DIFFERENT BASES IN INVESTIGATION OF $\sqrt[3]{2}$

Mitja Lakner[1], Peter Petek[2], Marjeta Škapin Rugelj[3]

The problem tackled is the nature of the continued fraction expansion of $\sqrt[3]{2}$: are the partial quotients bounded or not. Numerical experiments suggest an even stronger result in the lines of Kuzmin statistics. We apply different sets of bases for the ambient vector space V connected with the adjunction ring $\mathbb{Z}[\sqrt[3]{2}]$. As a result we get a criterion for continued fraction convergents in terms of their coefficient vectors from a lattice.

Keywords: bases, cubic root, continued fractions
MSC2010: 11A55, 11K50, 11R16.

1. Introduction

Stability of invariant circles in K.A.M. theory depends on the respective rotation number. The most stable circle has $\phi = \frac{-1+\sqrt{5}}{2}$, the golden mean ratio, as its rotation number, all partial quotients equal $b_i = 1$. As in [8] boundness of partial quotients would be a bonus in representation on a computer. Here however, experiments strongly suggest the opposite. Also cubic irrationals are interesting in studying quasiperiodic motion [3], [9]. Here we investigate $\sqrt[3]{2}$ and its adjunction ring. It is a common belief that the partial quotients of $\sqrt[3]{2}$ are not bounded, supported by extensive computations, but no proof.

Even more, computations suggest that their relative frequencies in the limit obey the Kuzmin law $P(b_n = k) = \log_2 \frac{(k+1)^2}{k(k+2)}$. In [12] several algebraic numbers were used in computations, among them $\sqrt[3]{2}$, $\sqrt[4]{2}$, $\sqrt[5]{2}$ and good accordance was found with Kuzmin's statistics, for $\sqrt[3]{2}$ even too good. So later in [6] and [2] larger samples were taken and the anomaly seemed to disappear. We played the same game, only we had the advantage of more sophistical computation tools that evolved in the years in between. The experimental results supporting of the stronger hypothesis instigated us to try towards some theoretical results.

Questions about "big" partial quotients still linger and tease us, although some explanation was given in a special case, using elliptic modular functions [13], [4]. In this paper we work towards the proof of

[1]Professor, University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, 1000 Ljubljana, Slovenia, e-mail: `mlakner@fgg.uni-lj.si`

[2]Professor, University of Ljubljana, Faculty of Education, Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia, e-mail: `Peter.Petek@guest.arnes.si`

[3]Teacher, University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, 1000 Ljubljana, Slovenia, e-mail: `mskapin@fgg.uni-lj.si`

Hypothesis. The partial quotients of $\sqrt[3]{2}$ are not bounded.

Only a very limited partial result is given, helping to recognize possible convergents and estimating the next partial quotient.

2. Adjunction ring $\mathbb{Z}[\sqrt[3]{2}]$

In the adjunction ring $\mathbb{Z}[\sqrt[3]{2}]$ we have the unit $\rho = 1+\sqrt[3]{2}+\sqrt[3]{4}$ and its inverse $\sigma = -1+\sqrt[3]{2}$, $\rho\sigma = 1$. Obviously the continued fraction expansions for $\sqrt[3]{2}$ and σ differ only in the starting partial quotient, so we may consider approximations to σ. In order to improve approximations, we construct a series of vector space bases of $\mathbb{R}^3$.

A multiplicative norm is defined in $\mathbb{Z}[\sqrt[3]{2}]$ as follows: for $x = a + b\sqrt[3]{2} + c\sqrt[3]{4}$, its norm is $N(x) = a^3 + 2b^3 + 4c^3 - 6abc = x \cdot x' \cdot x''$ where $x' = a + \omega b\sqrt[3]{2} + \omega^2 c\sqrt[3]{4}$, $x'' = a + \omega^2 b\sqrt[3]{2} + \omega c\sqrt[3]{4}$ and $\omega = e^{\frac{2\pi i}{3}}$ is 3^{rd} root of unity.

Division in general leads to the corresponding field $\mathbb{Q}[\sqrt[3]{2}]$. Carrying out the rationalization of the denominator as in

$$\frac{1}{x} = \frac{x' \cdot x''}{N(x)} = \frac{a^2 - 2bc + (2c^2 - ab)\sqrt[3]{2} + (b^2 - ac)\sqrt[3]{4}}{N(x)}$$

gives the elements of $\mathbb{Q}[\sqrt[3]{2}]$ in the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ where a, b, c are fractions.

3. The ambient vector space V

Instead of $(1, \sqrt[3]{2}, \sqrt[3]{4})$ we use the algebraic basis $(\rho, 1, \sigma)$ and elements $a + b\sqrt[3]{2} + c\sqrt[3]{4} = w$ of $\mathbb{Z}[\sqrt[3]{2}]$ are expressed by

$$\begin{aligned} w &= x \cdot \rho + y \cdot 1 + z \cdot \sigma \\ &= c \cdot \rho + (a - 2c + b) \cdot 1 + (b - c) \cdot \sigma. \end{aligned}$$

Now, let $V = \mathbb{R}^3$ be the 3-dimensional space endowed with the usual scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ and cross product $\mathbf{a} \times \mathbf{b}$. Vectors can be written as ordered triplets $V = \{\mathbf{v} = (x, y, z); x, y, z \in \mathbb{R}\}$ and we define a linear mapping $\eta : \mathbb{Z}[\sqrt[3]{2}] \to \mathbb{Z}^3 \subset V$ by $\eta(x \cdot \rho + y \cdot 1 + z \cdot \sigma) = (x, y, z)$, the resulting image consisting of all vectors with integer entries, multiplication inherited from $\mathbb{Z}[\sqrt[3]{2}]$. Taking into account that

$$x \cdot \rho + y \cdot 1 + z \cdot \sigma = (x + y - z) + (x + z)\sqrt[3]{2} + x\sqrt[3]{4}$$

we can define the norm function in the whole V:

$$\widetilde{N}(x, y, z) = (x + y - z)^3 + 2(x + z)^3 + 4x^3 - 6(x + y - z)(x + z)x \tag{1}$$

Multiplication with σ will prove very important and we observe that

$$\eta(\sigma \cdot w) = S\eta(w)$$

where the matrix

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix},$$

by the way, represents a hyperbolic toral authomorphism [5].

