GENERAL DECAY FOR NONLINEAR WAVE EQUATION WITH NONAUTONOMOUS DAMPING AND MEMORY EFFECT

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We consider the second order evolution equations with memory effect and time-dependent frictional dissipation, which have Neumann boundary conditions. With appropriate assumptions on nonlinear term, we obtain explicit decay rates for the solution in terms of the damping coefficient, in which the relaxation function decays exponentially or polynomially. The proof is based on some energy integral inequalities and it improves the previously related results via a different way.

Keywords: Integro-differential equations; decay rates; nonautonomous damping; memory effect.

MSC2010: 35B40, 35L70.

1. Introduction

In this work, we devote to the long time behavior of solution to the following integro-differential equation

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + \alpha(t)u_t + f(u(t)) = 0 \ in \ \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \gamma} = 0 \ on \ \Gamma \times (0, +\infty), \\ (u, u_t)|_{t=0} = (u_0, u_1) \ in \ \Omega. \end{cases} \tag{1.1}$$

Here, $u = u(t, x)$ is real-valued, and we have denoted by u_t the time derivative of u and by Δu the Laplacian of u with respect to space variable x. Also, $\alpha(t)u_t$ is time-dependent damping term, and $\Omega \subseteq R^n$ is bounded with smooth boundary Γ, also γ is the unit outward normal on Γ.

Under the framework of the unknown functions satisfying Dirichlet boundary conditions, the stabilization of wave equation with distributed damping and memory effect has been studied extensively (cf.,e.g.,[2, 4, 5, 13] and references therein). However, if replacing Dirichlet boundary conditions with Neumann boundary conditions, the solutions will display different behaviors due to the lack of Poincaré inequality. Actually, without u_t or $f(u)$, the solution energies

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to (1.1) do not necessarily tend to zero (this is different from the Dirichlet case), see [14, Remark 3.4(2)].

In the absence of memory terms, A. Cabot and P. Frankel in [1] considered the equation

$$u_{tt} + Au(t) + \alpha(t)u_t + f(u(t)) = 0,$$

where the linear self-adjoint operator A was nonnegative on a Hilbert space. Under some ellipticity-like conditions on A and appropriate assumptions on the nonlinear term f, they gave estimates of energy convergence speed, and the conclusion was enriched in [12]. In the case of $\alpha(t) = 1$, Ghisi et al in [6] proved that all solutions decay at least as fast as a suitable negative power of t. Also, they showed this decay rate was optimal. For more details, we refer the reader to [7, 8, 9]. For the special case of (1.1), [14] took into account the case of $\alpha(t) = 1$, and proved that the energy and solution decayed uniformly, whenever the memory kernel function g decayed exponentially or polynomially. Moreover, when g decayed exponentially, the decay estimate of the solutions was optimal in the sense of slow solutions. In this paper, we devote to the system (1.1), and expect that we can show the existence of slow solutions (this will extend the main result of [14]), but now pay much attention to the decay rate of energy in (1.1).

Our results are different from [6, 14] in setting and method. To be exact, instead of using Lyapunov method to construct disturbance energy, it is better to focus on the original energy and obtain the weighted integral inequality that the energy satisfies. The advantage of integral approach is clear, as a special case, we recover the result in [14, Theorem 3.1].

This paper is organized as follows: In Sect.2, we give some preliminaries and a wellposedness theorem for problem (1.1). Sect.3 is dedicated to the decay estimates of energy and solution. Throughout the paper, for the standard $L^2(\Omega)$, we use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to express the scalar product and norm, respectively. We denote by $c_0, c_1, c_2..., C_0, C_1, C_2...$, various positive constants, and by R the set of real numbers.

2. Preliminaries and Wellposedness

We impose the following basic assumptions on non-autonomous coefficient $\alpha(t)$, memory kernel function $g(t)$ and the nonlinear term $f(s)$.

Assumptions(A-1):

$(i)\alpha(t) \in W^{1,\infty}_{loc}(R^+, R^+)$ is a nonnegative map on R^+.

