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AN ANALYTICAL SOLUTION TO TIME-SPACE FRACTIONAL BLACK-SCHOLES OPTION PRICING MODEL

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The subject of this manuscript is devoted to a novel combination of the complex fractional transform (CFT) and residual power series method (RPSM) to acquire a quick solution to a time-space fractional Black-Scholes equation arising in the financial market. The fractional operator is considered as a local fractional derivative. By utilizing the initial conditions, the presented CFT-RPSM provides an accurate approximation in an easy manner and only with a few iterations. The applicability of the CFT-RPSM is shown on a test example and numerical results are presented through tables and figures. A comparison with an existing approximation method indicates that our approach is not only accurate but is also more straightforward in implementation. The current strategy can be easily applied to the other fractional models in physical science and engineering.

Keywords: Analytical solutions, Black Scholes model, Complex fractional transform, Local fractional derivative, Option pricing, RPSM

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1. Introduction

Financial markets have attracted attention of many scholars in the field of mathematics and economic over the ealier decades. In early 1970's, the first financial model to estimate the value of option pricing was given by M. Scholes, and F. Black [7]. This celebrated model known as Black-Scholes model has been generally adopted as an essential and standard instrument by finance practitioner. This model as a continuous-time mathematical model provides the basic methodology of option pricing.

Let $\psi(s,\tau)$ denote the option value at time τ and asset price s, the Black-Scholes equation (BSE) is given by

$$\frac{\partial\psi(s,\tau)}{\partial\tau} + rs\,\frac{\partial\psi(s,\tau)}{\partial s} + \frac{1}{2}\sigma^2 s^2\,\frac{\partial^2\psi(s,\tau)}{\partial s^2} - r\,\psi(s,\tau) = 0, \qquad (1)$$

where the coefficient parameter r represents the (risk-free) interest rate and σ denotes the volatility parameter. The related payoff functions for call and put

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options are

$$\psi_c(s,T) = \max(s-E,0), \quad \psi_p(s,T) = \max(E-s,0), \tag{2}$$

where E is the exercise (strike) price and T shows the maturity time or the expiration date. Over the past decades, numerous research studies have been conducted toward solving the classical Black-Scholes model (1), see [28] for its applications [28]. Moreover, several research studies have been devoted to proposing diverse analytical and computational approaches such as method of line [29], Cauchy Euler scheme [24], (explicit) difference procedures [5, 34], multivariate pade approximate approach [26], and Legendre wavelet scheme [9] to attain the approximate solution of BSE.

It is known that the classical BSE (1) can be converted to a parabolic PDE by the aid of following transformations

$$s = \eta\, e^x, \quad t = \sigma^2(T-\tau)/2, \quad \psi(s,\tau) = \eta\, v(x,t).$$

It is not a difficult task to show that under the former change of variables the resulting parabolic equation is obtained as

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2 v(x,t)}{\partial x^2} + (k-1)\dfrac{\partial v(x,t)}{\partial x} - k\, v(x,t) = 0, \quad k = \dfrac{2r}{\sigma^2}, \\ v(x,0) = v_0(x). \end{cases} \tag{3}$$

Over the past decades, lots of research papers have been devoted to fractional calculus including fractional differential equations. In fact and in particular PDEs are an essential tool for modelling many natural phenomena in science and engineering. The concept of fractional derivatives helps us to understand the nature of the models more deeply compared to the integer-order counterparts. Their aids also simplify the controlling design without any loss of hereditary behaviors of underlying events [20], see also [30, 31]. In this respect, many works have been conducted to solve diverse important mathematical models in science and engineering. Among others, we mention the previous works [6], [13]-[18], [27],[32, 33], and [35].

In this work, our aim is to solve the generalized version of BSE (3). To be more precise, we consider the space-time fractional BSE in the form [11, 25]

$$\begin{cases} {}^{LF}D_t^{\alpha} v(x,t) = {}^{LF}D_x^{2\beta} v(x,t) + (k-1)\, v_x(x,t) - k\, v(x,t) = 0, \quad 0 < \alpha, \beta \leqslant 1, \\ v(x,0) = v_0(x), \end{cases} \tag{4}$$

where ${}^{LF}D_t^{\alpha}$ and ${}^{LF}D_x^{2\beta}$ represent the local fractional derivative (LFD) with respect to t and x respectively. When $\beta = 1$ and the fractional derivative is taken in the sense of Caputo, the model (4) reduces to a time fractional BSE. In this case, this model has been considered in the literature and solved by some developed methods such as Adomian decomposition method [37], homotopy perturbation technique and homotopy analysis method [22], and power series method [10], to name a few.