4. **Interaction among different bases in V**

The term basis comes in several ways in mathematics. Our discussion needs it in two appearances:

- as the *basis* of a number system (ρ in our case),
- as the *basis* of a vector space (different bases of V here).

The first usage figures in our paper [11], here we are concerned with the second one. We can think of the basis $\mathcal{B}_0 = ((1,0,0),(0,1,0),(0,0,1))$ as the canonical one. And we shall also write $\mathcal{B}_0 = (\mathbf{s}_{-1},\mathbf{s}_0,\mathbf{s}_1)$ as we shall denote $\eta(\sigma^j) = \mathbf{s}_j$, and noting $S\mathbf{s}_j = \mathbf{s}_{j+1}$ we have

$\mathbf{s}_0 = (0,1,0)$, $\mathbf{s}_1 = (0,0,1)$, $\mathbf{s}_2 = (1,-3,-3)$, $\mathbf{s}_3 = (-3,10,6)$,
$\mathbf{s}_4 = (6,-21,-8)$, $\mathbf{s}_5 = (-8,30,3)$, $\mathbf{s}_6 = (3,-17,21)$,

and we may also need the ones with negative indices

$\mathbf{s}_{-1} = (1,0,0)$, $\mathbf{s}_{-2} = (3,3,1)$, $\mathbf{s}_{-3} = (12,10,3)$,
$\mathbf{s}_{-4} = (46,39,12)$, $\mathbf{s}_{-5} = (177,150,46)$, $\mathbf{s}_{-6} = (681,577,177)$,

as well as the inverse matrix

$$S^{-1} = \begin{bmatrix} 3 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

which we also meet in the Jacobi-Perron algorithm [1]. Further we define the series of bases $\mathcal{B}_j = (\mathbf{s}_{j-1},\mathbf{s}_j,\mathbf{s}_{j+1})$ for all integer j. These are good bases for our purposes as

Lemma 4.1. *Elements of $\mathbb{Z}^3$ have integer coefficients in each basis $\mathcal{B}_j$.*

Proof. Let v be an element from $\mathbb{Z}^3$. If we multiply expansion

$$v = \alpha\mathbf{s}_{j-1} + \beta\mathbf{s}_j + \gamma\mathbf{s}_{j+1}$$

by integer element matrix S^{-j}, we get vector with integer components

$$S^{-j}v = \alpha\mathbf{s}_{-1} + \beta\mathbf{s}_0 + \gamma\mathbf{s}_1 = (\alpha,\beta,\gamma).$$

□

Besides the series of bases $\mathcal{B}_j$ we also define the conjugate series $\mathcal{B}^*_j$ in the following manner.

For start $\mathbf{s}^*_0 = (1,0,0)$ and with the adjoint matrix

$$S^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -3 \end{bmatrix}$$

we define vectors $\mathbf{s}^*_{j+1} = S^*\mathbf{s}^*_j$ for positive and negative indices. So we have

$\mathbf{s}^*_{-3} = (46,12,3)$, $\mathbf{s}^*_{-2} = (12,3,1)$, $\mathbf{s}^*_{-1} = (3,1,0)$, $\mathbf{s}^*_0 = (1,0,0)$,
$\mathbf{s}^*_1 = (0,0,1)$, $\mathbf{s}^*_2 = (0,1,-3)$, $\mathbf{s}^*_3 = (1,-3,6)$.

Remark 4.2. Comparing with the vectors $\mathbf{s}_j = (x_j,y_j,z_j)$ we see that

$$\mathbf{s}^*_j = (x_{j-1},x_j,x_{j+1}) = (z_{j-2},z_{j-1},z_j).$$

The bases series $\mathcal{B}_j^* = (\mathbf{s}_{j-1}^*, \mathbf{s}_j^*, \mathbf{s}_{j+1}^*)$ again providing integer coefficients for $\mathbb{Z}[\sqrt[3]{2}]$. The two bases series shall be useful in further computations.

The action of linear transformation S is best understood in terms of its eigenvalues and eigenspaces. One eigenvalue is real, smaller then 1, and two are complex conjugate greater than 1

$$\lambda_1 = \sigma = -1 + \sqrt[3]{2},$$

$$\lambda_2 = \sigma' = -1 + \omega\sqrt[3]{2} = -1 - \tfrac{\sqrt[3]{2}}{2} + \tfrac{i}{2}\sqrt{3}\sqrt[3]{2},$$

$$\lambda_3 = \sigma'' = -1 + \omega^2\sqrt[3]{2} = -1 - \tfrac{\sqrt[3]{2}}{2} - \tfrac{i}{2}\sqrt{3}\sqrt[3]{2}.$$