$(ii)g : [0, +\infty) \to (0, +\infty)$ is a monotone nonincreasing and locally absolutely continuous function satisfying

$$l := 1 - \int_0^{+\infty} g(s)ds > 0, \tag{2.1}$$

and

$$g'(t) \leq -\xi g^{1+\frac{1}{r}}(t), \quad \forall t \geq 0, \tag{2.2}$$

where $r \in (1, \infty]$ with $\frac{1}{r} = 0$ if $r = \infty$, and $\xi > 0$ is a constant.
(iii) The nonlinear term $f(\cdot) \in W^{1,\infty}_{loc}(R)$ satisfies

$$f(s)s \geq 0, \quad \forall s \in R, \tag{2.3}$$

and the growth condition

$$|f'(s)| \leq C_0(1 + |s|^p), \quad \forall s \in R, \tag{2.4}$$

here, $p > 0$ if $n = 1, 2$, and $0 < p \leq \frac{2}{n-2}$ if $n \geq 3$.

The energy of the system (1.1) is defined as

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 + \frac{1}{2}\int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds \\ &+ \int_\Omega F(u(x))dx, \quad t \geq 0, \end{aligned} \tag{2.5}$$

with

$$F(s) = \int_0^s f(r)dr, \quad s \in R.$$

It is easy to get

$$E'(t) = -\alpha(t)\|u_t\|^2 + \frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds - \frac{1}{2}g(t)\|\nabla u(t)\|^2 \leq 0 \tag{2.6}$$

In fact, differentiating formula (2.5), we have

$$\begin{aligned} E'(t) =& \langle u_t, u_{tt}\rangle - \frac{1}{2}g(t)\|\nabla u(t)\|^2 + (1 - \int_0^t g(s)ds)\langle \nabla u(t), \nabla u_t(t)\rangle + \langle f(u(t)), u_t\rangle \\ &+ \frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds + \int_0^t g(t-s)\langle \nabla u(s) - \nabla u(t), -\nabla u_t(t)\rangle ds \\ =& \langle u_t, u_{tt}\rangle - \frac{1}{2}g(t)\|\nabla u(t)\|^2 + \langle -\Delta u(t), u_t(t)\rangle + \langle f(u(t)), u_t\rangle \\ &+ \frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds + \int_0^t g(t-s)\langle \Delta u(s), u_t(t)\rangle ds \\ =& \langle u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + f(u(t)), u_t\rangle - \frac{1}{2}g(t)\|\nabla u(t)\|^2 \\ &+ \frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds \\ =& -\alpha(t)\|u_t\|^2 - \frac{1}{2}g(t)\|\nabla u(t)\|^2 + \frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds, \end{aligned}$$

here the last identity holds because u is a solution of (1.1). As $\alpha(t) > 0, g(t) \geq 0$ and $g'(t) \leq 0$, the right-hand side above is negative. (2.6) means that $E(t)$ is nonincreasing function. Furthermore, $F(s)$ is nonnegative by (2.3) and $f(u) : H^1(\Omega) \to L^2(\Omega)$ is a locally Lipschitz continuous operator, by Assumptions(A-1)(iii). Therefore, we can get the following result regarding global existence and regularity (cf.[3, Proposition 4.3]).

Theorem 2.1. *Let Assumptions (A-1) hold. Then for given* $\{u_0, u_1\} \in V \times H^1(\Omega)$ *with* $V = \{u | u \in H^2(\Omega), \frac{\partial u}{\partial \gamma} = 0 \text{ on } \Gamma\}$, *problem (1.1) possesses a unique strong solution in the class*

$$u \in C(0, +\infty; V) \cap C^1(0, +\infty; H^1(\Omega)) \cap C^2(0, +\infty; L^2(\Omega)).$$

If $\{u_0, u_1\} \in H^1(\Omega) \times L^2(\Omega)$, *then problem (1.1) has a unique mild solution*

$$u \in C(0, +\infty; H^1(\Omega)) \cap C^1(0, +\infty; L^2(\Omega)).$$

We recall the following decay estimate(see, e.g., [10] [11, Lemma 1]), which plays an important role in the proof of our main result.

