U.P.B. Sci. Bull., Series A, Vol. 74, Iss. 4, 2012 ISSN 1223-7027

NEW PRECONDITIONERS FOR ELLIPTIC PROBLEMS IN MULTIRESOLUTION SPACE

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In this paper, we present two new preconditioners to solve the elliptic boundary value problems by Galerkin's method and discuss their qualitative and quantitative aspects. These preconditioners are constructed based on BPX (Bramble-Pasciak-Xu) preconditioners. We show that our preconditioners are optimal and also the convergence rates are lower than that of BPX preconditioner. Some numerical examples are given to show the effectiveness of the new preconditioners.

Keywords: BPX preconditioner, multi-resolution space, elliptic problems.

1. Introduction

Solving the problems of partial differential equations with finite element methods would lead to large systems. Solving these systems usually is impossible by direct methods and even if some of these systems are solved by usual indirect methods (Jacobi, Gauss Seidel and so on), they either would not converge or converge slowly, since these systems are usually ill-conditioned. This problem can be solved efficiently using some preconditioners like preconditioned conjugate gradient method ([9]). Multilevel techniques are used for preconditioning linear systems arising from Galerkin methods for elliptic value problems [6]). A class of additive Schwarz preconditioners including many well-known preconditioners like hierarchical basis, BPX multilevel, and domain decomposition preconditioners are suitable for implementation in parallel computers [4]. The BPX- preconditioner has been theoretically verified in two and three dimensional local mesh refinement settings ([1,3]). Also, the BPX preconditioner is optimal with respect to both problem and discretization parameters ([2,8]). Hence, the BPX is a suitable preconditioner to solve the systems arising from discretization of elliptic boundary value problems by Galerkin's methods.

The paper is organized as follows: In Section 2, we study some preliminaries on multiresolution space and introduce some operators needed in the next sections. Then, in Section 3, we present the BPX preconditioner along

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with our proposed preconditioners , the optimality of which would be proved. After that, the quantitative aspects of these preconditioners are shown in Section 4. Finally, in Section 5 some numerical experiments are given to confirm our theoretical results. The convergence factors and condition numbers of BPX preconditioner and our proposed preconditioners are compared in this section to show the optimality of the preconditioned systems and the advantages of the new preconditioners.

2. Preliminaries

Let Ω be a polygonal domain in $\mathbb{R}^n (n = 1,2,3)$ and consider the problem :

$$\begin{aligned} Lu &= f \quad in\,\Omega, \\ u &= 0 \quad on\,\partial\Omega, \end{aligned} \qquad (1)$$

where

$$Lu = -\sum_{i,j} \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial u}{\partial x_j}) + au$$

such that $\left(a_{ij}(x)\right)_{i,j}$ is a symmetric uniformly positive definite matrix and $a(x) \geq 0$ in Ω. In continue, we need to define the basic concepts of multi-scale methods for the space $L_2(\Omega)$. A multi-resolution analysis (MRA) is a sequence of nested spaces

$$V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots \subset L_2(\mathbb{R}) \qquad (2)$$

such that

- $\cap V_j = \{0\}$,
- $\overline{\cup V_j} = L_2(\mathbb{R})$,
- $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1},\ \ x \in \mathbb{R}$
- there is a function φ such that the translates $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $L_2(\mathbf{R})$, i.e. there exist $0 < C_1 \leq C_2$ such that for all finitely supported sequences (c_k), we have

$$C_1 \sum_{k\in\mathbb{Z}} | c_k|^2 \leq \|\sum_{k\in\mathbb{Z}} c_k \phi(x-k)\|^2_{L_2(\mathbb{R})} \leq C_2 \sum_{k\in\mathbb{Z}} | c_k|^2$$

Let us denote by $\varphi, \tilde{\varphi} \in L_2(\mathbb{R})$ a pair of functions that are refinable, i.e., there exist some masks $= (a_k)_{k\in\mathbb{Z}}$, $\tilde{\boldsymbol{a}} = (\tilde{a}_k)_{k\in\mathbb{Z}}$ such that

$$\varphi(x) = \sum_{k\in\mathbf{Z}} a_k \varphi(2x-k), \quad \tilde{\varphi}(x) = \sum_{k\in\mathbf{Z}} \tilde{a}_k \tilde{\varphi}(2x-k) \qquad (3)$$

for $x \in \mathbb{R}$. Furthermore, φ and $\tilde{\varphi}$ are called biorthogonal, i.e.