The eigenvectors being $\mathbf{h}$ and $\mathbf{g} \pm i\mathbf{k}$ where

$$\mathbf{h} = \tfrac{1}{6}(\sqrt[3]{2}, 2 - 2\sqrt[3]{2} + \sqrt[3]{4}, -\sqrt[3]{2} + \sqrt[3]{4}) \doteq (0.209987, 0.177926, 0.05458),$$

$$\mathbf{g} = \tfrac{1}{12}(-\sqrt[3]{2}, 4 + 2\sqrt[3]{2} - \sqrt[3]{4}, \sqrt[3]{2} - \sqrt[3]{4}) \doteq (-0.104993, 0.411037, -0.02729),$$

$$\mathbf{k} = \tfrac{\sqrt{3}}{12}(\sqrt[3]{2}, -2\sqrt[3]{2} - \sqrt[3]{4}, -\sqrt[3]{2} - \sqrt[3]{4}) \doteq (0.181854, -0.592829, -0.410976).$$

We can also compute the rotation angle $\theta = \pi - \arctan\frac{\sqrt{3}\sqrt[3]{2}}{2+\sqrt[3]{2}} \doteq 146.2^0$ and after some computation we can express $\mathbf{s}_j = \sigma^j\mathbf{h} + 2\rho^{\frac{j}{2}}(\mathbf{g}\cos(j\theta) - \mathbf{k}\sin(j\theta))$, $S\mathbf{g} = \sqrt{\rho}(\mathbf{g}\cos\theta - \mathbf{k}\sin\theta)$ and $S\mathbf{k} = \sqrt{\rho}(\mathbf{g}\sin\theta + \mathbf{k}\cos\theta)$.

The norm (1) takes zero value on the union of the eigenplane P spanned by vectors $\mathbf{g}$, $\mathbf{k}$ and the eigenline of $\mathbf{h}$. Except for the origin, there is no rational point (x, y, z) of zero norm.

The basic vectors $\mathbf{s}_j$ with increasing positive j are approaching the invariant plane and for negative j being almost colinear to the eigenvector $\mathbf{h}$.

On the other hand we can construct the eigenbasis $\mathcal{B}_e = (\mathbf{h}, \mathbf{g}, \mathbf{k})$ and the conjugate eigenbasis.

To make the conjugate eigenbasis, we compute the vector products

$$\mathbf{h}^* = \mathbf{g} \times \mathbf{k}, \quad \mathbf{g}^* = \mathbf{h} \times \mathbf{k}, \quad \mathbf{k}^* = \mathbf{h} \times \mathbf{g}$$

and they constitute the conjugate eigenbasis $\mathcal{B}_e^*$

$$\mathbf{h}^* = \tfrac{-\sqrt{3}}{36}(1 + \sqrt[3]{2} + \sqrt[3]{4}, 1, -1 + \sqrt[3]{2}) \doteq (-0.185104, -0.0481125, -0.0125055),$$

$$\mathbf{g}^* = \tfrac{-\sqrt{3}}{36}(-2 + \sqrt[3]{2} + \sqrt[3]{4}, -2, 2 + \sqrt[3]{2}) \doteq (-0.0407668, 0.096225, -0.156843),$$

$$\mathbf{k}^* = \tfrac{1}{12}(\sqrt[3]{2} - \sqrt[3]{4}, 0, \sqrt[3]{2}) \doteq (-0.02729, 0, 0.104993).$$

Later we shall need also the mixed product

$$[\mathbf{h}, \mathbf{g}, \mathbf{k}] = \frac{-\sqrt{3}}{36} = -M \doteq -0.0481125.$$

The plane P^* of the vectors $\mathbf{g}^*, \mathbf{k}^*$ is the invariant plane of S^*.

Lemma 4.3. *The vectors $\mathbf{h}^*, \mathbf{g}^* \pm i\mathbf{k}^*$ are eigenvectors of the matrix S^*.*

Proof. First we find the scalar products:

$$\langle S^*\mathbf{h}^*, \mathbf{g}\rangle = \langle \mathbf{g}\times\mathbf{k}, S\mathbf{g}\rangle = \langle \mathbf{g}\times\mathbf{k}, \sqrt{\rho}(\mathbf{g}\cos\theta - \mathbf{k}\sin\theta)\rangle = 0,$$

$$\langle S^*\mathbf{h}^*, \mathbf{k}\rangle = \langle \mathbf{g}\times\mathbf{k}, S\mathbf{k}\rangle = \langle \mathbf{g}\times\mathbf{k}, \sqrt{\rho}(\mathbf{g}\sin\theta + \mathbf{k}\cos\theta)\rangle = 0.$$

So it is clear $S^*\mathbf{h}^*$ is orthogonal to both $\mathbf{g}$ and $\mathbf{k}$ and therefore colinear to $\mathbf{h}^*$ itself, therefore an eigenvector with a real eigenvalue, the only one being σ, therefore $S^*\mathbf{h}^* = \sigma\mathbf{h}^*$. The other two cases demand a little more work to tell apart the two complex eigenvalues, of course unless we want to go into direct computation. □

We can also express the vectors $\mathbf{s}_j^*$ in terms of the conjugate eigenbasis

$$\mathbf{s}_j^* = -2\sqrt{3}\sqrt[3]{2}\left(\sigma^j\mathbf{h}^* + \rho^{\frac{j}{2}}(\mathbf{g}^*\cos(j\theta - \frac{\pi}{3}) - \mathbf{k}^*\sin(j\theta - \frac{\pi}{3}))\right)$$

and infer a connection between the basis and conjugate basis via the matrix

$$T = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

namely $\mathbf{s}_j^* = T\mathbf{s}_j$ and also connecting the eigenbasis with conjugate eigenbasis

$$T\mathbf{h} = -2\sqrt{3}\sqrt[3]{2}\mathbf{h}^*, \quad T\mathbf{g} = \frac{\sqrt{3}\sqrt[3]{2}}{2}(-\mathbf{g}^* - \sqrt{3}\mathbf{k}^*), \quad T\mathbf{k} = \frac{\sqrt{3}\sqrt[3]{2}}{2}(\sqrt{3}\mathbf{g}^* - \mathbf{k}^*).$$

We shall also need some scalar and cross products of the basis vectors.