Lemma 2.1. *Let* $E : R^+ \to R^+$ *be a nonincreasing function, and* $\phi : R^+ \to R^+$ *a strictly increasing* C^1*-function such that*

$$\phi(0) = 0, \ \phi(t) \to +\infty \text{ as } t \to +\infty.$$

Assume that there exist $q \geq 0$ *and* $w > 0$ *such that*

$$\int_S^{+\infty} E^{q+1}(t)\phi'(t)dt \leq \frac{1}{w}E^q(0)E(S), \quad \forall S \geq 0,$$

Then E *has the following decay property:*

$$if \ q = 0, \ then \ E(t) \leq E(0)e^{1-w\phi(t)}, \quad \forall t \geq 0,$$
$$if \ q > 0, \ then \ E(t) \leq E(0)\Big(\frac{q+1}{qw\phi(t)+1}\Big)^{\frac{1}{q}}, \quad \forall t \geq 0.$$

3. **The Main Results**

In this section, we are in a position to state the main result of this paper. It gives the decay rate of $E(t)$ and an upper estimate which are valid for all solutions to system (1.1). Such results are obtained by strengthening Assumptions (A-1) as follows.

Assumptions(A-2): Let $p > 0$. There exist constants $C_1 > 0, C_2 > 0$, such that

$$C_1|s|^{p+2} \leq F(s) \leq C_2 f(s)s, \quad \forall s \in R. \tag{3.1}$$

The first result concerns the case when the memory kernel g decays exponentially.

Theorem 3.1. *Let Assumptions (A-1) and (A-2) hold with* $r = \infty$. *Assume that*

$$\alpha'(t) \in L^1(R^+), \quad and \int_0^\infty \alpha(t)dt = +\infty. \tag{3.2}$$

Let $\{u_0, u_1\} \in H^1(\Omega) \times L^2(\Omega)$, *then, the unique solution* u *of problem* (1.1) *satisfies that for* $t \geq 0$,

$$E(t) \leq M(E(0))\Big(1 + \int_0^t \alpha(s)ds\Big)^{-(1+\frac{2}{p})}, \tag{3.3}$$

and

$$\|u(t)\| \leq M(E(0))\Big(1+\int_0^t \alpha(s)ds\Big)^{-\frac{1}{p}}, \tag{3.4}$$

where $M(\cdot)$ is a positive function on R, and is bounded on bounded sets.

Proof. We only show the proof for the strong solution. Actually, an approximation argument suffices to extend conclusions to mild solution.

From (2.6), we get that $\forall t \geq 0$,

$$E(t) \leq E(0). \tag{3.5}$$

Let $\beta \geq \frac{p}{p+2}$. For $0 \leq S < T$, multiplying (1.1) by $E^\beta(t)\alpha(t)u$, and integrating over $[S,T]$, we have

$$\int_S^T E^\beta(t)\alpha(t)\Big\langle u, u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(s)ds+\alpha(t)u_t+f(u(t))\Big\rangle dt = 0. \tag{3.6}$$

Integrating by parts yields

$$\begin{aligned}
&\int_S^T E^\beta(t)\alpha(t)\langle u, u_{tt}\rangle dt \\
=&\int_S^T \frac{d}{dt}\Big(E^\beta(t)\alpha(t)\langle u,u_t\rangle\Big)dt - \int_S^T E^\beta(t)\alpha(t)\|u_t\|^2 dt - \int_S^T E^\beta(t)\alpha'(t)\langle u,u_t\rangle dt \\
&-\beta\int_S^T E^{\beta-1}(t)E'(t)\alpha(t)\langle u,u_t\rangle dt \\
=&E^\beta(t)\alpha(t)\langle u,u_t\rangle|_S^T - \int_S^T E^\beta(t)\alpha(t)\|u_t\|^2 dt - \int_S^T E^\beta(t)\alpha'(t)\langle u,u_t\rangle dt \\
&-\beta\int_S^T E^{\beta-1}(t)E'(t)\alpha(t)\langle u,u_t\rangle dt.
\end{aligned} \tag{3.7}$$

Similarly, as $\langle u, -\Delta u\rangle = -\int_\Omega u\Delta u dx = -\int_\Gamma u\frac{\partial u}{\partial \gamma}d\Gamma + \int_\Omega \nabla u \nabla u dx = \|\nabla u\|^2$ by (1.1), we obtain

$$\int_S^T E^\beta(t)\alpha(t)\langle u, -\Delta u\rangle dt = \int_S^T E^\beta(t)\alpha(t)\|\nabla u\|^2 dt, \tag{3.8}$$

and

$$\begin{aligned}
&\int_S^T E^\beta(t)\alpha(t)\langle u, \int_0^t g(t-s)\Delta u(s)ds\rangle dt = -\int_S^T E^\beta(t)\alpha(t)\int_0^t g(t-s)\langle \nabla u(t), \nabla u(s)\rangle ds dt \\
=&-\int_S^T E^\beta(t)\alpha(t)\langle \nabla u(t), \int_0^t g(t-s)\nabla u(s)\rangle ds dt \\
=&-\int_S^T E^\beta(t)\alpha(t)\langle \nabla u(t), \int_0^t g(t-s)\Big(\nabla u(s)-\nabla u(t)\Big)\rangle ds dt \\
&-\int_S^T E^\beta(t)\alpha(t)\int_0^t g(s)ds\|\nabla u\|^2 dt.
\end{aligned} \tag{3.9}$$