$$\left(\varphi(x), \varphi(x-k)\right) = \delta_{0,k} \qquad k \in \mathbb{Z}$$

The functions φ and $\tilde{\varphi}$ are called dual scaling functions. Moreover, the scaled and translated version of a function $f: R \to R$ is defined by

$$f_{j,k}(x) = 2^{j/2} f(2^j x - k),$$

where $j, k \in \mathbb{Z}$. The pair of dual scaling functions generate a sequence of nested spaces defined by $V_j = clos_{L_2(\mathbb{R})} span\{\varphi_{j,k}, \quad k \in \mathbb{Z}\}$ and $\tilde{V}_j = clos_{L_2(R)} span\{\tilde{\varphi}_{j,k}, \quad k \in Z\}$

which are referred to as biorthogonal multi-resolution analysis (see [11]).
We now define a sequence of multi-resolution (MRA) spaces by rectangular mesh in the usual way. Let φ be a scaling function with compact support in $\mathbb{R}$ and in-addition, let $I_{j,m} = [m2^{-j}, (m+1)2^{-j}], \quad m \in \mathbb{Z}, j \in \mathbb{N}$. Also, suppose that

$$\varphi_{j,m}(x) = 2^{j/2}\varphi(2^j x - m), \quad m \in \mathbb{Z}.$$

The multi-resolution space in level j on x-coordinate in the domain $I = [0,1] \subset \mathbb{R}$ is defined as follows:

$$V_j^x = span\{\varphi_{j,m}, \bigcup_{m\in\{0,\dots,2^j-1\}} I_{j,m} = I\}.$$

We note that the unit square $\Omega = \bigcup_{m,n\in\{0,\dots,2^j-1\}} I_{j,m} \times I_{j,n}$. By tensor product, one can define the multi-resolution space in level j as $V_j = V_j^x \otimes V_j^y$, i.e. the basis functions in V_j are given by

$$\phi_{j,k}(x,y) = \varphi_{j,m}(x) \times \varphi_{j,n}(y), \quad k = \{(m,n) | m,n \in \{0,\dots,2^j - 1\}.$$

Then, a sequence of nested finite dimensional spaces are generated:

$$V_1 \subset V_2 \subset \cdots \subset V_J \equiv V, \quad J \geq 2.$$

The aim is finding a solution for the following system:

Given $f \in V$, find $u \in V$ satisfying

$$A(u,v) = f(v), \quad \forall v \in V.$$

where

$$A(u,v) = \sum_{i,j} \int_\Omega (a_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + auv) d\boldsymbol{x}$$

We assume that there are given symmetric positive definite forms $A_k(.,.)$ and $(.,.)_k$ defined on $V_k \times V_k$ for $k = 1,2,\cdots,J$. The norm corresponding to $(.,.)_k$ will be denoted by $\|.\|_k$.
The operator $A_k: V_k \to V_k$ is defined for $u \in V_k$ by

$$(A_k u, v) = A(u,v), \quad \forall v \in V_k. \tag{4}$$

We, moreover, need the following definitions for $k = 1,2,\cdots,J$:
(1) The projection $P_k: V \to V_k$ is defined for $u \in V$ by

$$A(P_k u, v) = A(u,v), \quad \forall v \in V_k. \tag{5}$$

(2) The projection $Q_k: V \to V_k$ is defined for $u \in V$ by

$$(Q_k u, v) = (u,v), \quad \forall v \in V_k. \tag{6}$$

So we will have:

$$Q_k A = A_k P_k, \tag{7}$$

$$Q_k Q_l = Q_l Q_k = Q_l, \quad for \quad l \leq k \tag{8}$$

$$(Q_k - Q_{k-1})(Q_l - Q_{l-1}) = 0, \tag{9}$$

where $Q_0 = 0$.

