.$$

Define $P : \mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp \to X^*$ via $P(a\otimes b) = \sigma^\sharp(a)\cdot D(b)$. Then for all $a\in\mathcal{A}^\sharp$,

$$w^* - \lim_i \big(\sum_{n=1}^\infty {\sigma^\sharp}^2(a)\sigma^\sharp(a_n^{(i,\alpha)})\cdot D(b_n^{(i,\alpha)}) - \sum_{n=1}^\infty \sigma^\sharp(a_n^{(i,\alpha)})\cdot D(b_n^{(i,\alpha)}\tau^\sharp(a))\big) \xrightarrow{\alpha} 0.$$

For $a \in \mathcal{A}^\sharp$, let $a' \in \mathcal{A}^\sharp$ such that $\tau^\sharp(a') = a$. Moreover, for each $x \in X$ we have

$$\begin{aligned}
\lim_\alpha \left\langle x, \sigma^\sharp(a) \cdot f_\alpha \right\rangle &= \lim_\alpha \left\langle x, \sigma^\sharp(\tau^\sharp(a')) \cdot f_\alpha \right\rangle, \\
&= \lim_\alpha \left\langle x, {\sigma^\sharp}^2(a') \cdot f_\alpha \right\rangle, \\
&= \lim_\alpha \lim_i \left\langle x, \sum_{n=1}^\infty {\sigma^\sharp}^2(a')\sigma^\sharp(a_n^{(i,\alpha)}) \cdot D(b_n^{(i,\alpha)}) \right\rangle, \\
&= \lim_\alpha \lim_i \left\langle x, \sum_{n=1}^\infty \sigma^\sharp(a_n^{(i,\alpha)}) \cdot D(b_n^{(i,\alpha)}\tau^\sharp(a')) \right\rangle, \\
&= \lim_\alpha \lim_i \left\langle x, \sum_{n=1}^\infty \sigma^\sharp(a_n^{(i,\alpha)})\sigma^\sharp(b_n^{(i,\alpha)}) \cdot D(\tau^\sharp(a')) \right\rangle \\
&\quad + \lim_\alpha \lim_i \left\langle x, \sum_{n=1}^\infty \sigma^\sharp(a_n^{(i,\alpha)}) \cdot D(b_n^{(i,\alpha)}) \cdot {\tau^\sharp}^2(a') \right\rangle, \\
&= \lim_\alpha \left\langle x \cdot \sigma^\sharp(\Delta \mathbf{m}_{i,\alpha}), D(\tau^\sharp(a')) \right\rangle + \lim_\alpha \left\langle x, f_\alpha \cdot {\tau^\sharp}^2(a') \right\rangle, \\
&= \left\langle x, D(\tau^\sharp(a')) \right\rangle + \lim_\alpha \left\langle x, f_\alpha \cdot {\tau^\sharp}^2(a') \right\rangle, \\
&= \langle x, D(a) \rangle + \lim_\alpha \left\langle x, f_\alpha \cdot \tau^\sharp(a) \right\rangle.
\end{aligned}$$

This means that

$$D(a) = \lim_\alpha \left(\sigma^\sharp(a) \cdot f_\alpha - f_\alpha \cdot \tau^\sharp(a)\right),$$

for all $a \in \mathcal{A}^\sharp$. Thus, D is approximate $(\sigma^\sharp, \tau^\sharp)$-inner and hence $\mathcal{A}^\sharp$ is approximate $(\sigma^\sharp, \tau^\sharp)$-amenable. It now follows from Corollary 2.1 that $\mathcal{A}$ is approximate (σ, τ)-amenable. The case $\tau^2 = \tau\sigma$ is similar. □

The next result may be considered as a converse version of Proposition 3.1.