Taking (3.7)-(3.9) into (3.6), we get

$$\begin{aligned}
&\int_S^T E^\beta(t)\alpha(t)\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|^2dt+\int_S^T E^\beta(t)\alpha(t)\langle f(u),u\rangle dt\\
=&-E^\beta(t)\alpha(t)\langle u,u_t\rangle|_S^T+\int_S^T E^\beta(t)\alpha(t)\|u_t\|^2dt+\int_S^T E^\beta(t)\alpha'(t)\langle u,u_t\rangle dt\\
&+\beta\int_S^T E^{\beta-1}(t)E'(t)\alpha(t)\langle u,u_t\rangle dt-\int_S^T E^\beta(t)\alpha^2(t)\langle u,u_t\rangle dt\\
&+\int_S^T E^\beta(t)\alpha(t)\langle\nabla u,\int_0^t g(t-s)\Big(\nabla u(s)-\nabla u(t)\Big)ds\rangle dt.
\end{aligned}\tag{3.10}$$

Letting $\theta\in(0,1)$ and noting (2.5), we obtain by the right hand inequality of (3.1)

$$\begin{aligned}
&\theta E^\beta(t)\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|^2+E^\beta(t)\langle f(u),u\rangle\\
\geq& E^\beta(t)\Big(\theta\big(1-\int_0^t g(s)ds\big)\|\nabla u\|^2+\frac{1}{C_2}\int_\Omega F(u(x))dx\Big)\\
\geq& c_1E^\beta(t)\Big(\frac12(1-\int_0^t g(s)ds)\|\nabla u\|^2+\int_\Omega F(u(x))dx\Big)\\
=& c_1E^\beta(t)\Big(E(t)-\frac12\|u_t\|^2-\frac12\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2ds\Big),
\end{aligned}$$

here $c_1=\min\{2\theta,\frac{1}{C_2}\}$. Combining with (3.10) leads to

$$\begin{aligned}
&c_1\int_S^T E^{\beta+1}(t)\alpha(t)dt\\
\leq&\frac12(c_1+2)\int_S^T E^\beta(t)\alpha(t)\|u_t\|^2dt+\frac{c_1}{2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2dsdt\\
&-(1-\theta)\int_S^T E^\beta(t)\alpha(t)\Big(1-\int_0^t g(s)ds\Big)\|\nabla u\|^2dt-E^\beta(t)\alpha(t)\langle u,u_t\rangle|_S^T\\
&+\int_S^T E^\beta(t)\alpha'(t)\langle u,u_t\rangle dt+\beta\int_S^T E^{\beta-1}(t)E'(t)\alpha(t)\langle u,u_t\rangle dt-\int_S^T E^\beta(t)\alpha^2(t)\langle u,u_t\rangle dt\\
&+\int_S^T E^\beta(t)\alpha(t)\langle\nabla u,\int_0^t g(t-s)\Big(\nabla u(s)-\nabla u(t)\Big)ds\rangle dt.
\end{aligned}\tag{3.11}$$

Our next task is to estimate the right-hand side of (3.11). Recalling (2.6), we have

$$\begin{aligned}
I_1:=&\int_S^T E^\beta(t)\alpha(t)\|u_t\|^2dt\leq\int_S^T E^\beta(t)\big(-E'(t)\big)dt=-\frac{1}{\beta+1}E^{\beta+1}(t)|_S^T\\
\leq&\frac{1}{\beta+1}E^{\beta+1}(S).
\end{aligned}\tag{3.12}$$

