A NOTE ON EXACT EXPLICIT TRAVELING WAVE SOLUTIONS FOR THE GENERALIZED B-EQUATION

Xijun Deng[1], Yuqiu Sun[2], Dahai Xu[3]

Liu [Liu Rui, Coexistence of multifarious exact nonlinear wave solutions for generalized b-equation, Inter. J. Bifurcation and Chaos, 20, pp. 3193-3208, 2010] investigated the coexistence of multifarious exact traveling wave solutions of the generalized b-equation, and also presented three conjectures and two questions. In this note, by using the method of complete discrimination system for polynomial, it is shown that some general exact explicit solutions of the generalized b-equation can be obtained directly. Moreover, the above conjectures and questions are confirmed/corrected and clarified, respectively.

Keywords: Generalized b-equation, method of complete discrimination system for polynomial, traveling wave solutions.

MSC2010: 35Q51, 35Q53, 37K10.

1. Introduction

Recently, Liu [1,2] studied the nonlinear wave solutions of the following generalized b-equation

$$u_t - u_{xxt} + (1+b)u^2u_x = bu_xu_{xx} + uu_{xxx} \tag{1}$$

He used the dynamical system approach combined with phase analysis to studied the coexistence and explicit expressions of various nonlinear wave solutions, which include smooth solitary wave solution, peakon wave solution, smooth periodic wave solution, singular wave solution. Also, the author [2] presented three conjectures and two questions.

The aim of this paper is to further solve the explicit travelling wave solutions of the generalized b-equation, and analyze the above conjectures and questions proposed by Liu [2].

This paper is organized as follows. In section 2 we give the general travelling wave solutions of the generalized b-equation, by applying the method of complete discrimination system for polynomial [3,4]. In section 3 we analyze the explicit nonlinear wave solutions of the generalized b-equation by Liu [2] and demonstrate that almost all solutions from the list by Liu [2] can be obtained by our general solutions. In section 4 we analyze the related conjectures and questions presented

[1] School of Information and Mathematics, Yangtze University, jingzhou, Hubei, 434023, China, e-mail: xijundeng@126.com

[2] College of Mathematics and Econometrics, Hunan University, Changsha 410082, China,

[3] School of Physical Science and Technology, Yangtze University, Jingzhou, Hubei, 434023, China

by Liu [2]. Not only his conjectures are confirmed or corrected, but his questions are clarified. Finally, some conclusions are summarized in section 5.

2. **Some general explicit solutions of the generalized b-equation**

In fact, some general explicit solutions for Eq.(1) can be obtained by using the method of complete discrimination system for polynomial [3,4]. Let us demonstrate it below. Making the travelling wave transformation in Eq.(1)

$$u(x,t) = \phi(\xi), \qquad \xi = x - ct. \tag{2}$$

and integrating with respect to ξ, we have the following first-order ordinary differential equation(ODE)

$$\phi'^2 = \frac{2(b+1)}{3(b+2)}\left[\phi^3 + \frac{3c}{b+1}\phi^2 + \frac{3(2c^2-2c-bc)}{b(b+1)}\phi + \frac{C_1}{(\phi-c)^{b-1}} + C_2\right], \tag{3}$$

where C_1, C_2 are arbitrary integral constants, $b \neq 0, -1, -2$. Noticing that in the case $C_1 \neq 0$, it seem to us that the exact explicit solutions for Eq.(1) can not be obtained. Thus we assume that $C_1 = 0$ throughout the whole paper. We would focus on the study of the first-order ODE as follows

$$\phi'^2 = \frac{2(b+1)}{3(b+2)}\left[\phi^3 + \frac{3c}{b+1}\phi^2 + \frac{3(2c^2-2c-bc)}{b(b+1)}\phi + C_2\right]. \tag{4}$$

For convenience we first assume that $\frac{2(b+1)}{3(b+2)} > 0$, and let

$$d_2 = \frac{3c}{b+1}, \quad d_1 = \frac{3(2c^2-2c-bc)}{b(b+1)}, \quad d_0 = C_2, \tag{5}$$

then Eq.(4) becomes

$$\int \frac{d\phi}{\sqrt{\phi^3 + d_2\phi^2 + d_1\phi + d_0}} = \pm\sqrt{\frac{2(b+1)}{3(b+2)}}(\xi - \xi_0) \tag{6}$$

Denote that $F(\phi) = \phi^3 + d_2\phi^2 + d_1\phi + d_0$, whose complete discrimination system is given by [3,4]

$$\triangle = -27\left(\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_2}{3}\right)^2 - 4\left(d_1 - \frac{d_2^2}{3}\right)^3, \quad D = d_1 - \frac{d_2^2}{3}, \tag{7}$$

According to the method of complete discrimination system for polynomial [3,4], we can obtain some corresponding explicit exact solutions for Eq.(1) in the following four cases.