In this research manuscript, we propose a combined analytical scheme to find approximate solutions of (4). For this purpose, we first apply the complex fractional transform (CFT) [23] to convert it to a integer-order PDE. Hence, we employ an iterative analytical approach, i.e., the residual power series method (RPSM) to gain the numerical solutions of (4). In fact, the RPSM is a novel recently proposed iterative strategdy to find the (analytical) Taylor expansion series form of solutions to systems of linear as well as nonlinear ODEs and PDEs. This technique was first propounded by O.A. Arqub [1]. This approach has now been successfully applied to different other problems. Among various solved model problems, let us mention the regular initial value proplems, the Lane-Emden equation, the fractional-order BVP, the non-composite and composite differential equations [2, 3, 4].

The RPSM approach actually utilizes the (generalized) Taylor series expansion together with residual error function to construct the power series solutions to diverse linear and nonlinear differential equations without employing any linearization, discretization, and perturbation. Let say the the solution of the transformed (integer-order) PDE is denoted by $u(x,t)$ and the associated initial condition is $u_0(x)$. The idea behind the underlying RPSM is to write $u(x,t)$ in terms of power series form about the initial point $t = t_0$. This implies that we have $u(x,t) = \sum_{n=0}^{\infty} g_n(x)\,(t-t_0)^n$. Practically, we confine ourselves to a truncated version with only i terms. Thus, we seek for approximate solutions in the forms

$$u_i(x,t) = u_0(x) + \sum_{n=0}^{i} g_n(x)\,(t-t_0)^n, \quad i = 1, 2, 3, \dots,$$

where $u_0(x) = u(x, t = 0)$. Hence, we insert the truncated series solutions into the PDE to get the residual error functions (REF) denoted by $\mathrm{Res}_i(x,t)$. By the fact that the REF belongs to C^∞, we have

$$\mathrm{Res}_\infty(x,t) := \lim_{i\to\infty} \mathrm{Res}_i(x,t).$$

In addition, during the construction of the RPSM, we use the following principal rules as

$$\frac{\partial^k}{\partial t^k}\mathrm{Res}_\infty(x,0) = 0, \quad \frac{\partial^k}{\partial t^k}\mathrm{Res}_i(x,0) = 0, \quad k = 1, 2, \dots, i.$$

For more details, we refer the readers to [1]-[4].

The plan of the remaining part of this study is as follows. In the next Section 2, some fact on the fractional calculus and the complex fractional transform are given. Section 3 as the main part of this work is devoted to the implemention of RPSM to the transformed PDE via CFT. This approach is called CFT-RPSM. The application of CFT-RPSM to an example is illustrated by numerical simulations. A comparison with an existing method is also carried out afterwards. Some conclusions are given in the final Section 4.

2. **Preliminaries and notations**

This section is devoted to review first some fundamental definitions of fractional calculus and their properties. Hence, we give some facts on complex fractional transform method used in the subsequent sections.

2.1. **Local fractional derivative**

Let us define the concept of local fractional derivative. In fact, the local fractional derivative (LFD) is based on the definition of Riemann-Liouville fractional derivative, see [21, 36] for more information.

Definition 2.1. *A function $g(t)$ is called fractional local continuous (FLC) at $t = t_0$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|t - t_0| < \delta$, then*

$$|g(t) - g(t_0)| < \epsilon^{\alpha}, \quad 0 < \alpha \leqslant 1,$$

and we write $\lim_{t\to t_0} g(t) = g(t_0)$.

If $g(t)$ be FLC on the interval (c, d), we denote is as $g(t) \in C_{\alpha}(c, d)$. Next, we have:

Definition 2.2. *Suppose that $g(t) \in C_{\alpha}(c, d)$. The local fractional derivative (LFD) of order α of function $g(t)$ is defined by*

$${}^{LF}D_t^{\alpha} g(t_0) = \frac{d^{\alpha} g(t)}{dt^{\alpha}}\Big|_{t=t_0} := \frac{\Delta^{\alpha}[g(t) - g(t_0)]}{(t - t_0)^{\alpha}}, \quad 0 < \alpha \leqslant 1,$$

where $\Delta^{\alpha}[g(t) - g(t_0)] \cong \Gamma(1 + \alpha)\,[g(t) - g(t_0)]$. Here, the function $\Gamma(\cdot)$ represents the well-known Gamma function.