Noting that $L^{p+2}(\Omega) \hookrightarrow L^2(\Omega)$, and applying the left hand inequality of (3.1), we have

$$\|u(t)\|^{p+2} \le c_0 E(t), \quad \forall t \ge 0. \tag{3.13}$$

Together with (2.5), we get

$$|\langle u, u_t\rangle| \le \|u_t\| \cdot \|u\| \le c_2 E^{\frac{1}{2}+\frac{1}{p+2}}(t),$$

from which we obtain

$$\begin{aligned} I_4 :&= -E^{\beta}(t)\alpha(t)\langle u, u_t\rangle|_S^T \le E^{\beta}(S)\|\alpha(t)\|_{L^\infty}\Big(\|u_t(T)\| \cdot \|u(T)\| + \|u_t(S)\| \cdot \|u(S)\|\Big) \\ &\le c_2 E^{\beta}(S)\|\alpha(t)\|_{L^\infty}\Big(E^{\frac{1}{2}+\frac{1}{p+2}}(T) + E^{\frac{1}{2}+\frac{1}{p+2}}(S)\Big) \\ &\le 2c_2 E^{\beta}(S)\|\alpha(t)\|_{L^\infty} E^{\frac{1}{2}+\frac{1}{p+2}}(S) = 2c_2\|\alpha(t)\|_{L^\infty} E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S). \end{aligned} \tag{3.14}$$

Also,

$$\begin{aligned} I_5 :&= \int_S^T E^{\beta}(t)\alpha'(t)\langle u, u_t\rangle dt \le c_2 \int_S^T E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(t)\big|\alpha'(t)\big| dt \\ &\le c_3 E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S), \end{aligned} \tag{3.15}$$

by $\alpha'(t) \in L^1(R^+)$ in (3.2), as well as

$$\begin{aligned} I_6 :&= \beta \int_S^T E^{\beta-1}(t)E'(t)\alpha(t)\langle u, u_t\rangle dt \le \beta c_2 \int_S^T E^{\beta-1}(t)\big(-E'(t)\big)\alpha(t)E^{\frac{1}{2}+\frac{1}{p+2}}(t)dt \\ &\le \beta c_2\|\alpha(t)\|_{L^\infty}\int_S^T E^{\beta-\frac{1}{2}+\frac{1}{p+2}}(t)\big(-E'(t)\big)dt \le \beta c_2\|\alpha(t)\|_{L^\infty} E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S). \end{aligned} \tag{3.16}$$

Using (2.6),(3.13) and Young's inequality, for any $\eta_1 > 0$, we further have

$$\begin{aligned} I_7 :&= -\int_S^T E^{\beta}(t)\alpha^2(t)\langle u, u_t\rangle dt \le \|\alpha(t)\|_{L^\infty}\int_S^T E^{\beta}(t)\alpha(t)\|u_t\| \cdot \|u\| dt \\ &\le c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}\int_S^T E^{\beta+\frac{1}{p+2}}(t)\alpha(t)\|u_t\| dt \\ &\le c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}\Big(\eta_1\int_S^T \alpha(t)E^{2\beta+\frac{2}{p+2}}(t)dt + \frac{1}{4\eta_1}\int_S^T \alpha(t)\|u_t\|^2 dt\Big) \\ &\le \eta_1 c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}\int_S^T \alpha(t)E^{2\beta+\frac{2}{p+2}}(t)dt + \frac{c_0^{\frac{1}{p+2}}}{4\eta_1}\|\alpha(t)\|_{L^\infty}E(S). \end{aligned} \tag{3.17}$$

By applying Young's inequality, for any $\eta_2 > 0$,

$$\begin{aligned} &\langle \nabla u, \int_0^t g(t-s)\Big(\nabla u(s) - \nabla u(t)\Big)ds\rangle \\ \le&\eta_2\|\nabla u\|^2 + \frac{1}{4\eta_2}\int_0^t g^{1-\frac{1}{r}}(s)ds\int_0^t g^{1+\frac{1}{r}}(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds. \end{aligned}$$

This yields that

$$
\begin{aligned}
I_8 :=& \int_S^T E^\beta(t)\alpha(t)\langle\nabla u, \int_0^t g(t-s)\Big(\nabla u(s)-\nabla u(t)\Big)ds\rangle dt \\
\le& \eta_2 \int_S^T E^\beta(t)\alpha(t)\|\nabla u\|^2 dt \qquad (3.18)\\
&+ \frac{1}{4\eta_2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g^{1-\frac{1}{r}}(s)ds\int_0^t g^{1+\frac{1}{r}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt.
\end{aligned}
$$