Proposition 3.2. *Let $\sigma, \tau \in$Hom$(\mathcal{A})$ such that τ is surjective and idempotent. Then, $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \sigma^\sharp\tau^\sharp)$-pseudo virtual diagonal, whenever $\mathcal{A}$ is approximate $(\sigma, \sigma\tau)$-amenable.*

Proof. From Corollary 2.1, $\mathcal{A}^\sharp$ is approximate $(\sigma^\sharp, \sigma^\sharp\tau^\sharp)$-amenable. Let e be an identity of $\mathcal{A}^\sharp$ and consider $D(a) = ad_{e\otimes e}^{(\sigma^\sharp, \sigma^\sharp\tau^\sharp)} : \mathcal{A}^\sharp \longrightarrow (\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$. Then, for all $a \in \mathcal{A}^\sharp$,

$$\Delta^{**}\big(D(\tau^\sharp(a))\big) = \Delta^{**}\big(\sigma^\sharp(\tau^\sharp(a)) \cdot e \otimes e - e \otimes e \cdot \sigma^\sharp(\tau^\sharp(\tau^\sharp(a)))\big) = \sigma^\sharp(\tau^\sharp(a)) - \sigma^\sharp({\tau^\sharp}^2(a)).$$

By assumption, we conclude that $\Delta^{**}\big(D(\tau^\sharp(a))\big) = 0$. Thus, $\Delta^{**}\big(D(a)\big) = 0$ for all $a \in \mathcal{A}^\sharp$ and hence $ad_{e\otimes e}^{(\sigma^\sharp, \sigma^\sharp\tau^\sharp)} : \mathcal{A}^\sharp \longrightarrow (\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$ is a $(\sigma^\sharp, \sigma^\sharp\tau^\sharp)$-derivation into $ker\Delta^{**}$. Therefore, there is the net $\{N_\alpha\}_\alpha$ in $ker\Delta^{**} \cong (ker\Delta)^{**}$ such that

$$ad_{e\otimes e}^{(\sigma^\sharp, \sigma^\sharp\tau^\sharp)} = \lim_\alpha ad_{N_\alpha}^{(\sigma^\sharp, \sigma^\sharp\tau^\sharp)}.$$

Now, as the standard argument, we can obtain that $\{e \otimes e - N_\alpha\}_\alpha \subseteq (\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$ is a $(\sigma^\sharp, \sigma^\sharp\tau^\sharp)$-pseudo virtual diagonal for $\mathcal{A}^\sharp$. □

Replacing τ with the identity map in the above proposition, we get the next corollary.

Corollary 3.1. *If $\mathcal{A}$ is approximate (σ, σ)-amenable, then $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \sigma^\sharp)$-pseudo virtual diagonal.*

Example 3.1. Let $\mathcal{A}$ be the Banach algebra $l^1(G)$, where G is a commutative locally compact group with identity e. Consider

$$\sigma, \tau : l^1(G) \longrightarrow l^1(G) \quad \sigma(f) = f, \quad \tau(f) = f * \delta_s, \quad (f \in l^1(G)),$$

that $s \neq e$ is an idempotent. Then, σ and τ are homomorphisms on $l^1(G)$ and $\sigma\tau = \tau$. By Theorem 3.2 of [7], $l^1(G)$ is approximate amenable, and hence Lemma 2.2 implies that it is approximate (σ, τ)-amenable. If $\{\mathbf{M}_\alpha\}_\alpha$ is a (σ, τ)-pseudo virtual diagonal for $\mathcal{A}$, then

$$\delta_g \mathbf{M}_\alpha - \mathbf{M}_\alpha(\delta_g * \delta_s) \longrightarrow 0 \quad (g \in G),$$

and so

$$\delta_g \Delta^{**}(\mathbf{M}_\alpha) - \Delta^{**}(\mathbf{M}_\alpha)\delta_{gs} \longrightarrow 0 \quad (g \in G).$$

Since $\Delta^{**}(\mathbf{M}_\alpha)\delta_g = \delta_g$ for each g in G, it follows that

$$\delta_e \Delta^{**}(\mathbf{M}_\alpha) \longrightarrow \Delta^{**}(\mathbf{M}_\alpha)\delta_{es} = \delta_s.$$

On the other hand, from the equality $\delta_{gs}\Delta^{**}(\mathbf{M}_\alpha) = \delta_{gs}$, we have

$$\delta_e \Delta^{**}(\mathbf{M}_\alpha) = \delta_{s^{-1}s}\Delta^{**}(\mathbf{M}_\alpha) = \delta_{s^{-1}s} = \delta_e.$$

Consequently, $\delta_s = \delta_e$, which is a contradiction. Therefore, the approximate (σ, τ)-amenability of $\mathcal{A}$ is not equivalent to the existence of a (σ, τ)-pseudo virtual diagonal for its, whenever σ and τ are arbitrary elements in $\mathrm{Hom}(\mathcal{A})$.