The main properties of the LFD are summarized as follow

a) ${}^{LF}D_t^{\alpha}(C) = 0, \quad (C \text{ is a constant}),$
b) ${}^{LF}D_t^{\alpha}(C\, g(t)) = C\, {}^{LF}D_t^{\alpha} g(t),$
c) ${}^{LF}D_t^{\alpha}(t^{\xi}) = \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-\alpha)} t^{\xi-\alpha}, \quad \text{for } \xi \geqslant \alpha > 0,$
d) ${}^{LF}D_t^{k\alpha}(g(t)) = \underbrace{{}^{LF}D_t^{\alpha}\,{}^{LF}D_t^{\alpha}\,{}^{LF}D_t^{\alpha}\,{}^{LF}D_t^{\alpha}}_{k\ times}(g(t)),$
e) ${}^{LF}D_t^{\alpha}[(f \circ g)(t)] = D_g^1 f \cdot {}^{LF}D_t^{\alpha} g.$

2.2. **The complex fractional transform**

Let assume that we have the following general nonlinear fractional PDE in the form

$$F(u, {}^{LF}D_t^{\alpha}, {}^{LF}D_x^{\beta}, {}^{LF}D_t^{2\alpha}, {}^{LF}D_x^{2\alpha}, \ldots) = 0, \quad 0 < \alpha, \beta \leqslant 1, \tag{5}$$

where $u = u(x, t)$ and ${}^{LF}D_t^{\alpha}$, ${}^{LF}D_x^{\beta}$ are the local fractional derivatives defined in the previous section. The original complex fractional transform (CFT) was first introduced in [23] in the form

$$u(x,t) = u(\zeta), \quad \zeta = \frac{at^\alpha}{\Gamma(1+\alpha)} + \frac{bx^\beta}{\Gamma(1+\beta)},$$

where a, b are two unknown constants to be specified later. Using this transformation, the given equation (5) is converted to a PDE with integer-order derivative. A modification of the former transformation is given by [19]

$$\tau = \frac{at^\alpha}{\Gamma(1+\alpha)}, \quad \chi = \frac{ax^\beta}{\Gamma(1+\beta)}. \tag{6}$$

By utilizing properties c) and d) of LFD, we arrive at

$$\begin{cases} {}^{LF}D_t^\alpha u = {}^{LF}D_t^\alpha u(\tau(t)) = D_\tau^1 \cdot {}^{LF}D_t^\alpha \tau = a\frac{\partial u}{\partial \tau}, \\ {}^{LF}D_x^\beta u = {}^{LF}D_x^\beta u(\chi(x)) = D_\chi^1 \cdot {}^{LF}D_x^\beta \chi = b\frac{\partial u}{\partial \chi}. \end{cases} \tag{7}$$

3. **The combined CFT-RPSM algorithm to solve the space-time BSE**

To proceed, we consider the generalized BSE (4). By using the modified CFT (6) and related relations (7) with $a, b = 1$, this equation is converted to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \chi^2} + (k-1)\frac{\partial v}{\partial \chi} - kv, \quad (\chi, \tau) \in \mathbb{R}^+ \times (0, T), \tag{8}$$

where T is a given final time and with initial condition

$$v(\chi, 0) = \max(e^\chi - 1, 0). \tag{9}$$

In according to the residual power series (RPS) algorithm, the solution of the BSE (8) is assumed to be written in the form of a Taylor series expansion about the initial point $\tau = \tau_0$ given by [8]

$$v(\chi, \tau) = \sum_{n=0}^{\infty} g_n(\chi)(\tau - \tau_0)^n, \quad \tau \geqslant \tau_0, \tag{10}$$

where the unknown coefficients $g_n(\chi)$ to be determined. Here, we set $\tau_0 = 0$ according to the initial condition. Obviously, $v(\chi, \tau)$ satisfies the given initial condition and one can write it as

$$g_0(\chi) = v(\chi, 0) = \max(e^\chi - 1, 0). \tag{11}$$

Thus, with the initial approximation $g_0(\chi)$ to $v(\chi, \tau)$, the series solution in (10) can be rewritten as

$$v(\chi, \tau) = g_0(\chi) + \sum_{n=1}^{\infty} g_n(\chi)\,\tau^k. \tag{12}$$

We now define the i-th truncated series related to $v(\chi, \tau)$ in (12) as

$$v_i(\chi, \tau) = g_0(\chi) + \sum_{n=1}^{i} g_n(\chi)\,\tau^n, \quad i = 1, 2, 3, \ldots. \tag{13}$$