Altogether, by taking (3.12), (3.14)-(3.17), (3.18) into (3.11), we get the estimate

$$
\begin{aligned}
c_1\int_S^T E^{\beta+1}(t)\alpha(t)dt \le& \frac{c_1+2}{2(\beta+1)}E^{\beta+1}(S) + \Big(\big(\beta+2\big)c_2\|\alpha(t)\|_{L^\infty}+c_3\Big)E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S)\\
&+\frac{1}{4\eta_1}c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}E(S)+\eta_1 c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}\int_S^T \alpha(t)E^{2\beta+\frac{2}{p+2}}(t)dt\\
&+\int_S^T\Big(\eta_2-(1-\theta)\big(1-\int_0^t g(s)ds\big)\Big)E^\beta(t)\alpha(t)\|\nabla u\|^2 dt \qquad (3.19)\\
&+\frac{c_1}{2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt\\
&+\frac{1}{4\eta_2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g^{1-\frac{1}{r}}(s)ds\int_0^t g^{1+\frac{1}{r}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt.
\end{aligned}
$$

Furthermore, we make use of Assumption (2.2) (with $r=\infty$) and (2.6), to get

$$
\begin{aligned}
&\int_S^T E^\beta(t)\alpha(t)\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt\\
\le&\frac{1}{\xi}\int_S^T E^\beta(t)\alpha(t)\int_0^t -g'(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt \qquad (3.20)\\
\le&\frac{2}{\xi}\int_S^T -E^\beta(t)\alpha(t)E'(t)dt \le \frac{2\|\alpha(t)\|_{L^\infty}}{\xi(\beta+1)}E^{\beta+1}(S).
\end{aligned}
$$

Therefore, with the help of (3.20) and (3.19) to obtain

$$
\begin{aligned}
&c_1\int_S^T E^{\beta+1}(t)\alpha(t)dt\\
\le&\Big(\frac{c_1+2}{2(\beta+1)}+\big(c_1+\frac{1}{2\eta_2}\big)\frac{\alpha(t)\|_{L^\infty}}{\xi(\beta+1)}\Big)E^{\beta+1}(S)+\Big(\big(\beta+2\big)c_2\|\alpha(t)\|_{L^\infty}+c_3\Big)E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S)\\
&+\frac{1}{4\eta_1}c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}E(S)+\eta_1 c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}\int_S^T \alpha(t)E^{2\beta+\frac{2}{p+2}}(t)dt\\
&+\int_S^T\Big(\eta_2-(1-\theta)\big(1-\int_0^t g(s)ds\big)\Big)E^\beta(t)\alpha(t)\|\nabla u\|^2 dt,
\end{aligned}
$$

where we have used the fact $\int_0^\infty g(s)ds < 1$ in (2.1). Take $\beta = \frac{p}{p+2}$, which satisfies $\beta + 1 = 2\beta + \frac{2}{p+2}$. For the fixed $\theta \in (0, 1)$, let η_2 small enough such that $\eta_2 - (1-\theta)\big(1 - \int_0^t g(s)ds\big) \le 0$. Then for the fixed θ, η_2, choosing η_1 small enough, we justify the existence of positive constants c_4, c_5, c_6 such that

$$\begin{aligned}\int_S^T E^{\beta+1}(t)\alpha(t)dt \le & c_4 E^{\beta+1}(S) + c_5 E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S) + c_6 E(S) \\ = & c_4 E^{\beta+1}(S) + c_5 E^{\frac{p}{2(p+2)}+1}(S) + c_6 E(S).\end{aligned}$$

Thus, by (3.5) and letting $T \to \infty$, we obtain

$$\int_S^\infty E^{\beta+1}(t)\alpha(t)dt \le \frac{1}{w} E^\beta(0)E(S),$$

where $\frac{1}{w} = c_4 + c_5 E^{-\frac{\beta}{2}}(0) + c_6 E^{-\beta}(0)$.

Then, applying Lemma 2.1 with $\phi(t) = \int_0^t \alpha(s)ds$, and $q = \frac{p}{p+2}$, we get

$$\begin{aligned}E(t) \le & E(0)\left(\frac{q+1}{qw\phi(t)+1}\right)^{\frac{1}{q}} = E(0)\left(\frac{q+1}{qw\int_0^t \alpha(s)ds + 1}\right)^{1+\frac{2}{p}} \\ \le & M(E(0))\left(1 + \int_0^t \alpha(s)ds\right)^{-(1+\frac{2}{p})},\end{aligned}$$

here $M(\cdot)$ is a positive function on R, and bounded on bounded sets. Recalling (3.13), it is easy to get (3.4). The proof is finished. □

In the following, we devote to the case of polynomially decaying memory kernel g.