The proof of the upcoming result is similar to the proof of Proposition 3.1.

Proposition 3.3. *Let σ and τ be surjective such tha $\sigma^2 = \sigma\tau$ or $\tau^2 = \tau\sigma$. If $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \tau^\sharp)$-pseudo diagonal, then $\mathcal{A}$ is approximate (σ, τ)-contractible.*

Corollary 3.2. *If $\mathcal{A}$ is approximate (σ, σ)-contractible, then $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \sigma^\sharp)$-pseudo diagonal.*

The following example shows that the concept of approximate (σ, τ)-contractibility for a Banach algebra $\mathcal{A}$ is not equivalent to the existence of a (σ, τ)-pseudo diagonal for it, whenever σ and τ are arbitrary elements of $\mathrm{Hom}(\mathcal{A})$.

Example 3.2. Let $\mathcal{A}$ be the Banach algebra of complex $n \times n$ diagonal matrices of dimension $2n$. For $i, j \in \mathbb{N}$, take e_{ij} to be the matrix unit of $\mathcal{A}$, i.e., the matrix with 1 in the $(i, j)^{th}$ position and 0, elsewhere. If

$$e_{ij}e_{kl} = \delta_{j,k}e_{il}, \qquad (i, j, k, l \in \mathbb{N}),$$

then $\mathcal{A}$ is unital with the matrix identity $e = \sum_{i=1}^n e_{ii}$. Since $\mathcal{A}$ is finite-dimensional, it is contractible and hence it is approximate contractible. Moreover,

$$\mathbf{M} = \frac{1}{n}\sum_{i,j=1}^n e_{ij} \otimes e_{ji},$$

is a diagonal for $\mathcal{A}$ and therefore it is a pseudo diagonal. Define $\sigma, \tau \in Hom(\mathcal{A})$ through

$$\sigma\left(\begin{bmatrix} a_1 & 0\dots & 0 \\ \vdots & & \vdots \\ 0 & \dots a_{n-1} & 0 \\ 0 & 0 & \dots a_n \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0\dots & 0 \\ \vdots & & \vdots \\ 0 & \dots a_{n-1} & 0 \\ 0 & 0 & \dots 0 \end{bmatrix},$$

$$\tau\left(\begin{bmatrix} a_1 & 0\dots & 0 \\ \vdots & & \vdots \\ 0 & \dots a_{n-1} & 0 \\ 0 & 0 & \dots a_n \end{bmatrix}\right) = \begin{bmatrix} 0 & 0\dots & 0 \\ \vdots & & \vdots \\ 0 & \dots 0 & 0 \\ 0 & 0 & \dots a_n \end{bmatrix}.$$

Then, both σ and τ are idempotents, but neither σ nor τ is surjective. Note that $\mathcal{A}$ is approximate (σ,τ)-contractible. Let

$$\mathbf{m}_\alpha = \sum_{m=1}^{\infty} \begin{bmatrix} x_{11}^{\alpha,m} & 0\dots & 0 \\ 0 & x_{22}^{\alpha,m}\dots & 0 \\ \vdots & & \vdots \\ 0 & 0 & \dots x_{nn}^{\alpha,m} \end{bmatrix} \otimes \begin{bmatrix} y_{11}^{\alpha,m} & 0\dots & 0 \\ 0 & y_{22}^{\alpha,m}\dots & 0 \\ \vdots & & \vdots \\ 0 & 0 & \dots y_{nn}^{\alpha,m} \end{bmatrix}.$$

