FIXED POINTS OF WEAKLY CYCLIC TYPE GENERALIZED CONTRACTIONS IN EXTENDED b-METRIC SPACE

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In the setting of extended b-metric space, we introduce new weakly cyclic generalized contraction conditions depending on continuous functions ω_θ and altering distance functions ξ_θ for self mappings, called "extended $(\xi_\theta, \omega_\theta)$-weakly cyclic generalized contraction conditions one and two". Via these contractions, we prove existence and uniqueness fixed points theorems. Moreover, we derive some results on b-metric spaces, and some examples to focus the attention on the importance of our work.

Keywords: Extended b-metric space, fixed point, cyclic representation, altering distance function, contraction.

1. Introduction

The fixed point area research is one of the major areas of mathematics, which has benefited from a systematic development in the last century; for the pioneer source, see Caccioppoli [3], which established "Banach Contraction Principle". This one assures the existence of a unique fixed point via contractions over a complete metric space. Since then, due to the importance of this theorem, numerous generalizations have been demonstrated in different directions.

By altering the distances between points, Khan *et al.* [9] brought an effective contribution to the theory of fixed point for self-mappings on complete metric spaces. As altering distance functions are monotone, increasing, and continuous, they became an attractive destination for researchers who took advantage of this concept to work on fixed point problems.

In 1997, by employing the definition of weak contractive mappings, Alber and Guerre-Delabriere [1] prolonged Banach contraction principle in Hilbert spaces. In 2001, Rhoades [14] generalized the weak contraction principle in the ambit of metric spaces. Shatanawi *et al.* [18] inspect new coincidence point theorems for a pair of weakly decreasing mappings satisfying (ψ, φ)-weakly contractive conditions in an ordered metric space, where ψ and φ are altering distance functions.

On other hand, in 2003, the notion of cyclic contraction was initiated by Kirk [10], who demonstrated some fixed point theorems. Furthermore, many scholars have developed this contraction. For more results, see [2, 17, 21].

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Other well-known extensions of the Banach fixed point principle is the generalization of the notion of metric space in many varied directions [5, 6, 7, 8, 12, 13]. In 2017, Kamran *et al.* [4] relaxed the triangle inequality of a metric space with the function θ and formally defined an extended b-metric space, which has contributed to the generation of many results.

In this work, we generalize [11], where we investigate some fixed point theorems throw extended $(\xi_\theta, \omega_\theta)$-weakly cyclic generalized contraction conditions one and two in the framework of complete extended b-metric space.

2. Preliminaries

In this section, we remember some notions for extended b-metric spaces, which we will need in the sequel.

Definition 2.1 ([4]). *Let Π be a non-empty set and $\theta\colon \Pi \times \Pi \longrightarrow [1, \infty)$. An extended b-metric is a function $d_\theta\colon \Pi \times \Pi \longrightarrow [0, \infty)$ such that for all $\mu, \nu, \omega \in \Pi$, we have:*
(1) $d_\theta(\mu, \nu) = 0$ *iff* $\mu = \nu$,
(2) $d_\theta(\mu, \nu) = d_\theta(\nu, \mu)$,
(3) $d_\theta(\mu, \omega) \leq \theta(\mu, \omega)\big[d_\theta(\mu, \nu) + d_\theta(\nu, \mu)\big]$.
Then the couple (Π, d_θ) is called an extended b-metric space.

If $\theta(\mu, \omega) = s$, $s \in [1, +\infty)$, then (Π, d_s) is named a b-metric space with parameter s.

Definition 2.2 ([4]). *Let (Π, d_θ) be an extended b-metric space. A sequence (μ_ι) in Π is:*
(i) *Cauchy sequence: if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$, with $d_\theta(\mu_\iota, \mu_\eta) < \epsilon$, for all ι, $\eta \geq N$.*
(ii) *convergent: if exists $\mu^\star \in \Pi$, for all $\epsilon > 0$, exists $N \in \mathbb{N}$, with $d_\theta(\mu_\iota, \mu^\star) < \epsilon$, for all $\iota \geq N$.*

The couple (Π, d_θ) is named complete extended b-metric space if every Cauchy sequence converges.

For our purposes, we demand to evoke the following definition,

Definition 2.3 ([9]). *A function $\xi_\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ is named an altering distance function if ξ_θ is continuous, strictly increasing on $\mathbb{R}^+$ and $\xi_\theta(\tau) = 0$ iff $\tau = 0$.*

We adopt [4] and [10] to generate the following definition:

Definition 2.4. *Let (Π, d_θ) be an extended b-metric space. Let $p \in \mathbb{N}$, $\aleph_1, \aleph_2, \ldots, \aleph_p$ be subsets of Π, $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$ and $\Upsilon\colon \mathcal{Z} \to \mathcal{Z}$. Then $\mathcal{Z}$ is a cyclic representation related to Υ, if $\aleph_i, (i = 1, 2, \ldots, p)$ are non empty, closed and $\Upsilon(\aleph_1) \subset \aleph_2, \ldots, \Upsilon(\aleph_{p-1}) \subset \aleph_p, \Upsilon(\aleph_p) \subset \aleph_1$.*

3. Main Results

Definition 3.1. *Let (Π, d_θ) be an extended b-metric space. Let $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$, where $\aleph_1, \aleph_2, \ldots, \aleph_p$ are non-empty and closed subsets of Π, for $p \in \mathbb{N}$. A self mapping $\Upsilon\colon \mathcal{Z} \to \mathcal{Z}$ is called an extended $(\xi_\theta, \omega_\theta)$-weakly cyclic generalized contraction condition one if these axioms hold:*

1) *$\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$ is a cyclic representation of $\mathcal{Z}$ related to Υ.*

2) *For any $(\mu, \nu) \in \aleph_i \times \aleph_{i+1}$, where $i = 1, 2, \ldots, p$ and $\aleph_{p+1} = \aleph_1$,*

$$\xi_\theta\left(\frac{1}{\alpha^5} d_\theta(\Upsilon\mu, \Upsilon\nu)\right) \leq \xi_\theta(\Delta_1(\mu, \nu)) - \omega_\theta(\Delta_2(\mu, \nu)), \qquad (\mathbf{3}.1)$$

where

$$\Delta_1(\mu,\nu)=\max\Big\{d_\theta(\mu,\nu),\frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu,\Upsilon\nu)+d_\theta(\nu,\Upsilon\mu)\Big),\frac{1}{2}\Big(d_\theta(\mu,\Upsilon\mu)+d_\theta(\nu,\Upsilon\nu)\Big)\Big\},\tag{3.2}$$

and

$$\Delta_2(\mu,\nu)=\min\Big\{d_\theta(\mu,\nu),\frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu,\Upsilon\nu)+d_\theta(\nu,\Upsilon\mu)\Big)\Big\}.\tag{3.3}$$

As well, θ is bounded by $\frac{1}{\alpha}$, where $\alpha\in(0,1)$, $\xi_\theta\colon\mathbb{R}^+\to\mathbb{R}^+$ is an altering distance function, $\omega_\theta\colon\mathbb{R}^+\to\mathbb{R}^+$ is continuous and $\omega_\theta(\tau)=0$ if and only if $\tau=0$.

Theorem 3.1. *Consider a complete extended b-metric space (Π,d_θ). If the self-mapping $\Upsilon\colon\mathcal{Z}\to\mathcal{Z}$ is extended $(\xi_\theta,\omega_\theta)$-weakly cyclic generalized contraction condition one, then Υ possesses a unique fixed point in $\bigcap_{i=1}^{p}\aleph_i$.*

Proof. Let $\mu_0\in\aleph_1$ (where $\aleph_i$ is non-empty for all i). Consider the sequence (μ_ι) in Π given by $\mu_{\iota+1}=\Upsilon\mu_\iota$, for all $\iota\in\mathbb{N}\cup\{0\}$.

- If $\mu_{\iota+1}=\mu_\iota$, then μ_ι is a fixed point of Υ.
- If $\mu_{\iota+1}\neq\mu_\iota$, then we shall give a proof.

Step 1 $\lim_{\iota\to\infty}d_\theta(\mu_\iota,\mu_{\iota+1})=0$: We have for all ι, $d_\theta(\mu_\iota,\mu_{\iota+1})>0$. From the first condition in Definition **3**.1, we obtain $i=i(\iota)$, $i=\{1,2,\dots,p\}$ for all ι, $(\mu_\iota,\mu_{\iota+1})\in\aleph_i\times\aleph_{i+1}$.

In the second condition of Definition **3**.1, putting $\mu=\mu_\iota$ and $\nu=\mu_{\iota+1}$, we get

$$\begin{aligned}\xi_\theta(d_\theta(\Upsilon\mu_\iota,\Upsilon\mu_{\iota+1}))&\le\xi_\theta\left(\frac{1}{\alpha^5}d_\theta(\Upsilon\mu_\iota,\Upsilon\mu_{\iota+1})\right)=\xi_\theta(\frac{1}{\alpha^5}d_\theta(\mu_{\iota+1},\mu_{\iota+2}))\\&\le\xi_\theta(\Delta_1(\mu_\iota,\mu_{\iota+1}))-\omega_\theta(\Delta_2(\mu_\iota,\mu_{\iota+1})),\end{aligned}\tag{3.4}$$

where

$$\begin{aligned}\Delta_1(\mu_\iota,\mu_{\iota+1})&=\max\Big\{d_\theta(\mu_\iota,\mu_{\iota+1}),\frac{1}{2\theta(\mu_\iota,\Upsilon\mu_{\iota+1})}\Big(d_\theta(\mu_\iota,\Upsilon\mu_{\iota+1})+d_\theta(\mu_{\iota+1},\Upsilon\mu_\iota)\Big),\\&\quad\frac{1}{2}\Big(d_\theta(\mu_\iota,\Upsilon\mu_\iota)+d_\theta(\mu_{\iota+1},\Upsilon\mu_{\iota+1})\Big)\Big\},\\&=\max\Big\{d_\theta(\mu_\iota,\mu_{\iota+1}),\frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}\Big(d_\theta(\mu_\iota,\mu_{\iota+2})+d_\theta(\mu_{\iota+1},\mu_{\iota+1})\Big),\\&\quad\frac{1}{2}\Big(d_\theta(\mu_\iota,\mu_{\iota+1})+d_\theta(\mu_{\iota+1},\mu_{\iota+2})\Big)\Big\}.\end{aligned}\tag{3.5}$$