Lemma 4.4. *The scalar products of basis vectors are as follows:*

$$\langle \mathbf{s}_n^*, \mathbf{s}_k\rangle = \langle \mathbf{s}_0^*, \mathbf{s}_{k+n}\rangle$$

with $\langle \mathbf{s}_0^*, \mathbf{s}_{-2}\rangle = 3$, $\langle \mathbf{s}_0^*, \mathbf{s}_{-1}\rangle = 1$, $\langle \mathbf{s}_0^*, \mathbf{s}_0\rangle = 0$, $\langle \mathbf{s}_0^*, \mathbf{s}_1\rangle = 0$, $\langle \mathbf{s}_0^*, \mathbf{s}_2\rangle = 1$, *and* $\langle \mathbf{s}_0^*, \mathbf{s}_3\rangle = -3$.

Proof. Since $\langle \mathbf{s}_n^*, \mathbf{s}_k\rangle = \langle S^{*n}\mathbf{s}_0^*, \mathbf{s}_k\rangle = \langle \mathbf{s}_0^*, S^n\mathbf{s}_k\rangle = \langle \mathbf{s}_0^*, \mathbf{s}_{k+n}\rangle$ we only need to read off the first component of $\mathbf{s}_j$ as $\mathbf{s}_0^* = (1, 0, 0)$. □

Lemma 4.5. *For two consecutive basis vectors we have the cross product*

$$\mathbf{s}_{-j}\times\mathbf{s}_{-j+1} = \mathbf{s}_j^*$$

and if we jump by one index: $\mathbf{s}_{-j-1}\times\mathbf{s}_{-j+1} = -\mathbf{s}_{j-1}^* + 3\mathbf{s}_j^*$.

Proof. Setting unknown coefficients α, β, γ: $\mathbf{s}_{-j}\times\mathbf{s}_{-j+1} = \alpha\mathbf{s}_{j-1}^* + \beta\mathbf{s}_j^* + \gamma\mathbf{s}_{j+1}^*$ and taking scalar products in turn with $\mathbf{s}_{-j-1}, \mathbf{s}_{-j}, \mathbf{s}_{-j+1}$ we get

$$1 = \alpha\cdot 3 + \beta\cdot 1 + \gamma\cdot 0$$

$$0 = \alpha\cdot 1 + \beta\cdot 0 + \gamma\cdot 0$$

$$0 = \alpha\cdot 0 + \beta\cdot 0 + \gamma\cdot 1$$

and from here $\alpha = \gamma = 0$ and $\beta = 1$. Likewise we prove the second formula. □

5. The shortest coefficient vector and convergents

The vector $(0, p, -q) = \eta(p - q\sigma)$ can as any vector be expanded with respect to any basis $\mathcal{B}_j$ and the coefficients are integers.

In [11], using the number system with basis ρ, we expressed $(0, p, -q)$ with a wider choice of vectors, but the coefficients were limited to 0,1,2 or 3: $(0, p, -q) = \sum_{j=k}^{n} a_j \mathbf{s}_j$. Whereas here, since each time only 3 vectors form the algebraic basis, we must allow all integer coefficients.

Should $\frac{p}{q}$ be a convergent to σ, we can control the size of the these coefficients, provided j has been chosen appropriately.

Definition 5.1. Let $\frac{p}{q}$ be a convergent to σ and $\mathbf{a} = (a_1, a_2, a_3)$ the coordinates of $\eta(p - q\sigma) = (0, p, -q) = a_1\mathbf{s}_{j-1} + a_2\mathbf{s}_j + a_3\mathbf{s}_{j+1}$ in the basis $\mathcal{B}_j$, and $|\rho^{-\frac{j}{4}}\mathbf{a}|$ length of *reduced coefficient vector*. Basis $\mathcal{B}_k$ is called *appropriate for the convergent* $\frac{p}{q}$, if the vector $\mathbf{a}' = \rho^{-\frac{k}{4}}\mathbf{a}$ is the shortest of all vectors $\rho^{-\frac{j}{4}}\mathbf{a}$, $j \in \mathbb{N}$.

Example 5.2. From the table of convergents [11, 15] for σ we take $p = 1251, q = 4813$ that is just preceding the relatively big partial quotient $b_{11} = 14$, so that $|\delta| < \frac{1}{14}$ in the estimate $p - q\sigma = \frac{\delta}{q}$.

Which j should we take to make the vector $\mathbf{a}'$ as small as possible? Here are some results in the Table 1. We see that the appropriate j and $\mathcal{B}_j$ to give the shortest $\mathbf{a}'$ is $j = 11$.

j	$\mathbf{a}$	$\lvert\mathbf{a}'\rvert$
8	(20,-69,-33)	5.34
9	(-9,27,20)	1.68
10	(0,-7,-9)	0.39
11	(-7,-9,0)	0.28
12	(-30,-21,-7)	0.65
13	(-111,-97,-30)	1.89

TABLE 1. Appropriate vector

Theorem 5.3. *Let $\frac{p}{q}$ be a convergent to σ, $\mathcal{B}_j$ its appropriate basis. Then for its reduced coefficient vector we have $|\mathbf{a}'| < 2.01$.*