If $\{\mathbf{m}_\alpha\}_\alpha$ is a (σ,τ)-pseudo diagonal for $\mathcal{A}$, then

$$\begin{aligned} e_{jj} = \sigma(e_{jj}) = \Delta(\mathbf{m}_\alpha)\cdot\sigma(e_{jj}) &= \sum_{m=1}^{\infty} \begin{bmatrix} x_{11}^{\alpha,m}y_{11}^{\alpha,m} & 0\dots & 0 \\ 0 & x_{22}^{\alpha,m}y_{22}^{\alpha,m}\dots & 0 \\ \vdots & & \vdots \\ 0 & 0 & \dots x_{nn}^{\alpha,m}y_{nn}^{\alpha,m} \end{bmatrix} .e_{jj} \\ &= \sum_{m=1}^{\infty} \begin{bmatrix} 0 & 0\dots & 0 \\ \vdots & & \vdots \\ 0 & \dots x_{jj}^{\alpha,m}y_{jj}^{\alpha,m} & \dots 0 \\ \vdots & & \vdots \\ 0 & 0 & \dots 0 \end{bmatrix}, \end{aligned}$$

for all $1 \le j < n$. Similarly,

$$e_{nn} = \tau(e_{nn}) = \tau(e_{nn})\cdot\Delta(\mathbf{m}_\alpha) = \sum_{m=1}^{\infty} \begin{bmatrix} 0 & 0\dots & 0 \\ 0 & 0 & 0 \\ \vdots & & \vdots \\ 0 & 0 & \dots x_{nn}^{\alpha,m}y_{nn}^{\alpha,m} \end{bmatrix}.$$

Hence, $\sum_{m=1}^{\infty} x_{jj}^{\alpha,m}y_{jj}^{\alpha,m} = 1$, for all $1 \le j \le n$. Thus, each $\Delta(\mathbf{m}_\alpha)$ is the identity matrix. Now, since $\tau(e_{11}) = 0$, we have that $\Delta(\mathbf{m}_\alpha\cdot\tau(e_{11})) = 0$, and

$$\Delta(\sigma(e_{11})\cdot\mathbf{m}_\alpha) = \sigma(e_{11})\cdot\Delta(\mathbf{m}_\alpha) = e_{11},$$

which is a contradiction. Therefore, there is no any (σ,τ)-pseudo diagonal for $\mathcal{A}$.

4. Approximate (σ,τ)-biflatness

In this section, we define the notions of (approximate) (σ,τ)-biflatness and (approximate) (σ,τ)-biprojectivity as a generalizations of the earlier notions of biflatness and biprojectivity.

Definition 4.1. *A Banach algebra $\mathcal{A}$ is called*

(i) *(σ,τ)-biflat if there is a bounded linear map $\theta : \mathcal{A} \longrightarrow (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $\Delta^{**}\circ\theta$ is the identity map on $\tau(\mathcal{A})$ and*

$$\sigma(a)\cdot\theta(\sigma(b)) = \theta(\sigma(ab)) = \theta(\sigma(a))\cdot\tau(b), \quad (a, b \in \mathcal{A});$$

(ii) *approximate (σ,τ)-biflat if there is a net $\{\theta_\alpha\}_\alpha$ of bounded linear maps from $\mathcal{A}$ to $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that each $\Delta^{**}\circ\theta_\alpha$ is the identity map on $\tau(\mathcal{A})$ and*

$$\lim_\alpha \sigma(a)\cdot\theta_\alpha(\sigma(b)) = \lim_\alpha \theta_\alpha(\sigma(ab)) = \lim_\alpha \theta_\alpha(\sigma(a))\cdot\tau(b), \quad (a, b \in \mathcal{A}).$$

We note that if σ and τ are the identity maps, then $(id_\mathcal{A}, id_\mathcal{A})$-biflatness is the same as biflatness [6, 11].