By the triangle inequality, we obtain

$$\Delta_1(\mu_\iota,\mu_{\iota+1})\le\max\Big\{d_\theta(\mu_\iota,\mu_{\iota+1}),\frac{1}{2}\Big(d_\theta(\mu_\iota,\mu_{\iota+1})+d_\theta(\mu_{\iota+1},\mu_{\iota+2})\Big)\Big\},\tag{3.6}$$

and

$$\begin{aligned}\Delta_2(\mu_\iota,\mu_{\iota+1})&=\min\Big\{d_\theta(\mu_\iota,\mu_{\iota+1}),\frac{1}{2\theta(\mu_\iota,\Upsilon\mu_{\iota+1})}\Big(d_\theta(\mu_\iota,\Upsilon\mu_{\iota+1})+d_\theta(\mu_{\iota+1},\Upsilon\mu_\iota)\Big)\Big\},\\&=\min\Big\{d_\theta(\mu_\iota,\mu_{\iota+1}),\frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}\Big(d_\theta(\mu_\iota,\mu_{\iota+2})+d_\theta(\mu_{\iota+1},\mu_{\iota+1})\Big).\end{aligned}$$

Thus

$$\Delta_2(\mu_\iota,\mu_{\iota+1})=\min\Big\{d_\theta(\mu_\iota,\mu_{\iota+1}),\frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}d_\theta(\mu_\iota,\mu_{\iota+2})\Big\}.\tag{3.7}$$

Now, we assume that

$$d_\theta(\mu_\iota,\mu_{\iota+1})<d_\theta(\mu_{\iota+1},\mu_{\iota+2}),\tag{3.8}$$

By (**3**.4), we obtain

$$\begin{aligned}\xi_\theta(d_\theta(\mu_{\iota+1},\mu_{\iota+2})) &\leq \xi_\theta\left(\frac{1}{\alpha^5}d_\theta(\mu_{\iota+1},\mu_{\iota+2})\right) \leq \xi_\theta(\Delta_1(\mu_\iota,\mu_{\iota+1})) - \omega_\theta(\Delta_2(\mu_\iota,\mu_{\iota+1}))\\ &\leq \xi_\theta(\Delta_1(\mu_\iota,\mu_{\iota+1})).\end{aligned}$$

Since ξ_θ is strictly increasing, we get

$$d_\theta(\mu_{\iota+1},\mu_{\iota+2}) \leq \Delta_1(\mu_\iota,\mu_{\iota+1}). \tag{3.9}$$

Presume that there exists ι so that $d_\theta(\mu_{\iota+1},\mu_{\iota+2}) > d_\theta(\mu_\iota,\mu_{\iota+1})$. Then, by (**3**.8) and (**3**.6), we get

$$\Delta_1(\mu_\iota,\mu_{\iota+1}) \leq d_\theta(\mu_{\iota+1},\mu_{\iota+2}). \tag{3.10}$$

From (**3**.9) and (**3**.10), we conclude that

$$\Delta_1(\mu_\iota,\mu_{\iota+1}) = d_\theta(\mu_{\iota+1},\mu_{\iota+2}). \tag{3.11}$$

On the other hand, from (**3**.6), we get that $d_\theta(\mu_{\iota+1},\mu_{\iota+2}) \leq d_\theta(\mu_\iota,\mu_{\iota+1})$, so the assumption was false, and then

$$d_\theta(\mu_{\iota+1},\mu_{\iota+2}) \leq d_\theta(\mu_\iota,\mu_{\iota+1}), \text{ for all } \iota. \tag{3.12}$$

Based on (**3**.12), (**3**.5), and (**3**.7), we obtain

$$\Delta_1(\mu_\iota,\mu_{\iota+1}) = d_\theta(\mu_\iota,\mu_{\iota+1}), \tag{3.13}$$

and

$$\Delta_2(\mu_\iota,\mu_{\iota+1}) = \frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}d_\theta(\mu_\iota,\mu_{\iota+2}). \tag{3.14}$$

Substituting (**3**.13) and (**3**.14) in (**3**.4), we get

$$\begin{aligned}\xi_\theta(d_\theta(\mu_{\iota+1},\mu_{\iota+2})) &\leq \xi_\theta\left(\frac{1}{\alpha^5}d_\theta(\mu_{\iota+1},\mu_{\iota+2})\right)\\ &\leq \xi_\theta(d_\theta(\mu_\iota,\mu_{\iota+1})) - \omega_\theta\left(\frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}d_\theta(\mu_\iota,\mu_{\iota+2})\right).\end{aligned} \tag{3.15}$$

From (**3**.12), we get that the sequence decreases monotonically, thus there exists $r \geq 0$, such that $\lim_{\iota\to\infty} d_\theta(\mu_\iota,\mu_{\iota+1}) = r$.

Letting $\iota \to \infty$, we get

$$\lim_{\iota\to\infty}\omega_\theta\left(\frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}d_\theta(\mu_\iota,\mu_{\iota+2})\right) = 0, \tag{3.16}$$

and also $r = \frac{r}{\alpha^5}$, which leads to $r = 0$.

So,

$$\lim_{\iota\to\infty} d_\theta(\mu_\iota,\mu_{\iota+1}) = 0. \tag{3.17}$$

Step 2 (μ_ι) is a Cauchy sequence: Now, we shall demonstrate that (μ_ι) is a Cauchy sequence in (Π, d_θ). Presume the converse, that (μ_ι) is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can obtain two subsequences $(\mu_{\eta(\varsigma)})$ and $(\mu_{\iota(\varsigma)})$ such that $\eta(\varsigma)$ is the smallest index for which

$$\eta_{(\varsigma)} > \iota_{(\varsigma)} > \varsigma, \qquad d_\theta(\mu_{\eta(\varsigma)},\mu_{\iota(\varsigma)}) \geq \epsilon. \tag{3.18}$$

This means that

$$d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)}) < \epsilon. \tag{3.19}$$

Now, from (**3**.18) and by triangle inequality, we get

$$\epsilon \leq d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) \leq \theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)})[d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)-1}) + d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)})].$$

From (**3**.19), we find $\epsilon \le d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) < \epsilon\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) + \theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)})d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)})$. Letting $\varsigma \to \infty$, using (**3**.17) and recalling that θ is bounded by $\frac{1}{\alpha}$, we obtain that there exists r so that

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) = r\epsilon, \tag{3.20}$$

and $1 < r < \frac{1}{\alpha}$.

For any positive integer ς, there exists $\jmath(\varsigma) \in \{1,2,\dots,p\}$ such that $\eta(\varsigma)-\jmath(\varsigma)-\iota(\varsigma) \equiv 1[p]$. Then $\mu_{\eta(\varsigma)-\jmath(\varsigma)}$ (for ς large enough $\eta(\varsigma) > \jmath(\varsigma)$) and $\mu_{\iota(\varsigma)}$ lie in different adjacently labeled sets $\aleph_i$ and $\aleph_{i+1}$ for specific $i \in \{1,2,\dots,p\}$.

Due to the triangle inequality, we obtain

$$\begin{aligned}
d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)}) &\le \theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)})[d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-1}) + d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)-\jmath(\varsigma)})],\\
&\le \theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)})d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-1}) + \theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)})\\
&\quad \theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)-\jmath(\varsigma)})[d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)-2}) + d_\theta(\mu_{\eta(\varsigma)-2},\mu_{\eta(\varsigma)-\jmath(\varsigma)})],\\
&\quad \vdots\\
&\le \theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)})d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-1}) + \cdots +\\
&\quad \prod_{c=0}^{\jmath(\varsigma)} \theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\eta(\varsigma)-c})d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)-1},\mu_{\eta(\varsigma)-\jmath(\varsigma)}).
\end{aligned}$$

Since θ is bounded by $\frac{1}{\alpha}$, we obtain

$$\begin{aligned}
d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)}) &\le \frac{1}{\alpha}d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-1}) + \frac{1}{\alpha^2}d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)-2}) + \cdots +\\
&\quad + \frac{1}{\alpha^{\jmath(\varsigma)+1}}d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)-1},\mu_{\eta(\varsigma)-\jmath(\varsigma)}).
\end{aligned}$$

Letting $\varsigma \to \infty$ and using (**3**.17), we get

$$d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)}) = 0. \tag{3.21}$$

Again by assumption (**3**.18) and by triangular inequality, we obtain

$$\epsilon \le d_\theta(\mu_{\eta(\varsigma)},\mu_{\iota(\varsigma)}) \le \theta(\mu_{\eta(\varsigma)},\mu_{\iota(\varsigma)})[d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)}) + d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)})].$$

Since θ is bounded by $\frac{1}{\alpha}$, we get $\epsilon \le \frac{1}{\alpha}d_\theta(\mu_{\eta(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)}) + \frac{1}{\alpha}d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)})$. By (**3**.21) and letting $\varsigma \to \infty$, we find

$$\alpha\epsilon \le \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}). \tag{3.22}$$

Also, we have

$$d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) \le \theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)})[d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\eta(\varsigma)}) + d_\theta(\mu_{\eta(\varsigma)},\mu_{\iota(\varsigma)})].$$

By (**3**.20) and (**3**.21) and by letting $\varsigma \to \infty$, we find

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) \le \frac{r\epsilon}{\alpha}. \tag{3.23}$$

From (**3**.22) and (**3**.23), we conclude that

$$\alpha\epsilon \le \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) \le \frac{r\epsilon}{\alpha}. \tag{3.24}$$

Now, for $d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})$, we use the triangular inequality, and we obtain

$$d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1}) \le \theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})[d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)},\mu_{\iota(\varsigma)+1})].$$

Taking the upper limit as $\varsigma \to \infty$ and employing (**3**.17) and (**3**.23), we obtain

$$d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1}) \le \frac{r\epsilon}{\alpha^2}. \tag{3.25}$$