Proof. We can write $p = q\sigma + \frac{\delta}{q}$ with $|\delta| < 1$

$$(0, p, -q) = p\mathbf{s}_0 - q\mathbf{s}_1 = p(\mathbf{h} + 2\mathbf{g}) - q(\sigma\mathbf{h} + 2\sqrt{\rho}(\mathbf{g}\cos\theta - \mathbf{k}\sin\theta))$$

and when we rearrange the terms

$$(0, p, -q) = \frac{\delta}{q}\mathbf{h} + q\sqrt[3]{2}\sqrt{3}(\sqrt{3}\mathbf{g} + \mathbf{k}) + \frac{2\delta}{q}\mathbf{g},$$

we see what happens to either term under action of S^{-j} with growing j. The first term grows exponentially with ρ^j in the direction of the eigenvector $\mathbf{h}$, the second decreases with $\rho^{-\frac{j}{2}}$ and rotates in the eigenplane, and the last term also decreases

$$\begin{aligned}\mathbf{a} = S^{-j}(0,p,-q) &= \rho^{j}\frac{\delta}{q}\mathbf{h} + \rho^{-\frac{j}{2}}q2\sqrt[3]{2}\sqrt{3}(\mathbf{g}\cos(j\theta+\frac{\pi}{6}) + \mathbf{k}\sin(j\theta+\frac{\pi}{6}))\\ &+ \rho^{-\frac{j}{2}}\frac{2\delta}{q}(\mathbf{g}\cos(j\theta) + \mathbf{k}\sin(j\theta)). \end{aligned} \tag{2}$$

But if we put $K_j = q\rho^{-\frac{3j}{4}}$ we get

$$\begin{aligned}\mathbf{a} = S^{-j}(0,p,-q) &= \rho^{\frac{j}{4}}\frac{\delta}{K_j}\mathbf{h} + \rho^{\frac{j}{4}}K_j2\sqrt[3]{2}\sqrt{3}(\mathbf{g}\cos(j\theta+\frac{\pi}{6}) + \mathbf{k}\sin(j\theta+\frac{\pi}{6}))\\ &+ \rho^{-\frac{5j}{2}}\frac{2\delta}{K_j}(\mathbf{g}\cos(j\theta) + \mathbf{k}\sin(j\theta)).\end{aligned}$$

First we represent the vector $\mathbf{a}' = \rho^{-\frac{j}{4}}\mathbf{a} = \mathbf{a}'' + \mathbf{a}'''$ with

$$\begin{aligned}\mathbf{a}'' &= \frac{\delta}{K_j}\mathbf{h} + 2K_j\sqrt[3]{2}\sqrt{3}(\mathbf{g}\cos\alpha_j + \mathbf{k}\sin\alpha_j),\\ \mathbf{a}''' &= \rho^{-\frac{5j}{4}}\frac{2\delta}{K_j}(\mathbf{g}\cos j\theta + \mathbf{k}\sin j\theta),\end{aligned}$$

where $\alpha_j = j\theta + \frac{\pi}{6}$. As for $\mathbf{a}''$ we write the square of its norm as a sum of three terms

$$\begin{aligned}|\mathbf{a}''|^2 &= \frac{\delta^2|\mathbf{h}|^2}{K_j^2} + 12\sqrt[3]{4}K_j^2|\mathbf{g}\cos\alpha_j + \mathbf{k}\sin\alpha_j|^2 + 4\delta\sqrt[3]{2}\sqrt{3}\langle\mathbf{h}, \mathbf{g}\cos\alpha_j + \mathbf{k}\sin\alpha_j\rangle\\ &= T_1 + T_2 + T_3.\end{aligned}$$

The last term is estimated independently of K_j

$$|T_3| < 4\cdot 1\cdot\sqrt[3]{2}\sqrt{3}\max_{\alpha\in\mathbb{R}}\langle\mathbf{h}, \mathbf{g}\cos\alpha + \mathbf{k}\sin\alpha\rangle < 0.894896. \tag{3}$$

Denote $F(K_j,\alpha_j) = T_1 + T_2 = \frac{\delta^2|\mathbf{h}|^2}{K_j^2} + 12\sqrt[3]{4}K_j^2|\mathbf{g}\cos\alpha_j + \mathbf{k}\sin\alpha_j|^2$ and to eliminate the dependence on α_j and δ we define another function

$$G(K_j) = \frac{|\mathbf{h}|^2}{K_j^2} + 12\sqrt[3]{4}K_j^2\max_{\alpha\in\mathbb{R}}|\mathbf{g}\cos\alpha + \mathbf{k}\sin\alpha|^2$$

or inserting the numerical values

$$G(K_j) < \frac{0.07873129}{K_j^2} + 12.95559953K_j^2 = H(K_j),$$

observing that $F(K_j,\alpha_j) < H(K_j)$. The values of both functions depend only on the choice of j, the variable K_j assumes discrete values from a geometric series as $K_j = q\rho^{-\frac{3j}{4}}$. The function $H(x) = \frac{a}{x^2} + bx^2$ as a function of continous variable $x > 0$ features just one minimum at $x_0 = \sqrt[4]{\frac{a}{b}}$ with value $H(x_0) = 2\sqrt{ab}$, but the discrete variable K_j shall almost certainly miss this minimum point. We shall further denote by x' the unique solution to the equation $H(x') = H(x'\rho^{-\frac{3}{4}})$. Indeed the equation $H(x') = H(x'\rho^{-\frac{3}{4}})$ reads

$$\frac{a}{x^2} + bx^2 = \frac{a}{x^2\rho^{-\frac{3}{2}}} + bx^2\rho^{-\frac{3}{2}}$$

and we can easily solve it $x' = \sqrt[4]{\frac{a}{b}}\rho^{\frac{3}{8}} = x_0\rho^{\frac{3}{8}}$, with $H(x') = \sqrt{ab}(\rho^{\frac{3}{4}} + \rho^{-\frac{3}{4}})$.