Proposition 4.1. *Suppose that there is $e \in \mathcal{A}$ such that e is an identity for $\sigma(\mathcal{A}) \cup \tau(\mathcal{A})$ and $\sigma(e) = e = \tau(e)$. Then, $\mathcal{A}$ is approximate (σ, τ)-biflat if and only if $\mathcal{A}$ has a (σ, τ)-pseudo virtual diagonal.*

Proof. Let $\mathcal{A}$ be approximate (σ, τ)-biflat and the net $\{\theta_\alpha\}_\alpha$ be as in Definition 4.1. Then, for fix θ_α, we have that

$$\theta_\alpha(\sigma(a)\sigma(e)) = \theta_\alpha(\sigma(e)\sigma(a)),$$

and therefore

$$\sigma(a) \cdot \theta_\alpha(\sigma(e)) - \theta_\alpha(\sigma(e)) \cdot \tau(a) \longrightarrow 0,$$

for all $a \in \mathcal{A}$. Now, the net $\{\mathbf{M}_\alpha\}_\alpha$ in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ is a (σ, τ)-pseudo virtual diagonal for $\mathcal{A}$ in which $\mathbf{M}_\alpha = \theta_\alpha(e)$.

Conversely, let $\{\mathbf{M}_\alpha\}_\alpha$ in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ be a (σ, τ)-pseudo virtaul diagonal for $\mathcal{A}$. For each $a \in \mathcal{A}$ define $\theta_\alpha : \mathcal{A} \longrightarrow (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ by $\theta_\alpha(a) = a \cdot \mathbf{M}_\alpha$. Then, θ_α satisfies the conditions of Definition 4.1 (ii). Hence, $\mathcal{A}$ is approximate (σ, τ)-biflat. □

Corollary 4.1. *Let $\mathcal{A}$ be a Banach algebra and $\sigma, \tau \in$Hom$(\mathcal{A})$. Then $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \tau^\sharp)$-pseudo virtual diagonal if and only if $\mathcal{A}^\sharp$ is approximate $(\sigma^\sharp, \tau^\sharp)$-biflat.*

Proposition 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$ and $\varphi, \psi \in \mathrm{Hom}(\mathcal{B})$. If $\mathcal{A}$ is approximate (σ, τ)-biflat and $\mathcal{B}$ is (φ, ψ)-biflat, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma \otimes \varphi, \tau \otimes \psi)$-biflat.*

Proof. Let the net $\{\theta_\alpha\}_\alpha$ of bounded linear maps from $\mathcal{A}$ to $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ be as in Definition 4.1 (ii). Assume that $\theta : \mathcal{B} \longrightarrow (\mathcal{B}\hat{\otimes}\mathcal{B})^{**}$ that $\Delta_{\mathcal{B}}\theta$ is the identity map on $\psi(\mathcal{B})$ and

$$\varphi(a) \cdot \theta(\psi(b)) = \theta(\varphi(ab)) = \theta(\varphi(a)) \cdot \psi(b), \quad (a, b \in \mathcal{B}).$$

Consider

$$\theta'_\alpha : \mathcal{A}\hat{\otimes}\mathcal{B} \longrightarrow ((\mathcal{A}\hat{\otimes}\mathcal{B})\hat{\otimes}(\mathcal{A}\hat{\otimes}\mathcal{B}))^{**},$$

where $\theta'_\alpha = \gamma^{**}(\theta_\alpha \otimes \theta)$ and

$$\gamma((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2),$$

for $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. It is routine to checked that each $\Delta^{**}\theta'_\alpha$ is the identity map on $\tau \otimes \psi(\mathcal{A}\hat{\otimes}\mathcal{B})$. On the other hand,

$$\begin{aligned}\lim_\alpha(\sigma \otimes \varphi)(a \otimes b) \cdot \rho'_\alpha((\sigma \otimes \varphi)(c \otimes d)) &= \lim_\alpha \rho'_\alpha((\sigma \otimes \varphi)(ac \otimes bd)) \\ &= \lim_\alpha \rho'_\alpha((\sigma \otimes \varphi)(a \otimes b)) \cdot (\tau \otimes \psi)(c \otimes d),\end{aligned}$$

for all a, c in $\mathcal{A}$ and b, d in $\mathcal{B}$. Consequently, $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma \otimes \varphi, \tau \otimes \psi)$-biflat. □