Conversely, by (**3**.18), the triangular inequality and since θ is bounded by $\frac{1}{\alpha}$, we obtain

$$\begin{aligned}\epsilon &\leq d_\theta(\mu_{\eta(\varsigma)}, \mu_{\iota(\varsigma)}) \leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)}, \mu_{\eta(\varsigma)-\jmath(\varsigma)}) + \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)}, \mu_{\iota(\varsigma)}) \\ &\leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)}, \mu_{\eta(\varsigma)-\jmath(\varsigma)}) + \frac{1}{\alpha^2} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)}, \mu_{\iota(\varsigma)+1}) + \frac{1}{\alpha^2} d_\theta(\mu_{\iota(\varsigma)+1}, \mu_{\iota(\varsigma)}).\end{aligned}$$

Taking the upper limit as $\varsigma \to \infty$, utilizing (**3**.17) and (**3**.21), we obtain

$$\epsilon\alpha^2 \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)}, \mu_{\iota(\varsigma)+1}). \tag{3.26}$$

From (**3**.25) and (**3**.26), we conclude that

$$\epsilon\alpha^2 \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)}, \mu_{\iota(\varsigma)+1}) \leq \frac{r\epsilon}{\alpha^2}. \tag{3.27}$$

Now, for $d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)})$, we use (**3**.18) and the triangle inequality. Since θ is bounded by $\frac{1}{\alpha}$, we obtain

$$\epsilon \leq d_\theta(\mu_{\eta(\varsigma)}, \mu_{\iota(\varsigma)}) \leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)}, \mu_{\eta(\varsigma)-\jmath(\varsigma)+1}) + \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)}).$$

Again by the triangle inequality and since θ is bounded by $\frac{1}{\alpha}$, we obtain

$$d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)}) \leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\eta(\varsigma)-\jmath(\varsigma)}) + \frac{1}{\alpha} d_\theta(\mu_{\iota(\varsigma)-\jmath(\varsigma)}, \mu_{\iota(\varsigma)}).$$

From (**3**.24) and (**3**.17), we obtain

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)}) \leq \frac{r\epsilon}{\alpha^2}. \tag{3.28}$$

From (**3**.28), we obtain

$$\epsilon\alpha \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)}) \leq \frac{r\epsilon}{\alpha^2}. \tag{3.29}$$

Now, for $d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)+1})$, we find

$$d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)+1}) \leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)}) + \frac{1}{\alpha} d_\theta(\mu_{\iota(\varsigma)}, \mu_{\iota(\varsigma)+1}).$$

Taking the upper limit as $\varsigma \to \infty$ and utilizing (**3**.17) and (**3**.29), we obtain

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)+1}) \leq \frac{r\epsilon}{\alpha^3}. \tag{3.30}$$

Also,

$$\begin{aligned}\epsilon &\leq d_\theta(\mu_{\eta(\varsigma)}, \mu_{\iota(\varsigma)}) \leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)}, \mu_{\eta(\varsigma)-\jmath(\varsigma)+1}) + \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)}) \\ &\leq \frac{1}{\alpha} d_\theta(\mu_{\eta(\varsigma)}, \mu_{\eta(\varsigma)-\jmath(\varsigma)+1}) + \frac{1}{\alpha^2} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)+1}) + \frac{1}{\alpha^2} d_\theta(\mu_{\iota(\varsigma)+1}, \mu_{\iota(\varsigma)}).\end{aligned}$$

Taking the upper limit as $\varsigma \to \infty$ and utilizing (**3**.17), we obtain

$$\epsilon\alpha^2 \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)+1}). \tag{3.31}$$

From (**3**.30) and (**3**.31), we obtain

$$\epsilon\alpha^2 \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1}, \mu_{\iota(\varsigma)+1}) \leq \frac{r\epsilon}{\alpha^3}. \tag{3.32}$$

Now, from the contraction condition, we find

$$\begin{aligned}\xi_\theta(d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1},\mu_{\iota(\varsigma)+1})) &\le \xi_\theta(\frac{1}{\alpha^5}d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)+1},\mu_{\iota(\varsigma)+1}))\\ &\le \xi_\theta(\Delta_1(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}))-\omega_\theta(\Delta_2(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}))\\ &\le \xi_\theta(\Delta_1(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)})), \qquad (\mathbf{3}.33)\end{aligned}$$

where

$$\begin{aligned}\Delta_1(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) = \max\Big\{&d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}),\frac{1}{2\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})}\\ &\Big(d_\theta(\mu_{\iota(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})+d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)+1})\Big),\\ &\frac{1}{2}\Big(d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)+1})+d_\theta(\mu_{\iota(\varsigma)},\mu_{\iota(\varsigma)+1})\Big)\Big\} \qquad (\mathbf{3}.34)\end{aligned}$$

and

$$\begin{aligned}\Delta_2(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) = \min\Big\{&d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}),\frac{1}{2\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})}\\ &\Big(d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})+d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)+1})\Big)\Big\}. \qquad (\mathbf{3}.35)\end{aligned}$$

Taking the upper limit as $\varsigma\to\infty$ in (**3**.34) and (**3**.35) as well using (**3**.17), (**3**.24), (**3**.27) and (**3**.29), we get

$$\begin{aligned}\alpha\epsilon &\le \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) \le \limsup_{\varsigma\to\infty}\triangle_1(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)})\\ &\le \max\Big\{\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}),\\ &\frac{1}{2}\Big(\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)+1})+\limsup_{\varsigma\to\infty} d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)-\jmath(\varsigma)+1})\Big),0\Big\}\\ &\le \max\Big\{\frac{r\epsilon}{\alpha},\frac{1}{2}(\frac{r\epsilon}{\alpha^2}+\frac{r\epsilon}{\alpha^2}),0\Big\}=\frac{r\epsilon}{\alpha^2}.\end{aligned}$$

Hence,

$$\alpha\epsilon \le \limsup_{\varsigma\to\infty}\triangle_1(\mu_{\eta(\varsigma)-\jmath(\varsigma)},\mu_{\iota(\varsigma)}) \le \frac{r\epsilon}{\alpha^2}. \qquad (\mathbf{3}.36)$$

Now, taking the upper limit as $\varsigma\to\infty$ in (**3**.33) and utilizing (**3**.32) and (**3**.36), we obtain

$$\xi(\frac{1}{\alpha^5}\alpha^2\epsilon)\le\xi(\frac{r\epsilon}{\alpha^2}). \qquad (\mathbf{3}.37)$$

We get a contradiction, (since we have $1<r<\frac{1}{\alpha}$. and $\frac{r}{\alpha^2}<\frac{1}{\alpha^3}$ and $\frac{r\epsilon}{\alpha^2}<\frac{\epsilon}{\alpha^3}$ where $\epsilon>0$). Thus (μ_ι) is a Cauchy sequence.

Step 3 Existence: By the completeness of (Π,d_θ), there exists some $\mu^\star\in\Pi$, such that

$$\lim_{\iota\to\infty}\mu_\iota=\mu^\star. \qquad (\mathbf{3}.38)$$

Now, from the first condition of cyclic representations in Definition **3**.1, and from the closedness of sets $\aleph_i$, from (**3**.38), we obtain $\mu^\star\in\aleph_3$. Continuing this process, we obtain

$$\mu^\star\in\bigcap_{i=1}^{p}\aleph_i. \qquad (\mathbf{3}.39)$$

Now, we show that $\mu^\star$ is fixed point for Υ.

From (**3**.39), we get that for all ι there exists $i(\iota) \in \{1, 2, \dots, p\}$ such that $\mu_\iota \in \aleph_{i(\iota)}$. Putting $\mu = \mu_\iota$ and $\nu = \mu^\star$ in (**3**.1), we obtain

$$\begin{aligned} \xi_\theta(d_\theta(\Upsilon\mu_\iota, \Upsilon\mu^\star)) &\leq \xi_\theta(\frac{1}{\alpha^5} d_\theta(\Upsilon\mu_\iota, \Upsilon\mu^\star)) = \xi_\theta(\frac{1}{\alpha^5} d_\theta(\mu_{\iota+1}, \Upsilon\mu^\star)) \\ &\leq \xi_\theta(\Delta_1(\mu_\iota, \mu^\star)) - \omega_\theta(\Delta_2(\mu_\iota, \mu^\star)) \leq \xi_\theta(\Delta_1(\mu_\iota, \mu^\star)), \end{aligned}$$

where

$$\Delta_1(\mu_\iota, \mu^\star) = \max\Big\{d_\theta(\mu_\iota, \mu^\star),$$

$$\frac{1}{2\theta(\mu_\iota, \Upsilon\mu^\star)}(d_\theta(\mu_\iota, \Upsilon\mu^\star) + d_\theta(\mu^\star, \Upsilon\mu_\iota)), \frac{1}{2}(d_\theta(\mu_\iota, \Upsilon\mu_\iota) + d_\theta(\mu^\star, \Upsilon\mu^\star))\Big\}.$$

Since $\theta(\mu_\iota, \Upsilon\mu^\star) \geq 1$, we get

$$\Delta_1(\mu_\iota, \mu^\star) \leq \max\Big\{d_\theta(\mu_\iota, \mu^\star), \frac{1}{2}(d_\theta(\mu_\iota, \Upsilon\mu^\star) + d_\theta(\mu^\star, \mu_{\iota+1})), \frac{1}{2}(d_\theta(\mu_\iota, \mu_{\iota+1}) + d_\theta(\mu^\star, \Upsilon\mu^\star))\Big\},$$

and

$$\Delta_2(\mu_\iota, \mu^\star) = \min\Big\{d_\theta(\mu_\iota, \mu^\star), \frac{1}{2\theta(\mu_\iota, \Upsilon\mu^\star)}(d_\theta(\mu_\iota, \Upsilon\mu^\star) + d_\theta(\mu^\star, \Upsilon\mu_\iota))\Big\}.$$

Letting $\iota \to \infty$,

$$\xi_\theta(d_\theta(\mu^\star, \Upsilon\mu^\star)) \leq \xi_\theta(\frac{1}{2} d_\theta(\mu^\star, \Upsilon\mu^\star)) - \omega_\theta(0) \leq \xi_\theta(\frac{1}{2} d_\theta(\mu^\star, \Upsilon\mu^\star)).$$

Since ξ_θ is monotonically increasing, we get $d_\theta(\mu^\star, \Upsilon\mu^\star) \leq \frac{1}{2} d_\theta(\mu^\star, \Upsilon\mu^\star)$, which implies $d_\theta(\mu^\star, \Upsilon\mu^\star) = 0$, thus $\Upsilon\mu^\star = \mu^\star$.