Case 1. $\Delta = 0, D < 0$. In this case, we have $F(\phi) = (\phi-\alpha)^2(\phi-\beta), \alpha \neq \beta$. If $\phi > \beta$, Eq.(1) admits three types of exact explicit solutions as follows

$$u^1(x,t) = \beta + (\alpha-\beta)\tanh^2\left[\sqrt{\frac{(b+1)(\alpha-\beta)}{6(b+2)}}(x-ct-\xi_0)\right], \quad \alpha > \beta, \tag{8}$$

$$u^2(x,t) = \beta + (\alpha-\beta)\coth^2\left[\sqrt{\frac{(b+1)(\alpha-\beta)}{6(b+2)}}(x-ct-\xi_0)\right], \quad \alpha > \beta, \tag{9}$$

$$u^3(x,t) = \beta + (\beta - \alpha)\tan^2\left[\sqrt{\frac{(b+1)(\beta-\alpha)}{6(b+2)}}(x - ct - \xi_0)\right], \quad \alpha < \beta. \tag{10}$$

Case 2. $\Delta = 0, D = 0$. In this case, we have $F(\phi) = (\phi - \alpha)^3$. We can get the rational form of exact solutions for Eq.(1)

$$u^4(x,t) = \alpha + \frac{\frac{6(b+2)}{b+1}}{(x - ct - \xi_0)^2}. \tag{11}$$

Case 3. $\Delta > 0, D < 0$. In this case, we have $F(\phi) = (\phi-\alpha)(\phi-\beta)(\phi-\gamma), \alpha < \beta < \gamma$. When $\alpha < \phi < \beta$, Eq.(1) has exact smooth periodic wave solutions

$$u^5(x,t) = \alpha + (\beta - \alpha)sn^2\left(\sqrt{\frac{(b+1)(\gamma-\alpha)}{6(b+2)}}(x - ct - \xi_0), m\right) \tag{12}$$

when $\phi > \gamma$, Eq.(1) has exact periodic blow-up solutions

$$u^6(x,t) = \frac{\gamma - \beta sn^2\left(\sqrt{\frac{(b+1)(\gamma-\alpha)}{6(b+2)}}(x - ct - \xi_0), m\right)}{cn^2\left(\sqrt{\frac{(b+1)(\gamma-\alpha)}{6(b+2)}}(x - ct - \xi_0), m\right)}, \tag{13}$$

where $m^2 = \frac{\beta-\alpha}{\gamma-\alpha}$.

Case 4. $\Delta < 0$. In this case, we have $F(\phi) = (\phi-\alpha)(\phi^2+p\phi+q), p^2-4q < 0$. We can obtain that Eq.(1) has the exact periodic blow-up solutions as follows

$$u^7(x,t) = \alpha - \sqrt{\alpha^2 + p\alpha + q} + \frac{2\sqrt{\alpha^2 + p\alpha + q}}{1 \pm cn\left(\sqrt{\frac{2(b+1)}{3(b+2)}}(\alpha^2 + p\alpha + q)^{\frac{1}{4}}(x - ct - \xi_0), m\right)}, \tag{14}$$

where $m^2 = \frac{1}{2}\left(1 - \frac{\alpha + \frac{p}{2}}{\sqrt{\alpha^2+p\alpha+q}}\right)$.