The next goal is to define the residual function $\mathrm{Res}_v(\chi,\tau)$ associated to the transformed PDE (8). It is given by

$$\mathrm{Res}_v(\chi,\tau) := \mathbb{D}_\tau v(\chi,\tau) - \mathbb{D}_{xx} v(\chi,\tau) - (k-1)\,\mathbb{D}_x v(\chi,\tau) + k\,v(\chi,\tau), \quad (14)$$

where $\mathbb{D}_s v := \frac{\partial v}{\partial s}$.
In this way, we introduce the i-th residual function by

$$\mathrm{Res}_{v,i}(\chi,\tau) := \mathbb{D}_\tau v_i(\chi,\tau) - \mathbb{D}_{xx} v_i(\chi,\tau) - (k-1)\,\mathbb{D}_x v_i(\chi,\tau) + k\,v_i(\chi,\tau). \quad (15)$$

Now, our task is to determine the first unknown coefficient $g_1(\chi)$ in (13). In this respect, we take $i=1$ in equation (15) to get

$$\mathrm{Res}_{v,1}(\chi,\tau) = \mathbb{D}_\tau v_1(\chi,\tau) - \mathbb{D}_{xx} v_1(\chi,\tau) - (k-1)\,\mathbb{D}_x v_1(\chi,\tau) + k\,v_1(\chi,\tau),$$

where

$$v_1(\chi,\tau) = g_0(\chi) + g_1(\chi)\,\tau.$$

After differentiation and substituting the resulting derivatives into the residual function $\mathrm{Res}_{v,1}(\chi,\tau)$ we get

$$\begin{aligned}\mathrm{Res}_{v,1}(\chi,\tau) = g_1(\chi) &- \{g_0''(\chi) + g_1''(\chi)\,\tau\} - (k-1)\,\{g_0'(\chi) + g_1'(\chi)\,\tau\} \\ &+ k\,\{g_0(\chi) + g_1(\chi)\,\tau\}\,.\end{aligned}$$

By setting $\tau = 0$, the residual function becomes

$$\mathrm{Res}_{v,1}(\chi,0) = g_1(\chi) - g_0''(\chi) - (k-1)\,g_0'(\chi) + k\,g_0(\chi).$$

Now, from $\mathrm{Res}_{v,1}(\chi,0) = 0$, using definition of $g_0(\chi)$ in (11), and after some simplifications we immediately conclude that

$$g_1(\chi) = k\,[e^\chi - \max(e^\chi - 1, 0)]\,. \quad (16)$$

Thus, in the first iteration, we get the RPS approximate solution as

$$v_1(\chi,\tau) = \max(e^\chi - 1, 0) + k\,[e^\chi - \max(e^\chi - 1, 0)]\,\tau. \quad (17)$$

In order to find the second unknown coefficient $g_2(\chi)$ in (15), let us insert $i=2$ into (15) to arrive at

$$\mathrm{Res}_{v,2}(\chi,\tau) = \mathbb{D}_\tau v_2(\chi,\tau) - \mathbb{D}_{xx} v_2(\chi,\tau) - (k-1)\,\mathbb{D}_x v_2(\chi,\tau) + k\,v_2(\chi,\tau),$$

where

$$v_2(\chi,\tau) = g_0(\chi) + g_1(\chi)\,\tau + g_2(\chi)\frac{\tau^2}{2!}.$$

After inserting $v_2(\chi,\tau)$ into the 2-th residual function $\mathrm{Res}_{v,2}(\chi,\tau)$ we render

$$\begin{aligned}\mathrm{Res}_{v,2}(\chi,\tau) &= g_1(\chi) + g_2(\chi)\,\tau - \left\{g_0(\chi) + g_1''(\chi)\,\tau + g_2''(\chi)\frac{\tau^2}{2!}\right\} \quad (18)\\ &- (k-1)\left\{g_0'(\chi) + g_1'(\chi)\,\tau + g_2'(\chi)\frac{\tau^2}{2!}\right\} + k\left\{g_0(\chi) + g_1(\chi)\,\tau + g_2(\chi)\frac{\tau^2}{2!}\right\}.\end{aligned}$$