However within the interval $[x'\rho^{-\frac{3}{4}}, x']$ there is exactly one K_j and this defines also the choice of j. To determine j we have

$$x'\rho^{-\frac{3}{4}} < K_j < x',$$

$$x'\rho^{-\frac{3}{4}} < q\rho^{-\frac{3j}{4}} < x'.$$

Taking logarithms $\ln x' - \frac{3}{4}\ln\rho < \ln q - \frac{3j}{4}\ln\rho < \ln x'$ and dividing by $(-\frac{3}{4}\ln\rho)$ we get

$$1 - \frac{4\ln x'}{3\ln\rho} > j - \frac{4\ln q}{3\ln\rho} > -\frac{4\ln x'}{3\ln\rho}, \tag{4}$$

$$1 + \frac{4(\ln q - \ln x')}{3\ln\rho} > j > \frac{4(\ln q - \ln x')}{3\ln\rho},$$

$$j = \left[1 + \frac{4(\ln q - \ln x')}{3\ln\rho}\right] \in \mathbb{N}.$$

Thus we have $F(K_j, \alpha_j) < H(K_j) < H(x')$.

Inserting numerical values gives $x_0 = 0.279205$, $x' = 0.462761$, $x'\rho^{-\frac{3}{4}} = 0.168457$, $H(x') = 3.142064$ and so $F(K, \alpha) < 3.142064$, which together with the estimate (3) yields

$$|\mathbf{a}''| < \sqrt{3.142064 + 0.894896} < 2.009219.$$

Let take $\varepsilon = 2.01 - 2.009219 = 0.000781$. We approximate $\mathbf{a}'''$

$$|\mathbf{a}'''| < \rho^{-\frac{5j}{4}} \frac{2 \cdot 1}{K_j} \max_{\theta\in\mathbb{R}} |\mathbf{g}\cos\theta + \mathbf{k}\sin\theta| < \rho^{-\frac{5j}{4}} \frac{1.649395}{x'\rho^{-\frac{3}{4}}} < \varepsilon$$

from where we get $j \geq 6$ and the desired inequality follows for these j.

If $j \leq 5$ we get from the inequality (4) condition on q

$$5 - \frac{4\ln q}{3\ln\rho} \geq j - \frac{4\ln q}{3\ln\rho} > -\frac{4\ln x'}{3\ln\rho}$$

and q has to be smaller than 73. There is only five convergents with such q and from the Table 2 we see, that computed $|\mathbf{a}'|$ satisfies our inequality. □

Remark 5.4. Over the first 10000 convergents we numerically find that $|\mathbf{a}'| < 1.753$. The adjacent Figure 1 shows the statistics in dots, $|\mathbf{a}'|$, for these convergents.

Also the other way round, if coefficients are small enough, we are dealing with a convergent. The following theorem is however a rather coarse one.

Theorem 5.5. *Let* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *be the coefficient vector for the basis* $\mathcal{B}_j$, $j \in \mathbb{N}$ *such that* $\langle\mathbf{a}, \mathbf{s}_j^*\rangle = 0$ *and* $|\mathbf{a}| < \frac{1}{3}\rho^{\frac{j}{4}}$. *Then the resulting vector* $a_1\mathbf{s}_{j-1} + a_2\mathbf{s}_j + a_3\mathbf{s}_{j+1} = (0, p, -q)$ *yields a continous fraction convergent* $\frac{p}{q}$.

p/q	$\lvert\mathbf{a}'\rvert$
1/3	1.151
1/4	0.581
6/23	0.928
7/27	0.870
13/50	0.415

Table 2. First five convergents

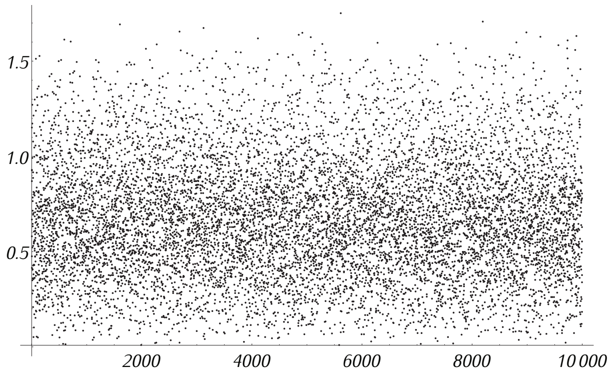

Figure 1. Length of vectors $\mathbf{a}'$

Proof. We multiply the equation (2) with the conjugate vectors $\mathbf{h}^*, \mathbf{g}^*, \mathbf{k}^*$ to obtain

$$\langle \mathbf{h}^*, \mathbf{a}\rangle = \rho^j \frac{\delta}{q}\langle \mathbf{h}^*, \mathbf{h}\rangle \tag{5}$$

$$\langle \mathbf{g}^*, \mathbf{a}\rangle = \rho^{-\frac{j}{2}} q 2\sqrt[3]{2}\sqrt{3}\langle \mathbf{g}^*, \mathbf{g}\rangle \cos(j\theta + \frac{\pi}{6}) + \rho^{-\frac{j}{2}}\frac{2\delta}{q}\langle \mathbf{g}^*, \mathbf{g}\rangle \cos(j\theta) \tag{6}$$

$$\langle \mathbf{k}^*, \mathbf{a}\rangle = \rho^{-\frac{j}{2}} q 2\sqrt[3]{2}\sqrt{3}\langle \mathbf{k}^*, \mathbf{k}\rangle \sin(j\theta + \frac{\pi}{6}) + \rho^{-\frac{j}{2}}\frac{2\delta}{q}\langle \mathbf{k}^*, \mathbf{k}\rangle \sin(j\theta) \tag{7}$$