3. Exact explicit travelling wave solutions of the generalized b-equation by Liu

Suppose the parameter $b > 1$ and for the constant wave speed $c \in (0, 1 + b)$, Liu [2] has found 12 exact nonlinear wave solutions. Now we present these solutions as follows:

$$u_1(x,t) = \frac{1}{b(b+1)}\left[p_0 - bc - 3p_0 sech^2\alpha_0\xi\right], \tag{15}$$

$$u_2(x,t) = \frac{1}{b(b+1)}\left[p_0 - bc + \frac{3p_0(2bc + b^2c - p_0)}{(\sqrt{3p_0}cosh\alpha_0\xi + \sqrt{2bc + b^2c + 2p_0}sinh|\alpha_0\xi|)^2}\right], \tag{16}$$

$$u_3(x,t) = \frac{1}{b(b+1)}\left[p_0 - bc + 3p_0 csch^2\alpha_0\xi\right], \tag{17}$$

$$u_4(x,t) = \frac{1}{b(b+1)}\left[2p_0 - bc + 3p_0\tan^2\alpha_0\xi\right], \tag{18}$$

$$\begin{aligned} u_5(x,t) &= a_1 + a_2 sn^2(\beta_0\xi, k_1), \quad c \in (0, \frac{1}{1+b}); \\ &= a_1 + a_3 sn^2(\gamma_0\xi, k_2), c \in (\frac{1}{1+b}, \frac{4(1+b)}{4+2b+b^2}) \end{aligned}$$

$$\begin{aligned} u_6(x,t) &= a_1 + a_3 sn^{-2}(\beta_0\xi, k_1), \quad c \in (0, \frac{1}{1+b}); \\ &= a_1 + a_2 sn^{-2}(\gamma_0\xi, k_2), c \in (\frac{1}{1+b}, \frac{4(1+b)}{4+2b+b^2}) \end{aligned}$$

$$u_7(x,t) = \frac{6(2+b)}{\left[\sqrt{6} + \sqrt{1+b}|x-(1+b)t|\right]^2} - 1, \tag{19}$$

$$u_8(x,t) = \frac{6(2+b)}{(1+b)(x-(1+b)t)^2} - 1, \tag{20}$$

$$u_9(x,t) = \frac{2}{4+2b+b^2}\left[2(1+b) + 3(2+b)\tan^2\sqrt{\frac{1+b}{4+2b+b^2}}(x - \frac{4(1+b)}{4+2b+b^2}t)\right], \tag{21}$$

$$u_1^0(x,t) = \frac{1}{b+1} - \frac{3(2+b)}{(1+b)^2} sech^2 \sqrt{\frac{1}{2(1+b)}}\left(x - \frac{1}{1+b}t\right), \tag{22}$$

$$u_3^0(x,t) = \frac{1}{b+1} + \frac{3(2+b)}{(1+b)^2} csch^2 \sqrt{\frac{1}{2(1+b)}}\left(x - \frac{1}{1+b}t\right), \tag{23}$$

$$u_4^0(x,t) = \frac{1}{(1+b)^2}\left[3 + 2b + 3(2+b)\tan^2\sqrt{\frac{1}{2(1+b)}}(x - \frac{1}{1+b}t)\right], \tag{24}$$

where

$$\xi = x - ct, \tag{25}$$

$$p_0 = \sqrt{b(2+b)(1+b-c)c}, \tag{26}$$

$$\alpha_0 = \sqrt{\frac{p_0}{2b(2+b)}}, \tag{27}$$

$$q_0 = \sqrt{3bc(2+b)(4+4b-4c-2bc-b^2c)}, \tag{28}$$

$$a_1 = -\frac{bc(4+b)+q_0}{2b(1+b)}, \tag{29}$$

$$a_2 = \frac{3bc(2+b)+q_0}{2b(1+b)}, \tag{30}$$

$$a_3 = \frac{q_0}{b(1+b)}, \tag{31}$$

$$k_1 = \frac{3bc(2+b)+q_0}{2q_0}, \tag{32}$$

$$k_2 = \frac{2q_0}{3bc(2+b)+q_0}, \tag{33}$$

$$\beta_0 = \sqrt{\frac{q_0}{6b(2+b)}}, \tag{34}$$

$$\gamma_0 = \sqrt{\frac{3bc(2+b)+q_0}{12b(2+b)}}. \tag{35}$$

By careful observation, we can find that almost all these exact solutions could be deduced from the general solutions described in section 2. Let us demonstrate it below.