Step 4 Uniqueness : Related to the uniqueness, suppose that there exists another fixed point $\nu^\star$ for Υ. Then by the cyclic representation condition in Definition **3**.1, we obtain that $\nu^\star \in \bigcap_{i=1}^{p} \aleph_i$.

As well, putting $\mu = \mu^\star$ and $\nu = \nu^\star$ in (**3**.1), we get

$$\xi_\theta(d_\theta(\Upsilon\mu^\star, \Upsilon\nu^\star)) = \xi_\theta(d_\theta(\mu^\star, \nu^\star)) \leq \xi_\theta\Big(\frac{1}{\alpha^5} d_\theta(\mu^\star, \nu^\star)\Big) \leq \xi_\theta(\Delta_1(\mu^\star, \nu^\star)) - \omega_\theta(\Delta_2(\mu^\star, \nu^\star)),$$

where

$$\Delta_1(\mu^\star, \nu^\star) = \max\Big\{d_\theta(\mu^\star, \nu^\star),$$

$$\frac{1}{2\theta(\mu^\star, \Upsilon\nu^\star)}(d_\theta(\mu^\star, \Upsilon\nu^\star) + d_\theta(\nu^\star, \Upsilon\mu^\star)), \frac{1}{2}(d_\theta(\mu^\star, \Upsilon\mu^\star) + d_\theta(\nu^\star, \Upsilon\nu^\star))\Big\},$$

and

$$\Delta_2(\mu^\star, \nu^\star) = \min\Big\{d_\theta(\mu^\star, \nu^\star), \frac{1}{2\theta(\mu^\star, \Upsilon\nu^\star)}(d_\theta(\mu^\star, \Upsilon\nu^\star) + d_\theta(\nu^\star, \Upsilon\mu^\star))\Big\}.$$

Since $\Upsilon\mu^\star = \mu^\star$, $\Upsilon\nu^\star = \nu^\star$, and $\theta(\mu^\star, \Upsilon\nu^\star) \geq 1$, we get

$$\Delta_1(\mu^\star, \nu^\star) = d_\theta(\mu^\star, \nu^\star); \quad \Delta_2(\mu^\star, \nu^\star) = \frac{1}{\theta(\mu^\star, \Upsilon\nu^\star)} d_\theta(\mu^\star, \nu^\star),$$

and

$$\xi_\theta(d_\theta(\mu^\star, \nu^\star)) \leq \xi_\theta(d_\theta(\mu^\star, \nu^\star)) - \omega_\theta\left(\frac{1}{\theta(\mu^\star, \Upsilon\nu^\star)} d_\theta(\mu^\star, \nu^\star)\right),$$

which implies that $\omega_\theta(\frac{1}{\theta(\mu^\star, \Upsilon\nu^\star)} d_\theta(\mu^\star, \nu^\star)) = 0$, then $d_\theta(\mu^\star, \nu^\star) = 0$. As a result, $\mu^\star = \nu^\star$. □

Corollary 3.1. *Consider the complete b-metric space* (Π, d_s). *Let* $\Pi = \bigcup_{i=1}^{p} \aleph_i$, *where* $\aleph_1, \aleph_2, \ldots, \aleph_p$ *are non-empty and closed subsets of* Π, *for* $p \in \mathbb{N}$. *Assume that*

1) $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$ *is a cyclic representation of* $\mathcal{Z}$ *related to* Υ.

2) For any $(\mu, \nu) \in \aleph_i \times \aleph_{i+1}$, *where* $i = 1, 2, \ldots, p$ *and* $\aleph_{p+1} = \aleph_1$,

$$\xi_s(s^5 d_s(\Upsilon\mu, \Upsilon\nu)) \leq \xi_s(\Delta_1(\mu, \nu)) - \omega_s(\Delta_2(\mu, \nu)),$$

where, $\Delta_1(\mu, \nu) = \max\Big\{d_s(\mu, \nu), \frac{1}{2s}\Big(d_s(\mu, \Upsilon\nu) + d_s(\nu, \Upsilon\mu)\Big), \frac{1}{2}\Big(d_s(\mu, \Upsilon\mu) + d_s(\nu, \Upsilon\nu)\Big)\Big\}$, $\Delta_2(\mu, \nu) = \min\Big\{d_s(\mu, \nu), \frac{1}{2s}\Big(d_s(\mu, \Upsilon\nu) + d_s(\nu, \Upsilon\mu)\Big)\Big\}$, $\xi_s \colon \mathbb{R}^+ \to \mathbb{R}^+$ *is an altering distance function,* $\omega_s : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and* $\omega_s(\tau) = 0$ *if and only if* $\tau = 0$.

Then the mapping Υ *possesses a unique fixed point in* $\bigcap_{i=1}^{p} \aleph_i$.

Proof. We achieve the result from Theorem **3**.1 by putting $\theta = s$, $s \geq 1$. □

Corollary 3.2. *Consider the complete extended b-metric space* (Π, d_s). *Let* $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$, *where* $\aleph_1, \aleph_2, \ldots, \aleph_p$ *are non-empty closed subsets of* Π, *for* $p \in \mathbb{N}$. *Assume that 1)* $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$ *is a cyclic representation of* $\mathcal{Z}$ *related to* Υ.

2) For any $(\mu, \nu) \in \aleph_i \times \aleph_{i+1}$, *where* $i = 1, 2, \ldots, p$ *and* $\aleph_{p+1} = \aleph_1$,

$$\frac{1}{\alpha^5} d_\theta(\Upsilon\mu, \Upsilon\nu) \leq \Delta_1(\mu, \nu) - \Delta_2(\mu, \nu),$$

where Δ_1 *and* Δ_2 *are defined in Definition* **3**.1, θ *is bounded by* $\frac{1}{\alpha}$ *and* $\alpha \in (0, 1)$.

Then the mapping Υ *possesses a unique fixed point in* $\bigcap_{i=1}^{p} \aleph_i$.

Proof. We achieve the desired result from Theorem **3**.1 by setting $\xi_\theta = \omega_\theta = I$, identity mapping. □

Example 3.1. Let $\Pi = \big[0, \frac{\pi}{4}\big]$ endowed with the metric $d_\theta \colon \Pi \times \Pi \to \mathbb{R}^+$, $d_\theta(\mu, \nu) = (\mu - \nu)^2$ and let $\theta \colon \Pi \times \Pi \to [1, \infty)$, $\theta(\mu, \nu) = \mu + \nu + 2$. Let $\aleph_1 = \big[0, \frac{\pi}{24}\big]$, $\aleph_2 = [0, \frac{\pi}{16}]$, $\aleph_3 = [0, \frac{\pi}{12}]$, $\aleph_4 = [0, \frac{5\pi}{48}]$, $\aleph_5 = [0, \frac{\pi}{8}]$, $\aleph_6 = [0, \frac{\pi}{4}]$. Define $\Upsilon \colon \mathcal{Z} \to \mathcal{Z}$, $\Upsilon(\mu) = \log(\frac{\mu^4}{12} + 1)$, Then:

(1) (Π, d_θ) is a complete extended b-metric space.

(2) $\mathcal{Z} = \bigcup_{i=1}^{6} \aleph_i$ is a cyclic representation of $\mathcal{Z}$ regarding to Υ, since
- $\aleph_i, i = \{1, \ldots, 6\}$ are non empty closed sets.
- $\Upsilon(\aleph_1) \subset \aleph_2, \Upsilon(\aleph_2) \subset \aleph_3, \Upsilon(\aleph_3) \subset \aleph_4, \Upsilon(\aleph_4) \subset \aleph_5, \Upsilon(\aleph_5) \subset \aleph_6, \Upsilon(\aleph_6) \subset \aleph_1$.

(3) We have $\theta(\mu, \nu)$ bounded by $\frac{1}{\alpha}$, where $\alpha = \frac{2}{4+\pi}$, $\alpha \in (0, 1)$,

$$0 \leq \mu \leq \frac{\pi}{4} \text{ and } \quad 0 \leq \nu \leq \frac{\pi}{4} \Rightarrow 2 \leq \mu + \nu + 2 \leq \frac{\pi + 4}{2} \Rightarrow \theta(\mu, \nu) \leq \frac{1}{\frac{2}{\pi+4}}.$$

(4) Υ satisfy the inequality of Theorem **3**.1 by taking $\nu = \frac{\mu}{n}$, for $n \in [4, +\infty[$ and some $\mu \in [0, \frac{\pi}{4}]$, because

$$d_\theta(\Upsilon\mu, \Upsilon\nu) = \Big(\log\Big(\frac{\mu^4}{12} + 1\Big) - \log\Big(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4 + 1\Big)\Big)^2; \quad d_\theta(\mu, \nu) = \Big(\mu - \frac{\mu}{n}\Big)^2,$$

$$\frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu,\Upsilon\nu)+d_\theta(\nu,\Upsilon\mu)\Big)$$
$$=\frac{1}{2(\mu+\log(\frac{1}{12}\left(\frac{\mu}{n}\right)^4+1)+2)}\Big(\Big(\mu-\log(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4+1)\Big)^2+\Big(\frac{\mu}{n}-\log(\frac{\mu^4}{12}+1)\Big)^2\Big),$$

and

$$\frac{1}{2}\Big(d_\theta(\mu,\Upsilon\mu)+d_\theta(\nu,\Upsilon\nu)\Big)=\frac{1}{2}\Big(\Big(\mu-\log(\frac{\mu^4}{12}+1)\Big)^2+\Big(\frac{\mu}{n}-\log(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4+1)\Big)^2\Big).$$