We define $\kappa(A)$, the condition number of A, to be $\kappa(A) = \lambda/\Lambda$ where λ and Λ are the largest and smallest eigenvalues of A, respectively. Now, for all $\phi \in V$

$$\Lambda \leq \frac{(A\phi,\phi)}{(\phi,\phi)} \leq \lambda.$$

Hence, if we define numbers c_0 and c_1 such that

$$c_0(\phi,\phi) \leq (A\phi,\phi) \leq c_1(\phi,\phi), \quad (10)$$

then $c_0 \leq \Lambda$ and $\lambda \leq c_1$ so that $\kappa(A) \leq c_1/c_0$.

3. Preconditioners

In this section, we introduce the BPX preconditioner moreover we present two new preconditioners that are optimal for elliptic problems. Throughout this paper, we assume that values C_1, C_2 and C_3 are positive constants and independent of level. In [3], the initial preconditioner has been introduced by

$$B = \sum_{k=1}^{J} \lambda_k^{-1}(Q_k - Q_{k-1}), \quad (11)$$

where λ_k denotes the spectral radius of A_k. Moreover, in general case, the preconditioner is stated by

$$\beta_o = \sum_{k=1}^{J} R_k Q_k, \quad (12)$$

where $R_k: V_k \to V_k$ is a symmetric positive definite operator. Also, in Remark 2.2 of [3] is given that for finite element applications with quasi uniform grid, it can be defined a suitable R_k, that for $u \in V$,

$$R_k Q_k u = \lambda_k^{-1} \sum_{l=1}^{J} (u, \overline{\psi}_k^l)\overline{\psi}_k^l \quad (13)$$

where $\{\overline{\psi}_k^l\}$ denote the normalized nodal basis functions. Hence, $R_k Q_k$ is computable without the solution of Gram matrix systems. In order to be optimal the preconditioner, following assumptions are needed [3]:

Assumption 1: For $k = 1,2,\cdots,J$, there exists a constant $C_1 > 0$ such that

$$\| (I - Q_{k-1})v \|^2 \leq C_1 \lambda_k^{-1} A(v,v) \quad \forall v \in V. \quad (14)$$

Assumption 2: Let C_2 and C_3 be constants independent of level k such that

$$C_2 \frac{\|u\|^2}{\lambda_k} \leq (R_k u, u) \leq C_3 (A_k^{-1} u, u) \quad \forall u \in V_k \quad (15)$$

holds.

The following lemmas state the optimality of preconditioners (11) and (12):

Lemma 1 Suppose that the Assumption (A1) holds and B is given by (11). Then

$$C_1^{-1} J^{-1} A(v,v) \leq A(BAv,v) \leq JA(v,v) \quad \forall v \in V.$$

Proof See [3].

Lemma 2 Suppose that the Assumptions (A1) and (A2) hold. Then,

$$C_1^{-1} C_2 J^{-1} A(v,v) \leq A\left(\beta_o Av,v\right) \leq C_3 JA(v,v) \quad \forall v \in V.$$

Proof see [3].

3.1 Biorthogonal Preconditioner

In this subsection, the first new preconditioner that we call biorthogonal preconditioner is defined. In order to introduce the biorthogonal preconditioner, we need the following definition:

Definition 3. A scaling function $\tilde{\varphi}$ in L^2 is dual to a scaling function φ, if it satisfies

$$\big(\varphi(x-m),\tilde{\varphi}(x-n)\big)=\delta_{m,n},\quad m,n\in\mathbb{Z}$$

where δ is Kronecker delta. Now, our first preconditioner is introduced by

$$\tilde{\beta}v=\sum_k \tilde{R}_k Q_k v=\sum_k \lambda_k^{-1}\sum_l \big(v,\tilde{\varphi}_{k,l}\big)\varphi_{k,l}\quad \forall\, v\in V \qquad (16)$$

where λ_k is the maximum eigenvalues of A_k and the operator $\tilde{R}_k\colon V_k\to V_k$ is defined as

$$\tilde{R}_k u=\lambda_k^{-1}\sum_l (u,\tilde{\varphi}_{k,l})\varphi_{k,l}. \qquad (17)$$

Remark 4. We recall that for any $u\in V$,

$$\lambda_k^{-1}(u,u)\le (A_k^{-1}u,u). \qquad (18)$$

On the other hand, by the preconditioner $\tilde{\beta}$ defined by (16) and biorthonormality, it is readily seen that

$$(\tilde{R}_k u,u)=\| u\|^2/\lambda_k. \qquad (19)$$

Therefore, by (18) and (19), the Assumption (A2) is satisfied by $C_2=C_3=1$, i.e

$$\| u\|^2/\lambda_k\le(\tilde{R}_k u,u)\le(A_k^{-1}u,u)\quad \forall u\in V_k. \qquad (20)$$

Moreover, in order to show the advantages of our preconditioner, we need the following definition ([5]).