Under Case 1, from (7) and $\Delta = 0, D < 0$, we can derive that

$$c \in (0, 1+b), \quad d_0 = \frac{3c^2b(b+1)(2c-2-b) - 2b^2c^3 \pm c(b+2)(b+1-c)p_0}{b^2(b+1)^3}. \tag{36}$$

Thus, we have $\alpha = \frac{-bc \pm p_0}{b(b+1)}$, $\beta = \frac{-bc \mp 2p_0}{b(b+1)}$. If $\alpha = \frac{-bc+p_0}{b(b+1)} > \beta = \frac{-bc-2p_0}{b(b+1)}$, then from (8)-(9) we can obtain that Eq.(1) admits the following travelling wave solutions

$$u_1^1(x,t) = \frac{1}{b(b+1)}\left[-2p_0 - bc + 3p_0 tanh^2\alpha_0(\xi - \xi_0)\right], \tag{37}$$

$$u_3^1(x,t) = \frac{1}{b(b+1)}\left[-2p_0 - bc + 3p_0 coth^2\alpha_0(\xi - \xi_0)\right], \tag{38}$$

Note that $\tanh^2 x + sech^2 x = 1$ and $\coth^2 x - csch^2 x = 1$, if setting $\xi_0 = 0$, it is easy to see that the solutions u_1^1 and u_3^1 agree well with u_1 and u_3, respectively. Moreover, assuming that $c = \frac{1}{1+b}$ in the solutions u_1, u_3, we accordingly have the solutions u_1^0, u_3^0.

If $\alpha = \frac{-bc-p_0}{b(b+1)} < \beta = \frac{-bc+2p_0}{b(b+1)}$, then from (10) we can derive that Eq.(1) has exact trigonometric periodic singular wave solution

$$u_4^1(x,t) = \frac{1}{b(b+1)}\left[2p_0 - bc + 3p_0 \tan^2\alpha_0(\xi - \xi_0)\right], \tag{39}$$

which coincides with the solution u_4 if taking $\xi_0 = 0$. Assuming that $c = \frac{1}{1+b}$ and $c = \frac{4(1+b)}{4+2b+b^2}$ in the solution u_4, we accordingly have the solutions u_4^0, u_9.

Note that $\tanh^2 x = \tanh^2|x|, \coth^2 x = \coth^2|x|$, we find that the following functions

$$u_{01}(x,t) = \frac{1}{b(b+1)}\left[p_0 - bc - 3p_0 sech^2(|\alpha_0\xi| + \eta_0)\right], \tag{40}$$

$$u_{03}(x,t) = \frac{1}{b(b+1)}\left[p_0 - bc + 3p_0 csch^2(|\alpha_0\xi| + \eta_0)\right], \tag{41}$$

are also the solutions of Eq.(1). By means of the following two identities

$$\frac{(A_2^2 - A_1^2)(1-\tanh^2\Theta)}{(A_2 + a_1\tanh|\Theta|)^2} = sech^2(|\Theta| + \Theta_0), \quad \Theta_0 = \tanh^{-1}\frac{A_1}{A_2}, A_2^2 > A_1^2, \tag{42}$$

and

$$\frac{(A_2^2 - A_1^2)(1-\tanh^2\Theta)}{(A_2 + A_1\tanh|\Theta|)^2} = -csch^2(|\Theta| + \Theta_0), \quad \Theta_0 = \coth^{-1}\frac{A_1}{A_2}, A_2^2 < A_1^2, \tag{43}$$

we have that the solution u_2 is equal to u_{01} if $c \in (0, \frac{1}{1+b}), \eta_0 = \tanh^{-1}\sqrt{\frac{2bc+b^2c+2p_0}{3p_0}}$, and u_2 is equal to u_{03} if $c \in (\frac{1}{1+b}, +\infty), \eta_0 = \coth^{-1}\sqrt{\frac{2bc+b^2c+2p_0}{3p_0}}$.

Under Case 2, from (7) and $\Delta = 0, D = 0$, we can derive that $c = b+1$ or $c = 0$, this implies that $\alpha = -1$ or $\alpha = 0$. Thus, Eq.(1) admits the exact rational form of solution

$$u_8^0(x,t) = -1 + \frac{\frac{6(b+2)}{b+1}}{(x-(1+b)t-\xi_0)^2}. \tag{44}$$

and

$$u_8^1(x,t) = \frac{\frac{6(b+2)}{b+1}}{(x-\xi_0)^2}. \tag{45}$$

Setting $\xi_0 = 0$, we can find that u_8^0 is equal to the solution u_8, while u_8^1 is a new special fractional solution to Eq.(1).