Now, from Figure 1 and Figure 2, we get

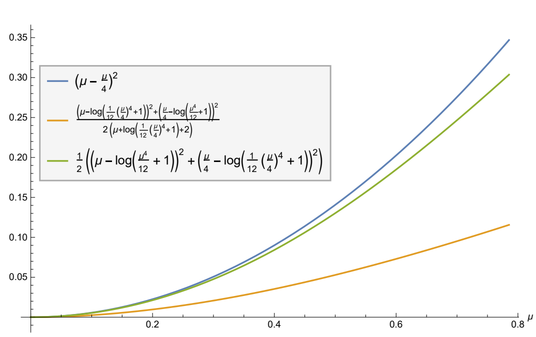

FIGURE 1. Comparison between functions when $n \lll$

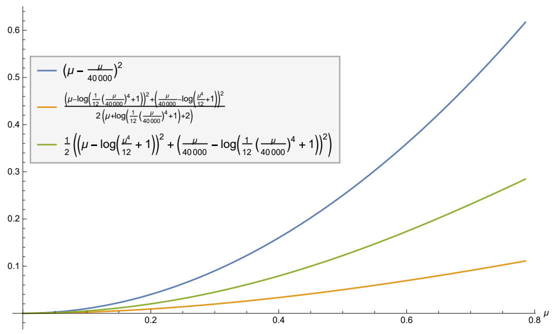

FIGURE 2. Comparison between functions when $n \ggg$.

$$\begin{aligned}\Delta_1(\mu,\nu) &= d_\theta(\mu,\nu)=\Big(\mu-\frac{\mu}{n}\Big)^2,\\ \Delta_2(\mu,\nu) &= \frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu,\Upsilon\nu)+d_\theta(\nu,\Upsilon\mu)\Big),\\ &= \frac{1}{2(\mu+\log(\frac{1}{12}\left(\frac{\mu}{n}\right)^4+1)+2)}\Big(\Big(\mu-\log(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4+1)\Big)^2+\Big(\frac{\mu}{n}-\log(\frac{\mu^4}{12}+1)\Big)^2\Big).\end{aligned}$$

Let $\xi_\theta(\tau)=6\Big(\frac{2}{\pi+4}\Big)^5\tau^{\frac{5}{4}}$ and $\omega_\theta(\tau)=\tau^4$ with $\omega_\theta(0)=0$. Then from Figure 3 and Figure 4, we get

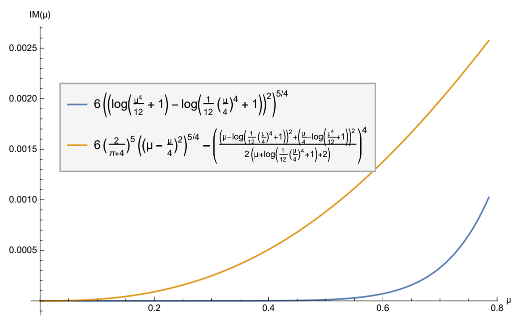

FIGURE 3. Comparison between functions when $n \lll$

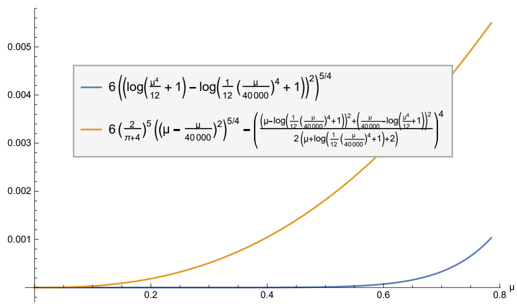

FIGURE 4. Comparison between functions when $n \ggg$.

$$\begin{aligned}\xi_\theta\Big((\frac{1}{\alpha})^5 d_\theta(\Upsilon\mu,\Upsilon\nu)\Big) &= 6\Big(\frac{2}{\pi+4}\Big)^5\Big(\frac{4+\pi}{2}\Big)^5\Big((\log(\frac{\mu^4}{12}+1)-\log(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4+1))^2\Big)^{\frac{5}{4}},\\ &\le 6\Big(\frac{2}{\pi+4}\Big)^5\Big((\mu-\frac{\mu}{n})^2\Big)^{\frac{5}{4}}-\Big(\frac{1}{2(\mu+\log(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4+1)+2)}\\ &\quad \Big(\Big(\mu-\log(\frac{1}{12}\Big(\frac{\mu}{n}\Big)^4+1)\Big)^2+\Big(\frac{\mu}{n}-\log(\frac{\mu^4}{12}+1)\Big)^2\Big)\Big)^4.\\ &= \xi_\theta(\Delta_1(\mu,\nu))-\omega_\theta(\Delta_2(\mu,\nu)).\end{aligned}$$

The example satisfies all the hypotheses of Theorem **3**.1. Hence, 0 is the unique fixed point of Υ in $\bigcap_{i=1}^{p}\aleph_i$.

Definition 3.2. *Let (Π, d_θ) be an extended b-metric space. Let $\mathcal{Z}=\bigcup_{i=1}^{p}\aleph_i$, where $\aleph_1,\aleph_2,\ldots,\aleph_p$ are non-void and closed subsets of Π, for $p\in\mathbb{N}$. A self mapping $\Upsilon\colon\mathcal{Z}\to\mathcal{Z}$ is called an extended $(\xi_\theta,\omega_\theta)$-weakly cyclic generalized contraction condition two, if the following conditions hold:*

1) $\mathcal{Z}=\bigcup_{i=1}^{p}\aleph_i$ is a cyclic representation of $\mathcal{Z}$ related to Υ.

2) For any $(\mu,\nu)\in\aleph_i\times\aleph_{i+1}$, where $i=1,2,\ldots,p$ and $\aleph_{p+1}=\aleph_1$,

$$\xi_\theta(d_\theta(\Upsilon\mu,\Upsilon\nu))\le\xi_\theta(\Delta_1(\mu,\nu))-\omega_\theta(\Delta_2(\mu,\nu)), \qquad (\mathbf{3}.40)$$

where

$$\Delta_1(\mu,\nu)=\max\Big\{d_\theta(\mu,\nu),\frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu,\Upsilon\nu)+d_\theta(\nu,\Upsilon\mu)\Big),\frac{1}{2}\Big(d_\theta(\mu,\Upsilon\mu)+d_\theta(\nu,\Upsilon\nu)\Big)\Big\}, \qquad (\mathbf{3}.41)$$

$$\Delta_2(\mu,\nu)=\min\Big\{d_\theta(\mu,\nu),\frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu,\Upsilon\nu)+d_\theta(\nu,\Upsilon\mu)\Big)\Big\}. \qquad (\mathbf{3}.42)$$

As well, $\xi_\theta\colon\mathbb{R}^+\to\mathbb{R}^+$ is an altering distance function, $\omega_\theta\colon\mathbb{R}^+\to\mathbb{R}^+$ is continuous and $\omega_\theta(\tau)=0$ if and only if $\tau=0$. Assume for $\mu_0\in\Pi$, where $\mu_k=\Upsilon^k\mu_0$, and $\iota,\eta,k\in\mathbb{N}\cup\{0\}$, we have

$$\lim_{\iota,\eta\to\infty}\theta(\mu_\iota,\mu_\eta)=1. \qquad (\mathbf{3}.43)$$

Theorem 3.2. *Consider an extended b-metric space (Π,d_θ), where Π is a non empty set. Assume that Π is complete and $p\in\mathbb{N}$ such that $\aleph_1,\aleph_2,\ldots,\aleph_p$ are non-void subset of Π and $\mathcal{Z}=\bigcup_{i=1}^{p}\aleph_i$. If the self-mapping $\Upsilon\colon\mathcal{Z}\to\mathcal{Z}$ is extended $(\xi_\theta,\omega_\theta)$-weakly cyclic generalized contraction condition two, then the mapping Υ possesses a unique fixed point in $\bigcap_{i=1}^{p}\aleph_i$.*

Proof. Let $\mu_0\in\aleph_1$ (where $\aleph_i$ is non-empty for all i). Consider the sequence (μ_ι) in Π given by, $\mu_{\iota+1}=\Upsilon\mu_\iota$, for all $\iota\in\mathbb{N}\cup\{0\}$.

- If $\mu_{\iota+1}=\mu_\iota$ then μ_ι is a fixed point of Υ.
- If $\mu_{\iota+1}\neq\mu_\iota$, then we shall prove.

Step 1 $\lim_{\iota\to\infty}d_\theta(\mu_\iota,\mu_{\iota+1})=0$**:** We have for all ι, $d_\theta(\mu_\iota,\mu_{\iota+1})>0$. From the first condition in Definition **3**.2, we obtain $i=i(\iota)$, $i=\{1,2,\ldots,p\}$ for all ι, $(\mu_\iota,\mu_{\iota+1})\in\aleph_i\times\aleph_{i+1}$.

In the second condition of Definition **3**.2, putting $\mu = \mu_\iota$ and $\nu = \mu_{\iota+1}$, we get

$$\begin{aligned} \xi_\theta(d_\theta(\Upsilon\mu_\iota, \Upsilon\mu_{\iota+1})) &= \xi_\theta(d_\theta(\mu_{\iota+1}, \mu_{\iota+2})) \\ &\leq \xi_\theta(\Delta_1(\mu_\iota, \mu_{\iota+1})) - \omega_\theta(\Delta_2(\mu_\iota, \mu_{\iota+1})), \end{aligned} \tag{3.44}$$

where

$$\begin{aligned} \Delta_1(\mu_\iota, \mu_{\iota+1}) &= \max\Big\{ d_\theta(\mu_\iota, \mu_{\iota+1}), \frac{1}{2\theta(\mu_\iota, \Upsilon\mu_{\iota+1})}\Big(d_\theta(\mu_\iota, \Upsilon\mu_{\iota+1}) + d_\theta(\mu_{\iota+1}, \Upsilon\mu_\iota)\Big), \\ &\quad \frac{1}{2}\Big(d_\theta(\mu_\iota, \Upsilon\mu_\iota) + d_\theta(\mu_{\iota+1}, \Upsilon\mu_{\iota+1})\Big)\Big\} \\ &= \max\Big\{ d_\theta(\mu_\iota, \mu_{\iota+1}), \frac{1}{2\theta(\mu_\iota, \mu_{\iota+2})}\Big(d_\theta(\mu_\iota, \mu_{\iota+2}) + d_\theta(\mu_{\iota+1}, \mu_{\iota+1})\Big), \\ &\quad \frac{1}{2}\Big(d_\theta(\mu_\iota, \mu_{\iota+1}) + d_\theta(\mu_{\iota+1}, \mu_{\iota+2})\Big)\Big\}. \end{aligned} \tag{3.45}$$