Now, from $\mathbb{D}_\tau \mathrm{Res}_{v,2}(\chi,\tau)=0$ we deduce

$$\mathrm{Res}_{v,2}(\chi,\tau)=g_2(\chi)-\{g_1''(\chi)+g_2''(\chi)\,\tau\}-(k-1)\,\{g_1'(\chi)+g_2'(\chi)\,\tau\}\\ +k\,\{g_1(\chi)+g_2(\chi)\,\tau\}\,.$$

By setting $\tau=0$ in the former relation we obtain

$$g_2(\chi)-g_1''(\chi)-(k-1)g_1'(\chi)+k\,g_1(\chi)=0.$$

By solving with respect to g_2 we get

$$g_2(\chi)=-k^2\,e^\chi+k^2\,\max(e^\chi-1,0). \tag{19}$$

Finally, the second approximate solution $v_2(\chi,\tau)$ is constituted as

$$v_2(\chi,\tau)=\max(e^\chi-1,0)+k\,\left[e^\chi-\max(e^\chi-1,0)\right]\tau\\ +k^2\left[\max(e^\chi-1,0)-e^\chi\right]\frac{\tau^2}{2!}. \tag{20}$$

In a similar way, we are able to find the third coefficient $g_3(\chi)$ in (13). By putting $i=3$ in relation (15) we have

$$\mathrm{Res}_{v,3}(\chi,\tau)=\mathbb{D}_\tau v_3(\chi,\tau)-\mathbb{D}_{xx}v_3(\chi,\tau)-(k-1)\,\mathbb{D}_x v_3(\chi,\tau)+k\,v_3(\chi,\tau),$$

where

$$v_3(\chi,\tau)=g_0(\chi)+g_1(\chi)\,\tau+g_2(\chi)\frac{\tau^2}{2!}+g_3(\chi)\frac{\tau^3}{3!}.$$

We then compute $\mathbb{D}_{\tau\tau}\mathrm{Res}_{v,3}(\chi,\tau)=0$ followed by inserting $\tau=0$. Solving the resulting relation in terms of $g_3(\chi)$ we get

$$g_3(\chi)=k^3\,e^\chi-k^3\max(e^\chi-1,0).$$

This implies that the third approximate RPS solution can be written as

$$v_3(\chi,\tau)=\max(e^\chi-1,0)+k\,\left[e^\chi-\max(e^\chi-1,0)\right]\tau\\ +k^2\left[\max(e^\chi-1,0)-e^\chi\right]\frac{\tau^2}{2!}\\ +k^3\,\left[e^\chi-\max(e^\chi-1,0)\right]\frac{\tau^3}{3!}.$$

Analogously, by continuing this process we can determine the fourth and fifth coefficients as

$$g_4(\chi)=-k^4\,e^\chi+k^4\,\max(e^\chi-1,0),\quad g_5(\chi)=k^5\,e^\chi-k^5\,\max(e^\chi-1,0).$$

Hence, the corresponding fourth and fifth approximate solutions obtained via RPS technique are given by

$$v_4(\chi,\tau)=\max(e^\chi-1,0)+k\left[e^\chi-\max(e^\chi-1,0)\right]\tau\\ +k^2\left[\max(e^\chi-1,0)-e^\chi\right]\frac{\tau^2}{2!}+k^3\left[e^\chi-\max(e^\chi-1,0)\right]\frac{\tau^3}{3!}\\ +k^4\,\left[\max(e^\chi-1,0)-e^\chi\right]\frac{\tau^4}{4!},$$

and

$$\begin{aligned} v_5(\chi,\tau) &= \max(e^{\chi}-1,0) + k\left[e^{\chi} - \max(e^{\chi}-1,0)\right]\tau \\ &+ k^2\left[\max(e^{\chi}-1,0) - e^{\chi}\right]\frac{\tau^2}{2!} + k^3\left[e^{\chi} - \max(e^{\chi}-1,0)\right]\frac{\tau^3}{3!} \\ &+ k^4\left[\max(e^{\chi}-1,0) - e^{\chi}\right]\frac{\tau^4}{4!} + k^5\left[e^{\chi} - \max(e^{\chi}-1,0)\right]\frac{\tau^5}{5!}. \end{aligned}$$