Definition 5. The convergence rate on the kth level is defined by a convergence factor δ_k satisfying

$$a((I-\beta_k A_k)u,u)\ \le\ \delta_k a(u,u)\quad \forall u\in V_k, \qquad (21)$$

for some $0<\delta_k<1$. The following lemma shows that our preconditioner $\tilde{\beta}$ is optimal.

Lemma 6. Under Assumptions (A1) and (A2),

$$C_1^{-1}C_2J^{-1}A(v,v)\ \le\ A(\tilde{\beta}Av,v)\le C_3JA(v,v)\quad \forall v\in V. \qquad (22)$$

Proof. Let $v\in V$. By Assumption (A2),

$$\begin{aligned}
A(\tilde{\beta}Av,v) &= A\Big(\sum_{k=1}^{J}\tilde{R}_kQ_kAv,v\Big)=\sum_{k=1}^{J}A(\tilde{R}_kQ_kAv,v)\\
&= \sum_{k=1}^{J}(\tilde{R}_kA_kP_kv,A_kP_kv) && \textit{by } (5) \textit{ and } (7) \textit{ and } (4)\\
&\le C_3\sum_{k=1}^{J}(P_kv,A_kP_kv) && \textit{by } (20)\\
&= C_3\sum_{k=1}^{J}A(P_kv,P_kv)=C_3JA(v,v). && \textit{by } (5) \textit{ and } (4)
\end{aligned}$$

Then, the upper inequality (22) holds. By (4), (5) and (16), we have:

$$A(\tilde{\beta}Av,v)=\sum_{k=1}^{J}(\tilde{R}_kA_kP_kv,A_kP_kv)$$

Now, we can write:

$$\begin{aligned}
C_1^{-1}J^{-1}A(v,v) &\leq A(BAv,v) && by\ Lemma(1)\\
&\leq \sum_{k=1}^{J}\lambda_k^{-1}\parallel Q_kAv\parallel^2=\sum_{k=1}^{J}\lambda_k^{-1}\parallel A_kP_kv\parallel && by\ (7)\\
&\leq C_2^{-1}\sum_k(\tilde{R}_kA_kP_kv,A_kP_kv) && by\ (20)\\
&= C_2^{-1}\sum_k A(\tilde{R}_kA_kP_kv,v) && by\ (5)\ and\ (4)\\
&= C_2^{-1}\sum_k A(\tilde{R}_kQ_kAv,v)=C_2^{-1}A(\tilde{\beta}Av,v).
\end{aligned}$$

3.2 Hierarchical Biorthogonal Preconditioner

In this subsection, we define our next preconditioner called hierarchical biorthogonal preconditioner. To this end, the operator $A_k:V_k\to V_k$ is defined for $u\in V_k$ by

$$(A_ku,v)=A(u,v)\qquad \forall v\in V_k. \tag{23}$$

Let $V_j=span\{\varphi_{j,l},\ l=1,2,\cdots\}$. Then, we define the space $\breve{V}_j$ by

$$\breve{V}_j=span\left\{\varphi_{j,2l-1},\ l=1,2,\cdots\right\}.$$

Now, we give the following definitions: For $k=1,2,\cdots,J$

(1): The projection $P_k:V\to\breve{V}_k$ is defined for $u\in V$ by

$$A(P_ku,v)=A(u,v),\quad \forall v\in\breve{V}_k. \tag{24}$$

(2) The projection $Q_k:V\to\breve{V}_k$ is defined for $u\in V$ by

$$(Q_ku,v)=(u,v),\quad \forall v\in\breve{V}_k. \tag{25}$$

So we will have:

$$(Q_kAu,v)=(A_kP_ku,v),\quad \forall v\in\breve{V}_k. \tag{26}$$

$$Q_kQ_l=Q_lQ_k=Q_l\quad for\quad l\leq k \tag{27}$$

$$(Q_k-Q_{k-1})(Q_l-Q_{l-1})=0 \tag{28}$$

where $Q_0=0$.