Under Case 3, from (7) and $\Delta > 0, D < 0$, we can derive that

$$c \in (0, 1+b), \quad d_0 \in (z_1, z_2), \tag{46}$$

where

$$z_1 = \frac{3c^2b(b+1)(2c-2-b) - 2b^2c^3 - c(b+2)(b+1-c)p_0}{b^2(b+1)^3}, \tag{47}$$

and

$$z_2 = \frac{3c^2b(b+1)(2c-2-b) - 2b^2c^3 + c(b+2)(b+1-c)p_0}{b^2(b+1)^3}. \tag{48}$$

Note that α, β, γ are determined by the selection of constants c and d_0. In addition, the constant d_0 is arbitrary. Therefore, we may always find the specific values of α, β, γ, which make the relevant parameters satisfy the condition $\Delta > 0, D < 0$. For example, when $c \in (0, \frac{1}{1+b})$, we can let $\alpha = a_1, \beta = c, \gamma = \frac{q_0 - bc(b+4)}{2b(b+1)}$, it is easy to verify that $\alpha < \beta < \gamma$, and the relevant parameters also satisfy the condition $\Delta > 0, D < 0$. Substituting it into the general solution (12), one can easily find that the obtained solution agrees well with the solution u_5; When $c \in (\frac{1}{1+b}, \frac{4(1+b)}{4+2b+b^2})$, let $\alpha = a_1, \beta = c, \gamma = \frac{q_0 - bc(b+4)}{2b(b+1)}$, we can also obtain the solution u_5. Similarly, the solution u_6 can be deduced from the general solution (13). However, it should be pointed out here that there is a minor error concerning k_1, k_2. That is, the above identities (32) and (33) should be corrected as

$$k_1^2 = \frac{3bc(2+b) + q_0}{2q_0}, \tag{49}$$

and

$$k_2^2 = \frac{2q_0}{3bc(2+b) + q_0}, \tag{50}$$

respectively.

In a word, we can observe that almost all solutions from the list by Liu can be obtained by our general solutions.

Remark 3.1. Under Case 4, from (7) and the condition $\Delta < 0$, we can also determine the concrete values of α, p, q and obtain numerous special exact travelling wave solutions for Eq.(1). For example, setting $d_0 = 0$, $\quad c \in (\frac{12(b+1)(b+2)}{15b+24}, +\infty)$ leads to

$\alpha = 0, \ \ p = \frac{3c}{b+1}, \ \ q = \frac{3c(2c-2-b)}{b(b+1)}$, then from the general solution (14) we can derive that Eq.(1) has new exact explicit periodic blow-up solutions

$$u_{10}(x,t) = -\sqrt{q} + \frac{2\sqrt{q}}{1 \pm cn\left(\sqrt{\frac{2(b+1)}{3(b+2)}}q^{\frac{1}{4}}(x-ct-\xi_0), m\right)}, \tag{51}$$

where $m^2 = \frac{1}{2}\left(1 - \frac{p}{2\sqrt{q}}\right)$.

Remark 3.2. Liu [2] claims in Proposition 3 that when $c = \frac{1}{1+b}$, "three types of exact nonlinear wave solutions" , namely smooth solitary wave solution u_1^0, hyperbolic singular wave solution u_3^0 and trigonometric periodic singular wave solution u_4^0 coexist for Eq.(1). However, we would like to point out here that it is not yet complete. And in fact other three types of exact explicit solutions including elliptic smooth periodic wave solution, elliptic periodic singular wave solution and periodic blow-up solution also coexist for Eq.(1). Let us show it below. For example, let $d_0 = 0, c = \frac{1}{1+b}$, we have $F(\phi) = \phi(\phi^2 + \frac{3}{(1+b)^2}\phi - \frac{3(b+3)}{(b+1)^3})$. This implies that $\alpha = -\frac{3+\sqrt{9+12(b+1)(b+3)}}{2(1+b)^2}, \ \ \beta = 0, \ \ \gamma = \frac{-3+\sqrt{9+12(b+1)(b+3)}}{2(1+b)^2}$. So from the general solution (12), we can know that when $u \in (-\frac{3+\sqrt{9+12(b+1)(b+3)}}{2(1+b)^2}, 0)$, Eq.(1) admits smooth periodic wave solution

$$u_{11}(x,t) = -\frac{3+\sqrt{9+12(b+1)(b+3)}}{2(1+b)^2}cn^2\left(\sqrt{\frac{\sqrt{12b^2+48b+45}}{6(b+1)(b+2)}}(x - \frac{1}{1+b}t - \xi_0), m\right), \tag{52}$$