Theorem 3.2. *Let Assumptions (A-1) and (A-2) hold with $r \in (1, +\infty)$ and $\{u_0, u_1\} \in H^1(\Omega) \times L^2(\Omega)$. For a fixed number β satisfies*

$$\beta \ge \frac{p}{p+2}, \text{ and } \beta > \frac{1}{r-1}, \tag{3.21}$$

the unique solution u of problem (1.1) *satisfies*

$$E(t) \le M(E(0))\Big(1 + \int_0^t \alpha(s)ds\Big)^{-\frac{1}{\beta}}, \quad t \ge 0, \tag{3.22}$$

and

$$\|u(t)\| \le M(E(0))\Big(1 + \int_0^t \alpha(s)ds\Big)^{-\frac{1}{\beta(p+2)}}, \quad t \ge 0, \tag{3.23}$$

where $M(\cdot)$ is a positive function on R, and is bounded on bounded sets.

Proof. Similar to the proof of Theorem 3.1, we can arrive at (3.19). Our goal is to obtain the estimates for the terms

$$J_1 := \frac{c_1}{2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|^2 dsdt,$$

and

$$J_2 := \frac{1}{4\eta_2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g^{1-\frac{1}{r}}(s)ds\int_0^t g^{1+\frac{1}{r}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt.$$

One can easily show that

$$\int_0^{+\infty} g^{1-\lambda}(s)ds < +\infty,\ for\ 0<\lambda<1-\frac{1}{r}.$$

Due to this fact and (2.1),(3.5), we see that

$$h(t) := \int_0^t g^{1-\lambda}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds \le 2\int_0^t g^{1-\lambda}(t-s)(\|\nabla u(s)\|^2+\|\nabla u(t)\|^2)ds$$
$$\le \frac{8}{l}E(0)\int_0^{+\infty} g^{1-\lambda}(s)ds.$$

This combines (2.2),(2.6) and Jensen's inequality, yields

$$\begin{aligned}\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds &= \int_0^t g^\lambda(t-s)g^{1-\lambda}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds\\ &= \int_0^t g^{(\frac{1}{r}+\lambda)\frac{\lambda}{\frac{1}{r}+\lambda}}(t-s)g^{1-\lambda}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds\\ &\le h(t)\left(\frac{1}{h(t)}\int_0^t g^{\frac{1}{r}+\lambda}(t-s)g^{1-\lambda}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds\right)^{\frac{\lambda}{\frac{1}{r}+\lambda}}\\ &= h^{1-\frac{\lambda}{\frac{1}{r}+\lambda}}(t)\left(\int_0^t g^{1+\frac{1}{r}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds\right)^{\frac{\lambda}{\frac{1}{r}+\lambda}}\\ &\le c_7(-E'(t))^{\frac{\lambda}{\frac{1}{r}+\lambda}}.\end{aligned}$$

Using Young's inequality, with exponents $1+r\lambda$ and $1+\frac{1}{r\lambda}$, we get for any $\eta_3>0$

$$\begin{aligned}E^\beta(t)\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2 ds &\le c_7E^\beta(t)(-E'(t))^{\frac{\lambda}{\frac{1}{r}+\lambda}}\\ &\le c_7\eta_3E^{\beta(1+r\lambda)}(t)+c_7c_{\eta_3}(-E'(t)).\end{aligned}\tag{3.24}$$

By (3.21), there exists $\lambda\in(0,1-\frac{1}{r})$ such that $\beta(1+r\lambda)=\beta+1$. For this fixed λ, (3.24) indicates

$$\begin{aligned}J_1 :=&\frac{c_1}{2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2 dsdt\\ \le&\frac{c_1c_7\eta_3}{2}\int_S^T \alpha(t)E^{\beta+1}(t)dt+\frac{c_1c_7c_{\eta_3}}{2}\int_S^T\alpha(t)\big(-E'(t)\big)dt\\ \le&\frac{c_1c_7\eta_3}{2}\int_S^T \alpha(t)E^{\beta+1}(t)dt+\frac{c_1c_7c_{\eta_3}}{2}\|\alpha(t)\|_{L^\infty}E(S).\end{aligned}\tag{3.25}$$