Now, our second preconditioner is introduced by

$$\breve{\beta}v=\sum_k\breve{R}_kQ_kv=\sum_k\lambda_k^{-1}\sum_l(v,\tilde{\varphi}_{k,2l-1})\varphi_{k,2l-1} \tag{29}$$

where λ_k is again the maximum eigenvalues of A_k and the projection $\breve{R}_k:\breve{V}_k\to\breve{V}_k$ is defined for $u\in\breve{V}_k$ by:

$$\breve{R}_ku=\lambda_k^{-1}\sum_l(u,\tilde{\varphi}_{k,2l-1})\varphi_{k,2l-1}. \tag{30}$$

Assumption 3: Let C_2 and C_3 be constants independent of level k such that

$$C_2\frac{\|u\|^2}{\lambda_k}\leq(\breve{R}_ku,u)\leq C_3(A_k^{-1}u,u)\quad \forall u\in\breve{V}_k \tag{31}$$

Remark 7. We recall that for any $u\in V$,

$$\lambda_k^{-1}(u,u) \le (A_k^{-1}u,u). \tag{32}$$

On the other hand, by the preconditioner $\breve{\beta}$ defined by (29) and biorthonormality, it is readily seen that

$$(\breve{R}_k u,u) = \frac{\|u\|^2}{\lambda_k}, \quad \forall u \in \breve{V}_k \tag{33}$$

Therefore, by (32) and (33), Assumption (A3) is satisfied by $C_2 = C_3 = 1$, i.e.

$$\frac{\|u\|^2}{\lambda_k} \le (\breve{R}_k u,u) \le (A_k^{-1}u,u) \quad \forall u \in \breve{V}_k. \tag{34}$$

The following lemma shows that the preconditioner $\breve{\beta}$ is optimal.

Lemma 8. Under Assumptions 1 and 3, there exist the constants C_1, C_2 and C_3 such that

$$C_1^{-1}C_2J^{-1}A(v,v) \quad \le \quad A(\breve{\beta}Av,v) \quad \le \quad JC_3A(v,v) \quad \forall v \in V \tag{35}$$

holds.

Proof. Let $v \in V$. We have

$$\begin{aligned}
A(\breve{\beta}Av,v) &= \textstyle\sum_k A(\breve{R}_k Q_k Av,v) = \sum_k A(\breve{R}_k Q_k Av, P_k v) && by\ (24)\\
&= \textstyle\sum_k (\breve{R}_k Q_k Av, A_k P_k v) && by\ (23)\\
&= \textstyle\sum_k (Q_k Av, \breve{R}_k A_k P_k v) = \sum_k (A_k P_k v, \breve{R}_k A_k P_k v) && by\ (26)\\
&\le C_3 \textstyle\sum_k (P_k v, A_k P_k v) && by\ (31)\\
&= \textstyle\sum_k C_3 A(P_k v, P_k v) = C_3 J A(v,v) && by\ (24)\ and\ (23)
\end{aligned}$$

Now, we can write

$$\begin{aligned}
C_1^{-1}J^{-1}A(v,v) &\le A(BAv,v) && by\ Lemma(1)\\
&\le \sum_{k=1}^{J} \lambda_k^{-1} \| Q_k Av \|^2 = \sum_{k=1}^{J} \lambda_k^{-1} \| A_k P_k v \|^2 && by\ (26)\\
&\le C_2^{-1} \sum_k (\breve{R}_k A_k P_k v, A_k P_k v) && by\ (34)\\
&= C_2^{-1} \sum_k A(\breve{R}_k A_k P_k v, v) && by\ (24)\ and\ (23)\\
&= C_2^{-1} \sum_k A(\breve{R}_k Q_k Av, v) = C_2^{-1} A(\breve{\beta}Av,v) && by\ (26).
\end{aligned}$$

4 Quantitative Aspects

In this section, we give a quantitative comparison between the BPX (β_o), $\tilde{\beta}$ and $\breve{\beta}$ preconditioners by their convergence factors. First, by Remark (4) and Lemma (6),

$$(1-J)A(v,v) \le A((I-\tilde{\beta}A)v,v) \le (1-C_1^{-1}J^{-1})A(v,v). \tag{36}$$

Also, by Lemma (8),

$$(1-J)A(v,v) \le A((I-\breve{\beta}A)v,v) \le (1-C_1^{-1}J^{-1})A(v,v). \tag{37}$$

Now, the following lemma gives a comparison between the preconditioners β_o and $\tilde{\beta}$, quantitatively.