By the triangle inequality, we obtain

$$\Delta_1(\mu_\iota, \mu_{\iota+1}) \leq \max\Big\{ d_\theta(\mu_\iota, \mu_{\iota+1}), \frac{1}{2}\Big(d_\theta(\mu_\iota, \mu_{\iota+1}) + d_\theta(\mu_{\iota+1}, \mu_{\iota+2})\Big)\Big\}. \tag{3.46}$$

Since

$$\begin{aligned} \Delta_2(\mu, \nu) &= \min\Big\{ d_\theta(\mu_\iota, \mu_{\iota+1}), \frac{1}{2\theta(\mu_\iota, \Upsilon\mu_{\iota+1})}\Big(d_\theta(\mu_\iota, \Upsilon\mu_{\iota+1}) + d_\theta(\mu_{\iota+1}, \Upsilon\mu_\iota)\Big)\Big\} \\ &= \min\Big\{ d_\theta(\mu_\iota, \mu_{\iota+1}), \frac{1}{2\theta(\mu_\iota, \mu_{\iota+2})}\Big(d_\theta(\mu_\iota, \mu_{\iota+2}) + d_\theta(\mu_{\iota+1}, \mu_{\iota+1})\Big), \end{aligned}$$

thus

$$\Delta_2(\mu_\iota, \mu_{\iota+1}) = \min\Big\{ d_\theta(\mu_\iota, \mu_{\iota+1}), \frac{1}{2\theta(\mu_\iota, \mu_{\iota+2})} d_\theta(\mu_\iota, \mu_{\iota+2})\Big\}. \tag{3.47}$$

Now, we assume that

$$d_\theta(\mu_\iota, \mu_{\iota+1}) < d_\theta(\mu_{\iota+1}, \mu_{\iota+2}), \tag{3.48}$$

By (**3**.44), we obtain

$$\begin{aligned} \xi_\theta(d_\theta(\mu_{\iota+1}, \mu_{\iota+2})) &\leq \xi_\theta(\Delta_1(\mu_\iota, \mu_{\iota+1})) - \omega_\theta(\Delta_2(\mu_\iota, \mu_{\iota+1})) \\ &\leq \xi_\theta(\Delta_1(\mu_\iota, \mu_{\iota+1})). \end{aligned}$$

Since ξ_θ is strictly increasing, we get

$$d_\theta(\mu_{\iota+1}, \mu_{\iota+2}) \leq \Delta_1(\mu_\iota, \mu_{\iota+1}). \tag{3.49}$$

Another hand, by (**3**.46) and (**3**.48), we get

$$\Delta_1(\mu_\iota, \mu_{\iota+1}) \leq d_\theta(\mu_{\iota+1}, \mu_{\iota+2}). \tag{3.50}$$

From (**3**.50) and (**3**.49), we conclude that

$$\Delta_1(\mu_\iota, \mu_{\iota+1}) = d_\theta(\mu_{\iota+1}, \mu_{\iota+2}). \tag{3.51}$$

Another side, (**3**.48) gives $d_\theta(\mu_\iota, \mu_{\iota+2}) > 0$,(since if $d_\theta(\mu_\iota, \mu_{\iota+2}) = 0$, we get $\mu_\iota = \mu_{\iota+2}$, that contradicts with our assumption), and since $\mu_\iota \neq \mu_{\iota+1}$, we obtain

$$\Delta_2(\mu_\iota, \mu_{\iota+1}) > 0. \tag{3.52}$$

Substituting (**3**.51) and (**3**.52) in (**3**.44), we get

$$\xi_\theta(d_\theta(\mu_{\iota+1}, \mu_{\iota+2})) \leq \xi_\theta(d_\theta(\mu_{\iota+1}, \mu_{\iota+2})) - \omega_\theta(\Delta_2(\mu_\iota, \mu_{\iota+1})) < \xi_\theta(d_\theta(\mu_{\iota+1}, \mu_{\iota+2})),$$

and this is a contradiction, so our assumption is not true. Then, for all $\iota \in \mathbb{N} \cup 0$, we have

$$d_\theta(\mu_{\iota+1}, \mu_{\iota+2}) \leq d_\theta(\mu_\iota, \mu_{\iota+1}). \tag{3.53}$$

From this result, ($\mathbf{3}$.45) and ($\mathbf{3}$.47), we obtain

$$\Delta_1(\mu_\iota,\mu_{\iota+1}) = d_\theta(\mu_\iota,\mu_{\iota+1}). \quad (\mathbf{3}.54)$$

$$\Delta_2(\mu_\iota,\mu_{\iota+1}) = \frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}d_\theta(\mu_\iota,\mu_{\iota+2}). \quad (\mathbf{3}.55)$$

Substituting ($\mathbf{3}$.54) and ($\mathbf{3}$.55) in ($\mathbf{3}$.44), we get

$$\xi_\theta(d_\theta(\Upsilon\mu_{\iota+1},\Upsilon\mu_{\iota+2})) \le \xi_\theta(d_\theta(\mu_\iota,\mu_{\iota+1})) - \omega_\theta\left(\frac{1}{2\theta(\mu_\iota,\mu_{\iota+2})}d_\theta(\mu_\iota,\mu_{\iota+2})\right). \quad (\mathbf{3}.56)$$

From ($\mathbf{3}$.53), we get that the sequence decreases monotonically, thus there exists $r \ge 0$, such that $\lim_{\iota\to\infty} d_\theta(\mu_\iota,\mu_{\iota+1}) = r$.

Now, from the relation

$$d_\theta(\mu_\iota,\mu_{\iota+2}) \le \theta(\mu_\iota,\mu_{\iota+2})\Big(d_\theta(\mu_\iota,\mu_{\iota+1}) + d_\theta(\mu_{\iota+1},\mu_{\iota+2})\Big),$$

we get $\limsup_{\iota\to\infty} d_\theta(\mu_\iota,\mu_{\iota+2}) = 2r\theta(\mu_\iota,\mu_{\iota+2})$.

Using the continuity of ξ_θ and ω_θ, we obtain $\xi_\theta(r) \le \xi_\theta(r) - \omega_\theta(r)$. Thus, $\omega_\theta(r) = 0$ and by definition of ω_θ, we get $r = 0$. As a result

$$\lim_{n\to\infty} d_\theta(\mu_\iota,\mu_{\iota+1}) = 0. \quad (\mathbf{3}.57)$$

Step 2 (μ_ι) **is a Cauchy sequence :** Now, we shall prove that (μ_ι) is a Cauchy sequence in (Π, d_θ). Presume the converse, that is (μ_ι) is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can obtain two sub-sequences $(\mu_{\eta(\varsigma)})$ and $(\mu_{\iota(\varsigma)})$ such that $\eta(\varsigma)$ is the smallest index for which

$$\eta_{(\varsigma)} > \iota_{(\varsigma)} > \varsigma, \qquad d_\theta(\mu_{\eta(\varsigma)},\mu_{\iota(\varsigma)}) \ge \epsilon. \quad (\mathbf{3}.58)$$

This means that

$$d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)}) < \epsilon. \quad (\mathbf{3}.59)$$

Now, from ($\mathbf{3}$.58) and by triangular inequality, we get

$$\epsilon \le d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) \le \theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)})[d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)-1}) + d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)})].$$

From ($\mathbf{3}$.59), we find $\epsilon \le d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) < \epsilon\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) + \theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)})d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)})$.

Letting $\varsigma\to\infty$, using ($\mathbf{3}$.57) and recalling that $\lim_{\varsigma\to\infty}\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) = 1$, we obtain

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\iota(\varsigma)},\mu_{\eta(\varsigma)}) = \epsilon. \quad (\mathbf{3}.60)$$

Putting $\mu = \mu_{\eta(\varsigma)-1}$ and $\nu = \mu_{\iota(\varsigma)-1}$ in ($\mathbf{3}$.40)

$$\begin{aligned}\xi_\theta(d_\theta(\Upsilon\mu_{\eta(\varsigma)-1},\Upsilon\mu_{\iota(\varsigma)-1})) &= \xi_\theta(d_\theta(\mu_{\eta(\varsigma)},\mu_{\iota(\varsigma)}))\\ &\le \xi_\theta(\Delta_1(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1})) - \omega(\Delta_2(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1}))\\ &\le \xi_\theta(\Delta_1(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1})), \quad (\mathbf{3}.61)\end{aligned}$$

where

$$\begin{aligned}\Delta_1(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1}) = \max\Big\{& d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1}), \frac{1}{2\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)})}\Big(d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)})\\ &+ d_\theta(\mu_{\iota(\varsigma)-1},\mu_{\eta(\varsigma)})\Big), \frac{1}{2}\Big(d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\eta(\varsigma)}), d_\theta(\mu_{\iota(\varsigma)-1},\mu_{\iota(\varsigma)})\Big)\Big\},\end{aligned} \quad (\mathbf{3}.62)$$

$$\begin{aligned}\Delta_2(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1}) = \min\Big\{& d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)-1}), \frac{1}{2\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)})}\\ &\Big(d_\theta(\mu_{\eta(\varsigma)-1},\mu_{\iota(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)-1},\mu_{\eta(\varsigma)})\Big)\Big\}.\end{aligned} \quad (\mathbf{3}.63)$$

Now, give the estimations of $d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1})$, $d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)})$ and $d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)})$. By applying triangle inequality in (**3**.58) using (**3**.43) and letting $\varsigma \to \infty$, we find

$$\epsilon \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)}). \tag{3.64}$$

Following the same method,we get

$$\epsilon \leq \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)+1}). \tag{3.65}$$