Moreover, by Assumption (2.2) and (2.6), we get

$$\begin{aligned}
J_2 :=&\frac{1}{4\eta_2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g^{1-\frac{1}{r}}(s)ds\int_0^t g^{1+\frac{1}{r}}(t-s)\|\nabla u(s)-\nabla u(t)\|^2dsdt\\
\le&\frac{1}{4\eta_2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g^{1-\frac{1}{r}}(s)ds\int_0^t -\frac{1}{\xi}g'(t-s)\|\nabla u(s)-\nabla u(t)\|^2dsdt\\
\le&\frac{1}{4\eta_2}\int_S^T E^\beta(t)\alpha(t)\int_0^t g^{1-\frac{1}{r}}(s)ds\Big(-\frac{2}{\xi}E'(t)\Big)dt \qquad (3.26)\\
\le&\frac{1}{2\eta_2\xi}\int_0^\infty g^{1-\frac{1}{r}}(s)ds\int_S^T E^\beta(t)\alpha(t)\big(-E'(t)\big)dt\\
\le&\frac{\|\alpha(t)\|_{L^\infty}}{2\eta_2\xi(1+\beta)}\int_0^\infty g^{1-\frac{1}{r}}(s)dsE^{\beta+1}(S).
\end{aligned}$$

Taking (3.25),(3.26) into (3.19), we arrive at

$$\begin{aligned}
&c_1\int_S^T E^{\beta+1}(t)\alpha(t)dt\\
\le&\Big(\frac{c_1+2}{2(\beta+1)}+\frac{\|\alpha(t)\|_{L^\infty}}{2\eta_2\xi(1+\beta)}\int_0^\infty g^{1-\frac{1}{r}}(s)ds\Big)E^{\beta+1}(S)\\
&+\Big(\big(\beta+2\big)c_2\|\alpha(t)\|_{L^\infty}+c_3\Big)E^{\beta+\frac{1}{2}+\frac{1}{p+2}}(S)+\Big(\frac{1}{4\eta_1}c_0^{\frac{1}{p+2}}+\frac{c_1c_7c_{\eta_3}}{2}\Big)\|\alpha(t)\|_{L^\infty}E(S)\\
&+\eta_1c_0^{\frac{1}{p+2}}\|\alpha(t)\|_{L^\infty}\int_S^T\alpha(t)E^{2\beta+\frac{2}{p+2}}(t)dt+\frac{c_1c_7\eta_3}{2}\int_S^T\alpha(t)E^{\beta+1}(t)dt\\
&+\int_S^T\Big(\eta_2-(1-\theta)\big(1-\int_0^t g(s)ds\big)\Big)E^\beta(t)\alpha(t)\|\nabla u\|^2dt.
\end{aligned}$$

Noting $2\beta+\frac{2}{p+2}\ge\beta+1$ (by $\beta\ge\frac{p}{p+2}$), we get

$$\int_S^T\alpha(t)E^{2\beta+\frac{2}{p+2}}(t)dt\le E^{\beta_1}(0)\int_S^T\alpha(t)E^{\beta+1}(t)dt,$$

where $\beta_1:=\beta-\frac{p}{p+2}$. Therefore, just as the proof of Theorem 3.1, we achive the estimate (3.22) and (3.23). This completes the proof. □

Corollary 3.1. *Let Assumptions (A-1) and (A-2) hold and* $\{u_0,u_1\}\in H^1(\Omega)\times L^2(\Omega)$. *If* $r>2(1+1/p)$, *then the estimates in* (3.3) *and* (3.4) *remain valid.*

Proof. Let $\beta=\frac{p}{p+2}$, it is easy to see that (3.21) is satisfied. Therefore, the estimates can be obtained by Theorem 3.2. □

Remark 3.1. (1) We can see from Corollary 3.1, the solution can still decays at the rate $\big(1+\int_0^t\alpha(s)ds\big)^{-\frac{1}{p}}$ as in (3.4), whenever $g(t)$ decays polynomially as $(1+t)^{-r}$, providing r is large enough.
(2) In the special case $\alpha(t)=1$, the decay rates of the energy and solution,

given in Theorem 3.1 and Theorem 3.2, recover those in [14, Theorem 3.1, Theorem 3.2].

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