Thus, we have proved the following results for the RPS solution of model (8):

Lemma 3.1. *The coefficients and solutions of the truncated series* (13) *for* $n \geqslant 1$ *satisfy*

$$g_n(\chi) = (-1)^{n+1}\,k^n\left[e^{\chi} - \max(e^{\chi}-1,0)\right], \tag{21}$$

$$\begin{aligned} v_n(\chi,\tau) &= \max(e^{\chi}-1,0)\left[1 - k\tau + k^2\frac{\tau^2}{2!} - k^3\frac{\tau^3}{3!} + \ldots + (-1)^n k^n \frac{\tau^n}{n!}\right] \\ &- e^{\chi}\left[-k\tau + k^2\frac{\tau^2}{2!} - k^3\frac{\tau^3}{3!} + \ldots + (-1)^n k^n \frac{\tau^n}{n!}\right]. \end{aligned} \tag{22}$$

Proof. The proofs can be done straightforwardly by induction on n. □

It is an easy job to obtain the following result as the final solution of (8):

Corollary 3.1. *In the limiting case as* $n \to \infty$ *we have*

$$v(\chi,\tau) = \lim_{n\to\infty} v_n(\chi,\tau) = e^{-k\tau}\max(e^{\chi}-1,0) + e^{\chi} - e^{\chi-k\tau}.$$

Now, we use the aforementioned transformations (6) to recover the approximate solution of the original space-time BSE of fractional order (4). Thus, we get

$$\begin{aligned} v(x,t) &= \max\left(e^{\frac{x^\beta}{\Gamma(1+\beta)}} - 1, 0\right) \\ &+ \left[\max\left(e^{\frac{x^\beta}{\Gamma(1+\beta)}} - 1, 0\right) - e^{\frac{x^\beta}{\Gamma(1+\beta)}}\right]\sum_{n=1}^{\infty}\frac{1}{n!}\left\{\frac{-kt^\alpha}{\Gamma(1+\alpha)}\right\}^n. \end{aligned} \tag{23}$$

It should be emphasize that our approximate solution coincides with the solution obtained via the coupled transformed method (CTM) [11], which confirms our results. However, our proposed technique is more straightforward than the CTM.

Corollary 3.2. *In view of Corollary 3.1 and the FCT* (6)*, the exact solution in the case of* $\alpha, \beta = 1$ *takes the following form*

$$v(x,t) = e^{-kt}\max(e^x - 1, 0) + e^x(1 - e^{-kt}). \tag{24}$$

To further validate our results, we make a comparison in the next experiments. Table 1 shows the numerical results obtained by the proposed technique via (24) with $\alpha, \beta = 1$. In this respect, the outcomes of the CTM [11] are also

Table 1. A comparison of numerical solutions with $\alpha, \beta = 1$, $T = 1, k = 2$, and various $x, t \in [0, 1]$

x	t	Present Method	CTM [11]	Exact [11]
	0.00	0.329679953964361	0.3296799	0.3296799
	0.25	0.613705370652102	0.6137053	0.6137053
0.2	0.50	0.978401224664489	0.9784012	0.9784012
	0.75	1.446679970577035	1.4466799	1.4466799
	1.00	2.047961782423406	2.0479617	2.0479617
	0.00	0.550671035882778	0.5506710	0.5506710
	0.25	0.834696452570520	0.8346964	0.8346964
0.24	0.50	1.199392306582907	1.1993923	1.1993923
	0.75	1.667671052495453	1.6676710	1.6676710
	1.00	2.268952864341824	2.2689528	2.2689528
	0.00	0.698805788087798	0.6988057	0.6988057
	0.25	0.982831204775539	0.9828312	0.9828312
0.6	0.50	1.347527058787926	1.3475270	1.3475270
	0.75	1.815805804700473	1.8158058	1.8158058
	1.00	2.417087616546843	2.4170876	2.4170876

tabulated in this table. Clearly, an excellent alignment between our results and those obtained by the CTM are seen.