Definition 4.2. *A Banach algebra $\mathcal{A}$ is called*

(i) *(σ, τ)-biprojective if there is a bounded linear map $\rho : \mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}\mathcal{A}$ such that each $\Delta\rho$ is the identity map on $\tau(\mathcal{A})$ and*

$$\sigma(a) \cdot \rho(\sigma(b)) = \rho(\sigma(ab)) = \rho(\sigma(a)) \cdot \tau(b), \quad (a, b \in \mathcal{A}).$$

(ii) *approximate (σ, τ)-biprojective if there is a net $\{\rho_\alpha\}_\alpha$ of bounded linear maps from $\mathcal{A}$ to $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that each $\Delta\rho_\alpha$ is the identity map on $\tau(\mathcal{A})$ and*

$$\lim_\alpha \sigma(a) \cdot \rho_\alpha(\sigma(b)) = \lim_\alpha \rho_\alpha(\sigma(ab)) = \lim_\alpha \rho_\alpha(\sigma(a)) \cdot \tau(b), \quad (a, b \in \mathcal{A}).$$

For more, see also Definition 3.2 of [18]. Reformulating the above results to avoid the use of an adjoined identity we have the next results.

Proposition 4.3. *Suppose that there is $e \in \mathcal{A}$ such that e is an identity for $\sigma(\mathcal{A}) \cup \tau(\mathcal{A})$ and $\sigma(e) = e = \tau(e)$. Then, $\mathcal{A}$ is approximate (σ, τ)-biprojective if and only if $\mathcal{A}$ has a (σ, τ)-pseudo diagonal.*

Corollary 4.2. *Let $\mathcal{A}$ be a Banach algebra and $\sigma, \tau \in$Hom$(\mathcal{A})$. Then $\mathcal{A}^\sharp$ has a $(\sigma^\sharp, \tau^\sharp)$-pseudo diagonal if and only if $\mathcal{A}^\sharp$ is approximate $(\sigma^\sharp, \tau^\sharp)$-biprojective.*

Proposition 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\sigma, \tau \in$Hom$(\mathcal{A})$ and $\varphi, \psi \in$Hom$(\mathcal{B})$. If $\mathcal{A}$ is approximate (σ, τ)-biprojective and $\mathcal{B}$ is (φ, ψ)-biprojective, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximate $(\sigma \otimes \varphi, \tau \otimes \psi)$-biprojective.*

Example 4.1. Let G be an infinite locally compact group and put $\mathcal{A} := L^1(G)^{**}$. Then $\mathcal{A}$ is a unital Banach algebra which is not approximate amenable by Theorem 3.3 in [7]. Consider E as the identity of $\mathcal{A}$ and $\sigma \in$Hom$(\mathcal{A})$. Suppose that $D : \mathcal{A} \longrightarrow X^*$ is a $(\sigma, -\sigma)$-derivation, where X is a $(\sigma, -\sigma)$-unital Banach $\mathcal{A}$-bimodule. For each $F \in \mathcal{A}$, let $H_1, H_2 \in \mathcal{A}$ and $f \in X^*$ such that $D(F) = \sigma(H_1) \cdot g \cdot (-\sigma(H_2))$. Then

$$\begin{aligned} D(F).(-\sigma)(E) &= \sigma(H_1) \cdot g \cdot \sigma(H_2E) \\ &= \sigma(H_1) \cdot g \cdot \sigma(H_2) \\ &= -D(F). \end{aligned}$$

Thus, for each F in $\mathcal{A}$,

$$D(F) = D(FE) = \sigma(F) \cdot D(E) + D(F) \cdot (-\sigma(E)) = -D(F),$$

and hence D is zero. Consequently, the Banach algebra $\mathcal{A}$ is $(\sigma, -\sigma)$-approximate amenable. Moreover, $\mathcal{A}$ has a $(\sigma, -\sigma)$-pseudo virtual diagonal and it is $(\sigma, -\sigma)$-approximate biflat by Proposition 3.2 and Corollary 4.1.

The preceding example shows that the class of approximately (σ, τ)-amenable Banach algebras is large than the class of approximate amenable Banach algebras.

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