Theorem 9. Let $\beta_o, \tilde{\beta}$ and $\breve{\beta}$ be three preconditioners defined by (12), (16) and (29) in the level k, respectively. Moreover, suppose that $\delta_k, \tilde{\delta}_k$ and $\breve{\delta}_k$ are the convergence factors related to the preconditioners $\beta_o, \tilde{\beta}$ and $\breve{\beta}$, respectively. Then,

$$\tilde{\delta}_k \le \delta_k \quad \text{and} \quad \breve{\delta}_k \le \delta_k$$

holds.

Proof. First, we show that $\tilde{\delta}_k \le \delta_k$. Since $\breve{\delta}_k \le \delta_k$ is proved similarly, we don't go through it. Let u be an arbitrary function in V_k. There occurs only two cases:

$$(i): 1/\lambda_k \parallel u \parallel^2 \le (R_k u, u) \quad \text{or} \quad (ii): 1/\lambda_k \parallel u \parallel^2 \ge (R_k u, u).$$

If (i) occurs, then $C_2 = 1$ but, if (ii) occurs, $C_2 \le 1$. Obviously, the values of C_2 can be chosen according to Assumption (A2). Hence, for any $u \in V_k$, we have $C_2 \le 1$. On the other hand, the constant C_1 is independent of level. Then, one can choose the finest level J such that $C_1^{-1} J^{-1} \le 1$ satisfies.

Then, $1 - C_1^{-1} C_2 J^{-1} \ge 1 - C_1^{-1} J^{-1}$ and so according to Remark (4), Lemma(6) and Definition (5) $\tilde{\delta}_k \le \delta_k$. holds.

5. Numerical Experiments

In this section, we present some numerical results with the new BPX-preconditioners given in Section 3 and compare them with the one in [3]. Like what has been shown in [3], the hypotheses of the previous section are satisfied. Next we present some examples and show the condition numbers of the preconditioned system to confirm that the theories of the previous sections are in a good agreement with the numerical results. We will employ the finite element discretization of the following problems and produce the grids in a custom way.

In the following examples, we consider the elliptic boundary value problems on a unit square and a slit domain (i.e. the set of points in the interior of a unite square excluding the line $\{(1/2, y) | y \in [1/2, 1)\}$), respectively. In the following examples, we compare the condition numbers and the convergence rates of preconditioned systems. In Tables 1, 2 and 3, $cond(\beta_o A), cond(\tilde{\beta} A)$ and $cond(\breve{\beta} A)$ denote the condition numbers of the preconditioned systems with BPX-preconditioner and the two new preconditioners defined in Section 3, respectively. Also, $\delta_o, \tilde{\delta}$ and $\breve{\delta}$ denote the convergence rates of the preconditioned systems with $\beta_o, \tilde{\beta}$ and $\breve{\beta}$ preconditioners, respectively.

In all the following examples, let $\{\varphi_{j,l}\}_{l=0}^{2^j-1}$ be the second order B-spline (hat) functions in $[0,1]$ and $V_j = span\{\varphi_{j,l}, l = 0, \cdots, 2^{j-1}\}$. We have (see e.g. [11]):

$$\varphi_{j-1,l} = 1/2\varphi_{j,l-1} + \varphi_{j,l} + 1/2\varphi_{j,l+1}.$$

Also, for the dual B-spline space $\tilde{V}_j$,

$$\tilde{\varphi}_{j-1,l} = -1/4\tilde{\varphi}_{j,l-2} + 1/2\tilde{\varphi}_{j,l-1} + 3/2\tilde{\varphi}_{j,l} + 1/2\tilde{\varphi}_{j,l+1} - 1/4\tilde{\varphi}_{j,l+2}.$$

Example 1 Consider the Poisson's problem as follows:

$$\begin{aligned} -\Delta u &= f, \quad in\, \Omega = [0,1] \times [0,1] \\ u &= 0 \qquad on\, \partial\Omega. \end{aligned} \tag{38}$$