Now, we turn our attention to the fractional cases and take two values of $\alpha = 0.75, \beta = 1$ and $\alpha = 1, \beta = 0.75$. The graphical picture of approximate solutions for $k = 2$ and with $i = 10$ number of iterations are plotted in Fig. 1 on the space-time $(x, t) \in [0, 1] \times [0, 1]$. Finally, the convergence of the presented combined technique is shown numerically through Table 2. In this experiments, we set $i = 5, 10, 15, 20$, and $i = 25$ and again $k = 2$. The numerical solutions at various points $x_p = p/10$ for $p = 0, 1, \ldots, 10$ and the final $t = T = 1$ are reported in Table 2. Here, we used $\alpha, \beta = 0.75$. Obviously, by increasing i, the number of fixed digits are increased.

4. **Conclusions**

In this study, we have proposed a combined approach based on the complex fractional transform (CFT) and the residual power series method (RPSM) to provide an accurate estimation of solutions of a class of time-space fractional Black-Scholes (BS) equation arising in the financial market. The presented CFT-RPSM technique can be considred as a directly applicable approach, which is free of any sort of discretization, linearization, or any other extra imposed assumptions. By utilizing the given initial conditions, the approximate analytical solutions of the governing equations are straightforwardly evaluated. The application of the CFT-RPSM to a test example indicates that one obtains

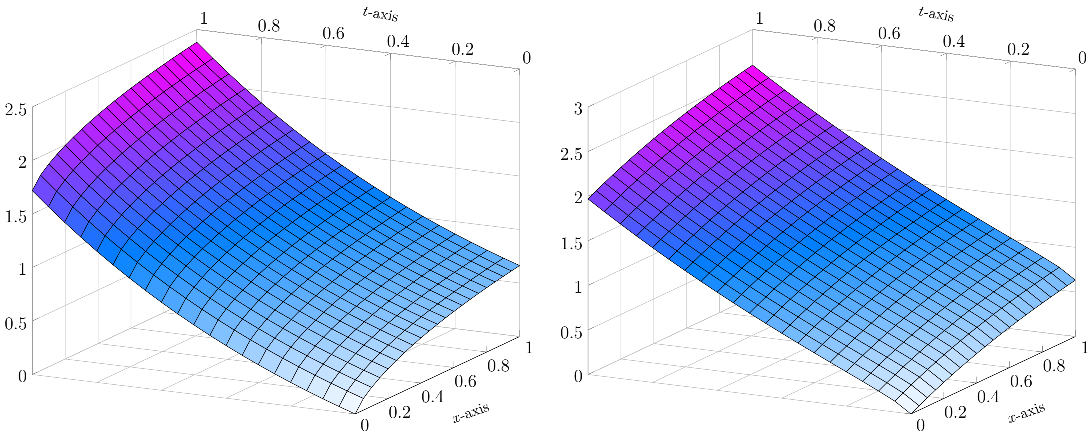

FIGURE 1. Visualization of approximate solution using $\alpha = 0.75, \beta = 1$ (left) and $\alpha = 1, \beta = 0.75$ (right) with $k = 2$ and $i = 10$ number of iterations.

TABLE 2. A comparison of numerical solutions with $\alpha, \beta = 0.75$, $T = 1, k = 2$, and various $x \in [0, 1], t = T$.

x	$i = 5$	$i = 10$	$i = 15$	$i = 20$	$i = 25$
0.0	0.99816599	0.88641051	0.886520218968664	0.886520208260417	0.886520208260637
0.1	1.21164130	1.09988583	1.099995533328814	1.099995522620567	1.099995522620786
0.2	1.38276062	1.27100514	1.271114847383046	1.271114836674800	1.271114836675019
0.3	1.55251712	1.44076165	1.440871352594751	1.440871341886504	1.440871341886724
0.4	1.72668996	1.61493449	1.615044190496329	1.615044179788082	1.615044179788302
0.5	1.90790668	1.79615121	1.796260911388124	1.796260900679877	1.796260900680097
0.6	2.09781142	1.98605594	1.986165647402933	1.986165636694686	1.986165636694906
0.7	2.29764056	2.18588508	2.185994784996265	2.185994774288018	2.185994774288238
0.8	2.50843807	2.39668259	2.396792295468982	2.396792284760735	2.396792284760955
0.9	2.73115323	2.61939775	2.619507454469871	2.619507443761624	2.619507443761844
1.0	2.96669116	2.85493568	2.855045384244428	2.855045373536181	2.855045373536401

a better approximation with only a few iterations. Numerical simulations and MATLAB plots have been presented to demonstrate the main results derived in this study. A comparison to other existing computational procedure, i.e., the CTM [11] shows that our approach is easy to implement and computationally efficient.

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