Now, by the triangle inequality, we have

$$\begin{aligned} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) &\leq \theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1})[d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\eta(\varsigma)}) + d_\theta(\mu_{\eta(\varsigma)}, \mu_{\iota(\varsigma)-1})], \\ &\leq \theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1})d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\eta(\varsigma)}) + \theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) \\ &\quad \theta(\mu_{\eta(\varsigma)}, \mu_{\iota(\varsigma)-1})[d_\theta(\mu_{\eta(\varsigma)}, \mu_{\iota(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)}, \mu_{\iota(\varsigma)-1})]. \end{aligned}$$

Letting $\varsigma \to \infty$ and using (**3**.57), (**3**.60) and (**3**.43), we get

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) \leq \epsilon. \tag{3.66}$$

Therefore from (**3**.64) and (**3**.66) we obtain

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) = \epsilon. \tag{3.67}$$

Also, $d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)}) \leq \theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)})[d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\iota(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)}, \mu_{\eta(\varsigma)})]$. Letting $\varsigma \to \infty$ and using (**3**.57), (**3**.60) and (**3**.43), we get

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)}) \leq \epsilon \tag{3.68}$$

Therefore from (**3**.65) and (**3**.68) we obtain

$$\limsup_{\varsigma\to\infty} d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)}) = \epsilon \tag{3.69}$$

Taking the upper limit as $\varsigma \to \infty$ in (**3**.62) and (**3**.63) and using (**3**.60), (**3**.67) and (**3**.69)

$$\begin{aligned} \limsup_{\varsigma\to\infty} \Delta_1(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) &= \max\Big\{ \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}), \limsup_{\varsigma\to\infty} \Big(\frac{1}{2\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)})} \\ &\Big(d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)})\Big)\Big), \limsup_{\varsigma\to\infty} \frac{1}{2}\Big(d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\eta(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\iota(\varsigma)})\Big)\Big\} \\ &= \max\{\epsilon, \frac{1}{2}(\epsilon + \epsilon), 0\}. \end{aligned}$$

Hence

$$\limsup_{\varsigma\to\infty} \Delta_1(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) = \epsilon. \tag{3.70}$$

$$\begin{aligned} \limsup_{\varsigma\to\infty} \Delta_2(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) &= \min\Big\{ \limsup_{\varsigma\to\infty} d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}), \\ \limsup_{\varsigma\to\infty} \Big(\frac{1}{2\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)})}&\Big(d_\theta(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)}) + d_\theta(\mu_{\iota(\varsigma)-1}, \mu_{\eta(\varsigma)})\Big)\Big)\Big\} = \min\Big\{\epsilon, \frac{1}{2}(\epsilon+\epsilon)\Big\}. \end{aligned}$$

and

$$\limsup_{\varsigma\to\infty} \Delta_2(\mu_{\eta(\varsigma)-1}, \mu_{\iota(\varsigma)-1}) = \epsilon. \tag{3.71}$$

Now, taking the upper limit as $\varsigma \to \infty$ in (**3**.61) using (**3**.60), (**3**.70) and (**3**.71), we get

$$\xi_\theta(\epsilon) \leq \xi_\theta(\epsilon) - \omega_\theta(\epsilon) < \xi_\theta(\epsilon),$$

which is a contradiction. Thus (μ_ι) is a Cauchy sequence.

Step 3 Existence: By completeness of (Π, d_θ), there exists some $\mu^\star \in \Pi$, such that

$$\lim_{\iota\to\infty} \mu_\iota = \mu^\star. \tag{3.72}$$

Now, from Definition (**3**.2) and as $\mu_0 \in \aleph_1$, we have $(\mu_{\iota p})_{\iota\geq 0} \subseteq \aleph_1$, since $\aleph_1$ is closed and by (**3**.72) we get $\mu^\star \in \aleph_2$. Also, from Definition **3**.2, we have $(\mu_{\iota p+1})_{\iota\geq 0} \subseteq \aleph_2$. Since $\aleph_2$ is closed, from (**3**.72), we obtain $\mu^\star \in \aleph_3$. Continuing this process, we obtain

$$\mu^\star \in \bigcap_{i=1}^{p} \aleph_i. \tag{3.73}$$

Now, we demonstrate that $\mu^\star$ is fixed point for $\varUpsilon$. From (**3**.73), we get that for all ι there exists $i(\iota) \in \{1, 2, \ldots, p\}$ such that $\mu_\iota \in \aleph_{i(\iota)}$.

Putting $\mu = \mu_\iota$ and $\nu = \mu^\star$ in Definition **3**.2, we obtain

$$\xi_\theta(d_\theta(\varUpsilon\mu_\iota, \varUpsilon\mu^\star)) = \xi_\theta(d_\theta(\mu_{\iota+1}, \varUpsilon\mu^\star)) \leq \xi_\theta(\Delta_1(\mu_\iota, \mu^\star)) - \omega_\theta(\Delta_2(\mu_\iota, \mu^\star)) \leq \xi_\theta(\Delta_1(\mu_\iota, \mu^\star)),$$

where

$$\begin{aligned}\Delta_1(\mu_\iota, \mu^\star) &= \max\Big\{d_\theta(\mu_\iota, \mu^\star), \frac{1}{2\theta(\mu_\iota, \varUpsilon\mu^\star)}(d_\theta(\mu_\iota, \varUpsilon\mu^\star) + d_\theta(\mu^\star, \varUpsilon\mu_\iota)),\\ &\quad \frac{1}{2}(d_\theta(\mu_\iota, \varUpsilon\mu_\iota) + d_\theta(\mu^\star, \varUpsilon\mu^\star))\Big\},\\ \Delta_2(\mu_\iota, \mu^\star) &= \min\Big\{d_\theta(\mu_\iota, \mu^\star), \frac{1}{2\theta(\mu_\iota, \varUpsilon\mu^\star)}(d_\theta(\mu_\iota, \varUpsilon\mu^\star) + d_\theta(\mu^\star, \varUpsilon\mu_\iota))\Big\}.\end{aligned}$$

Letting $\iota \to \infty$, we find $\xi_\theta(d_\theta(\mu^\star, \varUpsilon\mu^\star)) \leq \xi_\theta(\frac{1}{2}d_\theta(\mu^\star, \varUpsilon\mu^\star)) - \omega_\theta(0)$. Since, ξ_θ is monotonically increasing, we get $d_\theta(\mu^\star, \varUpsilon\mu^\star) \leq \frac{1}{2}d_\theta(\mu^\star, \varUpsilon\mu^\star)$, which gives $d_\theta(\mu^\star, \varUpsilon\mu^\star) = 0$ and $\varUpsilon\mu^\star = \mu^\star$.

Step 4 Uniqueness: Regarding the uniqueness, suppose that there exists further fixed point $\nu^\star$ for $\varUpsilon$. Then by the cyclic representation condition in Definition **3**.2, we obtain that $\nu^\star \in \bigcap_{i=1}^{p} \aleph_i$. As well, putting $\mu = \mu^\star$ and $\nu = \nu^\star$ in (**3**.40), we get

$$\xi_\theta(d_\theta(\varUpsilon\mu^\star, \varUpsilon\nu^\star)) = \xi_\theta(d_\theta(\mu^\star, \nu^\star)) \leq \xi_\theta(\Delta_1(\mu^\star, \nu^\star)) - \omega_\theta(\Delta_2(\mu^\star, \nu^\star)),$$

where

$$\begin{aligned}\Delta_1(\mu^\star, \nu^\star) &= \max\Big\{d_\theta(\mu^\star, \nu^\star), \frac{1}{2\theta(\mu^\star, \varUpsilon\nu^\star)}(d_\theta(\mu^\star, \varUpsilon\nu^\star) + d_\theta(\nu^\star, \varUpsilon\mu^\star)),\\ &\quad \frac{1}{2}(d_\theta(\mu^\star, \varUpsilon\mu^\star) + d_\theta(\nu^\star, \varUpsilon\nu^\star))\Big\},\\ \Delta_2(\mu^\star, \nu^\star) &= \min\Big\{d_\theta(\mu^\star, \nu^\star), \frac{1}{2\theta(\mu^\star, \varUpsilon\nu^\star)}(d_\theta(\mu^\star, \varUpsilon\nu^\star) + d_\theta(\nu^\star, \varUpsilon\mu^\star))\Big\}.\end{aligned}$$

Since $\varUpsilon\mu^\star = \mu^\star$ and $\varUpsilon\nu^\star = \nu^\star$, thus

$$\Delta_1(\mu^\star, \nu^\star) = d_\theta(\mu^\star, \nu^\star); \quad \Delta_2(\mu^\star, \nu^\star) = \frac{d_\theta(\mu^\star, \nu^\star)}{\theta(\mu^\star, \nu^\star)}.$$

and $\xi_\theta(d_\theta(\mu^\star, \nu^\star)) \leq \xi_\theta(d_\theta(\mu^\star, \nu^\star)) - \omega_\theta\left(\frac{d_\theta(\mu^\star,\nu^\star)}{\theta(\mu^\star,\nu^\star)}\right)$. Which implies that $\omega_\theta\left(\frac{d_\theta(\mu^\star,\nu^\star)}{\theta(\mu^\star,\nu^\star)}\right) = 0$, then $d_\theta(\mu^\star, \nu^\star) = 0$. As a result, we get $\mu^\star = \nu^\star$. ☐

Corollary 3.3. *Let* (Π, d_θ) *be a complete extended b-metric space. Let* $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$*, where* $\aleph_1, \aleph_2, \ldots, \aleph_p$ *are non-empty and closed subsets of* Π*, for* $p \in \mathbb{N}$*. Assume that:*

1) $\mathcal{Z} = \bigcup_{i=1}^{p} \aleph_i$ *is a cyclic representation of* $\mathcal{Z}$ *regarding to* Υ.