When $u \in (\frac{-3+\sqrt{9+12(b+1)(b+3)}}{2(1+b)^2}, +\infty)$, Eq.(1) admits periodic singular wave solution

$$u_{12}(x,t) = \frac{-3+\sqrt{9+12(b+1)(b+3)}}{2(1+b)^2}cn^{-2}\left(\sqrt{\frac{\sqrt{12b^2+48b+45}}{6(b+1)(b+2)}}(x - \frac{1}{1+b}t - \xi_0), m\right), \tag{53}$$

where $m^2 = \frac{3+\sqrt{12b^2+48b+45}}{2\sqrt{12b^2+48b+45}}$.

Notice that when $c = \frac{1}{1+b}$, we always have $D = d_1 - \frac{d_2^2}{3} < 0$. So from (7) we can know that the condition $\Delta < 0$ holds provided that the parameter d_0 is chosen large enough. According to Case 4, we can derive that Eq.(1) always admits exact periodic blow-up solutions like the solution (14).

4. Discussions on the conjectures and questions proposed by Liu

With the aid of bifurcation phase portraits, Liu [2] proposed 3 conjectures and 2 questions. Below we would analyze and discuss them one by one.

4.1. Discussions on Conjecture 1

Conjecture 1 claims that: "When $b > 1$ and the wave speed $c = \frac{1}{1+b}$ or $c > 1 + b$, Eq.(1.1) has no peakon wave solution."

From (5) and (7) , we can obtain that

$$D = d_1 - \frac{d_2^2}{3} = \frac{3c(b+2)[c-(b+1)]}{b(b+1)^2}, \tag{54}$$

When $b > 1$, it is easy to verify that $D < 0$ for $c = \frac{1}{1+b}$, and $D > 0, \triangle < 0$ for $c > 1 + b$. In view of the discussions as Remark 2 in Section 2, we can find that Eq.(1) has no peakon wave solution. Since $D > 0, \triangle < 0$ corresponds to Case 4, for which apparently has also no peakon solution. Hence, Conjecture 1 is confirmed.

4.2. Discussions on Conjecture 2

Conjecture 2 claims that: "When the wave speed $c \geq 1 + b$, Eq.(1) has no explicit smooth solitary wave solution."

If $b > 0$, it follows from (54) and from the wave speed $c \geq 1 + b$ that $D \geq 0$, which certainly leads to that Eq.(1) has no explicit smooth solitary wave solution. However, if $b \in (-1, 0)$ and $c \geq 1 + b$, then from (54) we can obtain that $D \leq 0$. According to Case 1, by properly selecting the constant d_0, Eq.(1) has explicit smooth solitary wave solution like the solution (8). Therefore, the above-mentioned Conjecture 2 is not strict. And it should be described as follows: "When $b > 0$ and the wave speed $c \geq 1 + b$, Eq.(1) has no explicit smooth solitary wave solution."

4.3. Discussions on Conjecture 3

Conjecture 3 claims that: "When the wave speed $c \neq 1 + b$, Eq.(1) has no fractional solution."

Note that we have obtained in section 2 that Eq.(1) has fractional solution u_8^1. Consequently, it implies that Conjecture 3 is also not correct. In fact, from (54) we can know that Eq.(1) would admit fractional solution if and only if $c = 0$ or $c = 1 + b$. I think the Correct statement is the following: "When the wave speed $c \neq 1 + b$ and $c \neq 0$, Eq.(1) has no fractional solution."

4.4. Discussions on Question 1

Question 1 Claims that: "Our derivations were based on $b > 1$. But for $b \leq 1$, we do not know how to derive the solutions $u_1(x,t), u_2(x,t), \ldots, u_9(x,t)$, $u_1^0(x,t), u_3^0(x,t), u_4^0(x,t)$."

Actually, it is easy to see that the solutions $u_1, \ldots, u_4$ belong to Case 1($D < 0, \triangle = 0$), while the solutions u_5, u_6 belong to Case 3($D < 0, \triangle > 0$). We can observe from (54) that $D < 0$ as long as $c \in (0, 1 + b)$ and $b > 0$. Moreover, when $b > 0$, we also have the inequalities (5) described in [2]. This certainly implies that when $b > 0$, the derivations on the solutions $u_1, \ldots, u_6$ still hold well.