The aim is finding $u \in V \subset H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in V.$$

Table 1

A comparison between the condition numbers and the convergence factors of the preconditioned systems

$level$	$cond(A)$	$cond(\beta_o A)$	$cond(\tilde{\beta} A)$	$cond(\breve{\beta} A)$	δ_o	$\tilde{\delta}$	$\breve{\delta}$
2	3.1530	2.6951	2.0993	2.6502	0.7603	0.3480	0.4980
3	12.8211	4.6962	3.8184	4.8120	0.8796	0.4653	0.6153
4	51.7144	19.3627	7.0325	10.6267	0.9777	0.5340	0.76740
5	207.3402	77.8565	14.0490	31.8941	0.9892	0.5992	0.84392

Table 1 shows a comparison between the condition numbers BPX ($cond(\beta_o A)$), biorthogonal ($cond(\tilde{\beta} A)$) and hierarchical biorthogonal ($cond(\breve{\beta} A)$) and also the corresponding convergence factors $\delta_o, \tilde{\delta}$ and $\breve{\delta}$. As it can be seen, the condition numbers increase when the levels increase. Moreover, this table shows the optimality of our preconditioners. In addition, the convergence factors of the preconditioned systems with $\tilde{\beta}$ and $\breve{\delta}$ are lower than δ_o and also we observe that the biorthogonal preconditioner $\tilde{\beta}$ works better than β_o and $\breve{\beta}$.

Example 2 We consider the following problem:

$$\begin{aligned} -\Delta u + u &= f \quad in\ \Omega = [0,1] \times [0,1] \\ u &= 0 \quad on \qquad \partial\Omega \end{aligned}$$

Table 2

A comparison between the condition numbers and the convergence factors of the preconditioned systems

$level$	$cond(A)$	$cond(\beta_o A)$	$cond(\tilde{\beta} A)$	$cond(\breve{\beta} A)$	δ_o	$\tilde{\delta}$	$\breve{\delta}$
2	2.4091	3.1495	2.2580	3.0912	0.4105	0.2221	0.3821
3	9.2623	4.2137	3.7665	4.179	0.7790	0.4814	0.7014
4	6.9324	13.8576	6.9680	9.6945	0.8386	0.5410	0.7810
5	7.6741	55.4903	13.8970	37.3086	0.89790	0.6288	0.8088

Table 2 shows the optimality of the given preconditined system. We note that $\tilde{\beta}$ and $\breve{\beta}$ are preferred over standard BPX preconditioner β_o.

Example 3 We consider the following problem:

$$\begin{aligned} -\Delta u + u &= f \quad in\ \Omega = [0,1] \times [0,1] \\ u &= 0 \qquad on\, \partial\Omega \end{aligned} \tag{39}$$

where Ω is a slit domain. ⍰

Table 3

A comparison between the condition numbers and the convergence factors of the preconditioned systems

$level$	$cond(A)$	$cond(\beta_o A)$	$cond(\tilde{\beta} A)$	$cond(\breve{\beta} A)$	δ_o	$\tilde{\delta}$	$\breve{\delta}$
2	5.1987	3.2353	3.2344	3.2352	0.4437	0.3073	.3993
3	42.7332	15.0379	9.68209	14.9408	.7942	0.5404	.7004
4	145.9780	48.4062	21.5451	38.7047	0.8816	.6944	.79044
5	307.9780	83.4062	40.5451	69.7047	0.9516	.7144	.9244

Table 3 shows the convergence factors of the preconditioned system in a slit domain. As we observe, the convergence factors of $\tilde{\beta}A$ and $\breve{\beta}A$ are lower than $\beta_o A$.

6 Conclusion

Multilevel subspace decomposition provides tools for the construction of preconditioners. The aim of this paper was to improve the BPX preconditioner in the multiresolution space. We presented two preconditioners based on biorthogonal and hierarchical spaces. The optimality of these preconditioners was proved and their advantages were shown quantitatively and qualitatively.

Acknowledgments

The writers are very grateful to Dr. H. Ravand, the faculty member of Vali-e-Asr University of Rafsanjan, Iran, for his kind cooperation in reading the manuscript of the article and providing linguistic hints.

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