Inserting the conditions of the theorem, we can estimate $\frac{|\delta|}{q}$ using (5):

$$\frac{|\delta|}{q} < \rho^{-\frac{3j}{4}}\frac{|\mathbf{h}^*|}{3M}. \tag{8}$$

Using (6) and (7) we get

$$q \cdot 2\sqrt[3]{2}\sqrt{3}\cos(j\theta + \frac{\pi}{6})M = \rho^{\frac{j}{2}}\langle \mathbf{g}^*, \mathbf{a}\rangle - 2\frac{\delta}{q}M\cos(j\theta),$$

$$q \cdot 2\sqrt[3]{2}\sqrt{3}\sin(j\theta + \frac{\pi}{6})M = -\rho^{\frac{j}{2}}\langle \mathbf{k}^*, \mathbf{a}\rangle - 2\frac{\delta}{q}M\sin(j\theta).$$

We square and add up the last two equations to eliminate the sines and cosines

$$q^2(2\sqrt[3]{2}\sqrt{3}M)^2 = \rho^j(\langle \mathbf{g}^*, \mathbf{a}\rangle^2 + \langle \mathbf{k}^*, \mathbf{a}\rangle^2) + \rho^{\frac{j}{2}}\frac{4\delta}{q}M(\langle \mathbf{k}^*, \mathbf{a}\rangle\cos(j\theta) - \langle \mathbf{g}^*, \mathbf{a}\rangle\sin(j\theta)) + \frac{4\delta^2}{q^2}M^2$$

and estimate $q^2(2\sqrt[3]{2}\sqrt{3}M)^2 < \rho^{\frac{3j}{2}}\frac{|\mathbf{g}^*|^2+|\mathbf{k}^*|^2}{9} + \frac{4|\mathbf{h}^*|}{9}(|\mathbf{g}^*|+|\mathbf{k}^*|) + \rho^{-\frac{3j}{2}}\frac{4|\mathbf{h}^*|^2}{9}$.
Using (8) and last inequality we can estimate

$$|\delta| < \frac{|\mathbf{h}^*|}{18M^2\sqrt[3]{2}\sqrt{3}}\sqrt{|\mathbf{g}^*|^2+|\mathbf{k}^*|^2+4\rho^{-\frac{3j}{2}}|\mathbf{h}^*|(|\mathbf{g}^*|+|\mathbf{k}^*|)+4\rho^{-3j}|\mathbf{h}^*|^2}$$

which is smaller than 0.48 for $j \geq 2$. For $j = 1$ condition $|\mathbf{a}| < \frac{1}{3}\rho^{\frac{1}{4}} < 0.47$ implies nonexistance of such integer vector. So we show $|\delta| < \frac{1}{2}$, that is enough for our conclusion. □

From the proof we can also infer the implications of an even smaller $|\mathbf{a}|$ on the $|\delta|$ and consecutively the next partial quotient B.

Lemma 5.6. *If in the above theorem* $|\mathbf{a}| < \frac{\Delta}{3}\rho^{\frac{j}{4}}, \Delta < 1$, *we can estimate the next partial quotient* B *to the convergent* $\frac{p}{q}$ *by*

$$B > \frac{2}{\Delta^2} - 2.$$

Proof. From the well known estimate [10]

$$\frac{1}{(B+2)q^2} < \left|\frac{p}{q} - \sigma\right| < \frac{1}{Bq^2}$$

we find $\frac{1}{B+2} < |\delta|$ or $B > \frac{1}{|\delta|} - 2$.
But from the above proof, if instead of $\frac{1}{3}$, we put $\frac{\Delta}{3}$, we have $|\delta| < \Delta^2 \cdot 0.48 < \Delta^2 \cdot \frac{1}{2}$ and the estimate from lemma follows. □

Example 5.7. In our numerical experiment we found for $j = 750$, $\Delta = 0.03906$, next $B = 4941 = b_{619}$ (well known big partial quotient [11, 15]) with lemma suggesting $B > 1308$.

5.1. The problem of the shortest lattice vector

Our case the lattice $\Lambda_j = \{\mathbf{a} \in \mathbb{Z}^3, \mathbf{a} \perp \mathbf{s}_j^*\}$. Gauss reduction process mimics the euclidian algorithm. Let's have a basis $\mathbf{z}_1, \mathbf{z}_2$, such that $|\mathbf{z}_1| < |\mathbf{z}_2|$. Choose k so that

$$-\frac{1}{2}|\mathbf{z}_1|^2 < \langle \mathbf{z}_2 - k\mathbf{z}_1, \mathbf{z}_1\rangle \leq \frac{1}{2}|\mathbf{z}_1|^2$$

so $k \in \mathbb{Z}$ is the nearest integer to $\frac{\langle \mathbf{z}_2, \mathbf{z}_1\rangle}{|\mathbf{z}_1|^2}$.

Now, set the new $\mathbf{z}_2 := \mathbf{z}_2 - k\mathbf{z}_1$ and compare: if $|\mathbf{z}_1| < |\mathbf{z}_2|$ the process terminates, our shortest vector is $\mathbf{z}_1$, else we interchange $\mathbf{z}_1 \leftrightarrow \mathbf{z}_2$ and start again. In some steps we get the shortest lattice vector [14].