Note that the solutions $u_1^0, u_3^0, u_4^0(u_9)$ are derived by the solutions u_1, u_3, u_4, respectively. Hence, by comparing the solutions $u_1^0, u_3^0, u_4^0(u_9)$ with the general solutions (8)-(10), we can obtain that it should satisfy the condition $b+2 > 0$. However our derivations concerning with the general solutions in Section 2 are based on the condition $\frac{b+1}{b+2} > 0$. These facts implies that when $b > -1$, u_1^0, u_3^0, u_4^0, u_9 are still the real solutions of Eq.(1).

The solution u_8 corresponds to $D = 0, \triangle = 0$. In fact, in this case we have from (4) that

$$\phi'^2 = \frac{2(b+1)}{3(b+2)}(\phi+1)^3. \tag{55}$$

Obviously, Eq.(55) always have the solution like u_8 as long as $b \neq -1$. This illustrates that u_8 is still a real solution to Eq. (1) for $b \neq -1$.

Thus, Question 1 is clarified.

4.5. Discussions on Question 2

Question 2 Claims that: " For $b < -1$, we do not know whether there is any other explicit real solution except $u_i^*(x,t)(i = 1,2,3)$ and $u_8(x,t)$."

Without loss of generality, we assume that $c = \frac{2+b}{2}$. Below it will be shown that Eq.(1) admits all other types of explicit real solutions except $u_i^*(x,t)(i = 1,2,3)$ and $u_8(x,t)$.

First, we consider the case $b < -2$. From (5) and $c = \frac{2+b}{2}$, we have $d_1 = 0$ and $d_2 > 0$, and it implies that $D < 0$. Note that exact solutions of Eq.(1) depend completely on the values of $\triangle, D$. So from (7), we can observe that one can flexibly controls the value of the constant d_0 to make $\triangle = 0$ or $\triangle > 0$ or $\triangle < 0$. This implies that Eq.(1) also admits the explicit solutions like (10),(12)-(14) except $u_i^*(x,t)(i = 1,2,3)$ and $u_8(x,t)$. For example, if setting $d_0 = -\frac{4}{27}d_2^3$, then we have $F(\phi) = (\phi + \frac{2d_2}{3})^2(\phi - \frac{d_2}{3})$. This implies that $\alpha = -\frac{2d_2}{3}, \beta = \frac{d_2}{3}$. So from (10), we can obtain that Eq.(1) also has the following trigonometric periodic wave solution

$$u(x,t) = \frac{b+2}{2(b+1)}\left[1 + 3\tan^2\frac{1}{2}(x - \frac{2+b}{2}t)\right]. \tag{56}$$

Similarly, setting $d_0 = -\frac{2}{27}d_2^3$ yields that $D < 0, \triangle > 0$. Thus, according to Case 3, Eq.(1) admits the solutions like (12) and (13); Setting $d_0 = -\frac{6}{27}d_2^3$ yields that $\triangle < 0$. Thus, according to Case 4, Eq.(1) admits the solution like (14).

As for the case $b \in (-2,-1)$, notice that $\frac{b+1}{b+2} < 0$, then by performing the transformation $\psi = -\phi$ in Eq.(4), we can analyze it similarly.

Therefore, Question 2 is also clarified.

5. Conclusions

Let us shortly formulate the results of our paper. We have demonstrated that using the method of complete discrimination system for polynomial one can find some general explicit solutions of the generalized b-equation. We also demonstrated that almost all explicit solutions from the list by Liu [2] can be obtained by our general solutions. Moreover, we have confirmed or corrected some related conjectures presented by Liu [2] and clarified some questions presented by Liu [2].

Acknowledgements

This work was partially supported by the Technology Creative Project of Excellent Middle and Young Team of Hubei Province (Grants No.T201204).

REFERENCES

[1] R. Liu, Several new types of solitary wave solutions for the generalized Camassa-Holm and Degasperis-Procesi equations, Commun. Pur. Appl. Anal., **9**(2010), 77-90.

[2] R. Liu, Coexistence of multifarious exact nonlinear wave solutions for generalized b-equation, Int. J. Bifurcation and Chaos, **20**(2010), 3193-3208.

[3] C.S. Liu, Trial equation method and its applications to nonlinear evolution equations, Acta. Phys. Sinica, **54**(2005), 2505-2509.

[4] C.S. Liu, Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations, Comput. Phys. Commun., **181**(2010), 317-324.