2) *For any* $(\mu, \nu) \in \aleph_i \times \aleph_{i+1}$, *where* $i = 1, 2, \ldots, p$ *and* $\aleph_{p+1} = \aleph_1$,

$$d_\theta(\Upsilon\mu, \Upsilon\nu) \leq \Delta_1(\mu, \nu) - \omega_\theta(\Delta_2(\mu, \nu)),$$

where, $\Delta_1(\mu, \nu) = \max\left\{d_\theta(\mu, \nu), \frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu, \Upsilon\nu)+d_\theta(\nu, \Upsilon\mu)\Big), \frac{1}{2}\Big(d_\theta(\mu, \Upsilon\mu)+d_\theta(\nu, \Upsilon\nu)\Big)\right\}$, $\Delta_2(\mu, \nu) = \min\left\{d_\theta(\mu, \nu), \frac{1}{2\theta(\mu,\Upsilon\nu)}\Big(d_\theta(\mu, \Upsilon\nu) + d_\theta(\nu, \Upsilon\mu)\Big)\right\}$, $\omega_\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and* $\omega_\theta(\tau) = 0$ *if and only if* $\tau = 0$. *Assume that for* $\mu_0 \in \Pi$, *we have* $\mu_k = \Upsilon^k\mu_0$ *and* $\lim_{\iota,\eta\to\infty} \theta(\mu_\iota, \mu_\eta) = 1$.

Then the mapping Υ *possesses a unique fixed point in* $\bigcap_{i=1}^{p} \aleph_i$.

Proof. We obtain the result from Theorem **3**.2 by setting $\xi_\theta = I$. □

Example 3.2. Let $\Pi = [-1, 1]$ endowed with $d_\theta\colon \Pi \times \Pi \to \mathbb{R}^+$, $d_\theta(\mu, \nu) = |\mu - \nu|$ and let $\theta\colon \Pi\times\Pi \to [1, \infty)$, $\theta(\mu, \nu) = |\mu|+|\nu|+1$. Let $\aleph_1 = [\frac{-\sqrt{3}}{2}, 1]$, $\aleph_2 = [\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$, $\aleph_3 = [\frac{-\sqrt{3}}{2}, \frac{\pi}{4}]$, $\aleph_4 = [\frac{-\sqrt{3}}{2}, \frac{\sqrt{2}}{2}]$, $\aleph_5 = [\frac{-\pi}{4}, \frac{\sqrt{3}}{2}]$, $\aleph_6 = [\frac{-\pi}{4}, \frac{\pi}{4}]$, $\aleph_7 = [\frac{-\pi}{4}, \frac{\sqrt{2}}{2}]$, $\aleph_8 = [\frac{-\sqrt{2}}{2}, \frac{\sqrt{3}}{2}]$, $\aleph_9 = [\frac{-\sqrt{2}}{2}, \frac{\pi}{4}]$, $\aleph_{10} = [\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and $\aleph_{11} = [-1, 1]$. Define $\Upsilon\colon \mathcal{Z} \to \mathcal{Z}$, $\Upsilon(\mu) = \sin\mu$. Then:

1. (Π, d_θ) is a complete extended b-metric space.

2. $\mathcal{Z} = \bigcup_{i=1}^{11} \aleph_i$ is a cyclic representation of $\mathcal{Z}$ regarding to Υ, since

a) $\aleph_i, i = \{1, \ldots, 11\}$ are non empty closed sets.

b) $\Upsilon(\aleph_1) \subset \aleph_2, \Upsilon(\aleph_2) \subset \aleph_3, \Upsilon(\aleph_3) \subset \aleph_4, \Upsilon(\aleph_4) \subset \aleph_5, \Upsilon(\aleph_5) \subset \aleph_6, \Upsilon(\aleph_6) \subset \aleph_7$, $\Upsilon(\aleph_7) \subset \aleph_8, \Upsilon(\aleph_8) \subset \aleph_9, \Upsilon(\aleph_9) \subset \aleph_{10}, \Upsilon(\aleph_{10}) \subset \aleph_{11}, \Upsilon(\aleph_{11}) \subset \aleph_1$.

3. Let $\mu_0 \in \aleph_1$, we have $\lim_{\iota,\eta\to\infty} \theta(\mu_\iota, \mu_\eta) = 1$, taking into account that for $\mu_0 \in \aleph_1$, $\mu_\iota = \sin^\iota \mu_0$ and $\mu_\eta = \sin^\eta \mu_0$

$$\begin{aligned} \lim_{\iota,\eta\to\infty} \theta(\mu_\iota, \mu_\eta) &= \lim_{\iota,\eta\to\infty} (|\sin^\iota(\mu_0)| + |\sin^\eta(\mu_0)| + 1) \\ &= \lim_{\iota,\eta\to\infty} (|\sin(\mu_0)|^\iota + |\sin(\mu_0)|^\eta + 1) = 1, \end{aligned}$$

because $\lim_{\iota\to\infty} |\sin(\mu_0)|^\iota = \lim_{\iota\to\infty} e^{\iota \ln|\sin(\mu_0)|} = 0$.

4. Υ satisfies the inequality of Theorem **3**.2 by taking $\nu = \frac{\mu}{n}$, for $n \in [2, \infty[$ and some $\mu \in [-1, 1]$.

$$d_\theta(\Upsilon\mu, \Upsilon\nu) = |\sin(\mu) - \sin(\nu)| = |\sin(\mu) - \sin(\frac{\mu}{n})|; \quad d_\theta(\mu, \nu) = |\mu - \frac{\mu}{n}|.$$

$$\frac{1}{2\theta(\mu, \Upsilon\nu)}\Big(d_\theta(\mu, \Upsilon\nu) + d_\theta(\nu, \Upsilon\mu)\Big) = \frac{1}{2(|\mu| + |\sin(\frac{\mu}{n})| + 1)}\Big(|\mu - \sin(\frac{\mu}{n})| + \Big|\frac{\mu}{n} - \sin(\mu)\Big|\Big).$$

$$\frac{1}{2}\Big(d_\theta(\mu, \Upsilon\mu) + d_\theta(\nu, \Upsilon\nu)\Big) = \frac{1}{2}\Big(\Big|\mu - \sin(\mu)\Big| + \Big|\frac{\mu}{n} - \sin(\frac{\mu}{n})\Big|\Big).$$

Now from Figure 5 and Figure 6, we get

$$\Delta_1(\mu, \nu) = d_\theta(\mu, \nu) = \Big|\mu - \frac{\mu}{n}\Big|,$$

$$\Delta_2(\mu, \nu) = \frac{1}{2\theta(\mu, \Upsilon\nu)}\Big(d_\theta(\mu, \Upsilon\nu)+d_\theta(\nu, \Upsilon\mu)\Big) = \frac{1}{2(|\mu| + |\sin(\frac{\mu}{n})| + 1)}\Big(|\mu-\sin(\frac{\mu}{n})|+\Big|\frac{\mu}{n}-\sin(\mu)\Big|\Big).$$

Let $\xi_\theta(\tau) = \pi^3\tau$ and $\omega_\theta(\tau) = \frac{\tau}{\pi^2}$ with $\omega_\theta(0) = 0$, then from Figure 7 and Figure 8, we get

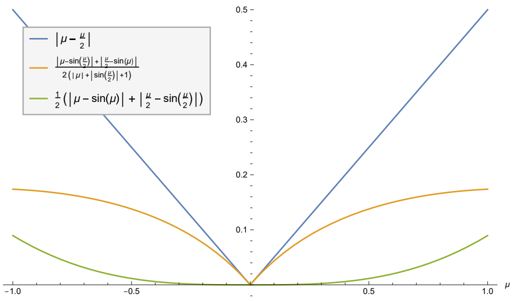

FIGURE 5. Comparison between functions when $n \lll$

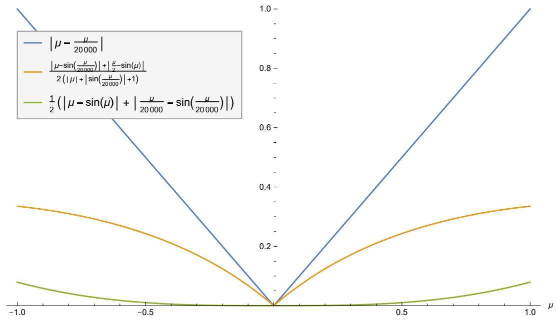

FIGURE 6. Comparison between functions when $n \ggg$.

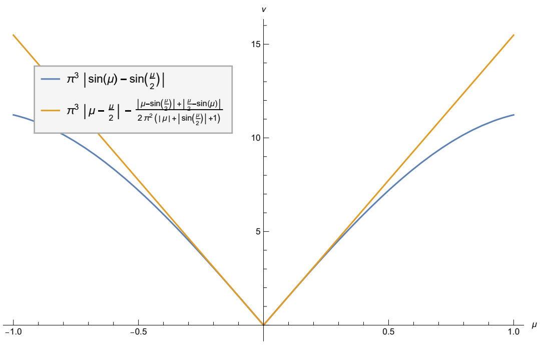

FIGURE 7. Comparison between functions when $n \lll$

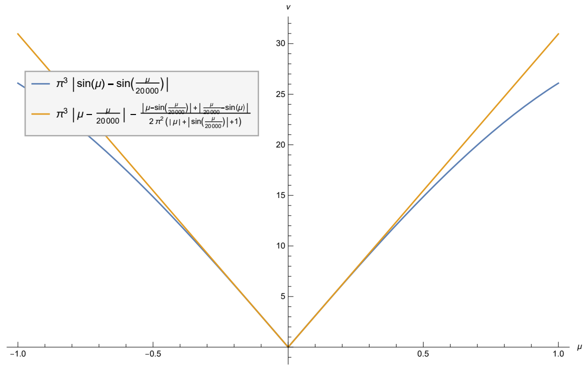

FIGURE 8. Comparison between functions when $n \ggg$.

$$\begin{aligned}\xi_\theta(d_\theta(\Upsilon\mu,\Upsilon\nu)) &= \pi^3|\sin(\mu)-\sin(\frac{\mu}{n})| \\ &\le \pi^3\Big|\mu-\frac{\mu}{n}\Big| - \frac{1}{2\pi^2(|\mu|+|\sin(\frac{\mu}{n})|+1)}\Big(|\mu-\sin(\frac{\mu}{n})|+\Big|\frac{\mu}{n}-\sin(\mu)\Big|\Big) \\ &= \xi_\theta(\Delta_1(\mu,\nu))-\omega_\theta(\Delta_2(\mu,\nu)).\end{aligned}$$

This example shows that all hypotheses of Theorem **3**.2 are satisfied. Hence, 0 is the unique fixed point of Υ in $\bigcap_{i=1}^{11} \aleph_i$.

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