Remark 5.8. In higher dimensions the so called LLL-algorithm [14], [7] is similar to Gramm-Schmidt orthogonalization to generalize the Gauss process.

Example 5.9. Here is how we carried out this process for $j = 7$, i.e. $\mathbf{z}_1 = \mathbf{s}_{-6}$, $\mathbf{z}_2 = \mathbf{s}_{-7}$ and we have the shortest vector $\mathbf{a} = (-7, 1, 0)$. Combining $-7\mathbf{s}_6 + 1\mathbf{s}_7 = (0, 59, -227)$ we read off the quotient $\frac{59}{227}$ which does appear in the sequence of approximants (Table 3).

n	$\mathbf{z}_1$	$\mathbf{z}_2$	$\|\mathbf{z}_1\|^2$	$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle$	k
1	(681,577,177)	(2620,2220,681)	828019	3185697	4
2	(-104,-88,-27)	(681,577,177)	19289	-126379	-7
3	(-47,-39,-12)	(-104,-88,-27)	3874	8644	2
4	(-10,-10,-3)	(-47,-39,-12)	209	896	4
5	(-7,1,0)	(-10,-10,-3)	50	60	1
6	(-3,-11,-3)	(-7,1,0)	139	10	0

Table 3. Shortest lattice vector

We carried out the shortest vector algorithm for $j = 2$ until $j = 1000$. The resulting $|\mathbf{a}'|$ were, as shown in the dotted Figure 2, all below 1. Only 21 of them did not result in continued fraction approximants (marked with squares).

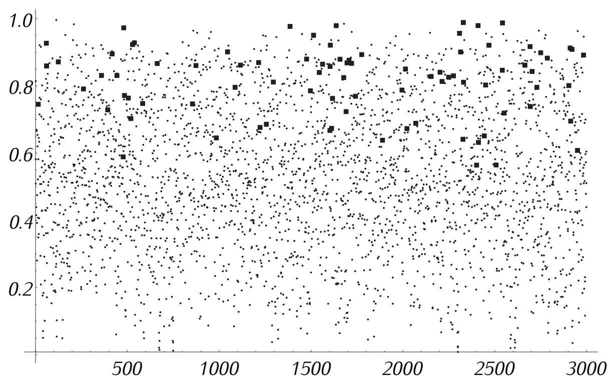

Figure 2. Length of vectors $\mathbf{a}'$ resulting from the shortest vector algorithm

6. χ^2-test of distribution of partial quotients

We applied χ^2-test to compare observed frequencies of partial quotients of $\sqrt[3]{2}$ with theoretical frequencies $P(b_n = k) = \log_2 \frac{(k+1)^2}{k(k+2)}$. Using [15] we computed 75 000 partial quotients and divided them into R groups consisting of numbers 1, 2, 3, …, $R-1$ and of all numbers over $R-1$. Let O_i be the observed frequency of the i^{th} group and E_i its expected frequency. The value of the test statistic is $X^2 = \sum_{i=1}^{R} \frac{(O_i - E_i)^2}{E_i}$. If the partial quotients the hypothesized distribution, X^2 has, approximately, a χ^2 distribution with $R-1$ degrees of freedom. Since all the P-values for $R \leq 100$ are above 0.05 we can not reject the hypothesis that the partial quotients of $\sqrt[3]{2}$ follow the distribution law of Kuzmin.

REFERENCES

[1] *L. Bernstein*, The Jacobi-Perron Algorithm. Its Theory and Application, Lecture Notes in Mathematics 207 (1971) Springer-Verlag.

[2] *A. D. Bruno*, Expansion of algebraic numbers in continued fractions, Zhurnal Vycch. Mat. i Mat. Fiziki 4 (1964), 211-221.

[3] *Shi-gang Chen and You-qin Wang*, Cubic irrational numbers and critical scaling law for quasiperiodic motion, Physics Letters A vol. 153 (1991), 2-3, 113-116.

[4] *R. F. Churchhouse, and S.T.E. Muir*, Continued Fractions, Algebraic Numbers and Modular Invariants. J.Inst. Maths Applics (1969), 318-328.

[5] *B. Devaney*, An introduction to chaotic dynamical systems, Westview Press 2003.

[6] *S. Lang and H. Trotter*, Continued fractions for some algebraic numbers, J. für Mathematik 255 (1972), 112-134, Addendum 219-220.

[7] *A. K. Lenstra, H. W. Lenstra and L. Lovasz*, Factoring Polynomials with rational coefficients, Math. Ann. 261 (1986), 515-534.

[8] *S. Lucas*, On using bounded continued fractions to represents reals, ANZIAM J. 45 (E) (2004), C921-C934.

[9] *J. Mao and R. H. G. Heileman*, Breakup of Kolmogorov-Arnold-Moser tori of cubic irrational winding number, Phys. Rev. A39, 344-346.

[10] *O. Perron*, Die Lehre von den Kettenbrüchen, Teubner 1913.

[11] *P. Petek, M. Lakner and M. Škapin Rugelj*, In the search of convergents to $\sqrt[3]{2}$, Chaos, Solitons and Fractals, 41(2) (2009), 811-817.

[12] *R. D. Richtmyer, M. Devaney and N. Metropolis*, Continued fractions expansions of algebraic numbers, Numerische Mathematik 4 (1962), 68-84.

[13] *H. M. Stark*, An explanation of some exotic continued fraction found by Brillhart, Computers in Number Theory (1971), 21-35.

[14] *J. Steuding*, Diophantine Analysis, Chapman Hall 2005

[15] Wolfram Research Inc., Mathematica. Version 8.0, Champaign 2010.