U.P.B. Sci. Bull., Series A, Vol. 79, Iss. 4, 2017 ISSN 1223-7027

CHARACTERIZATION OF *-VARIETIES AND THEIR CORES

Ghorban Khalilzadeh Ranjbar[1], Tooraj Amiri [2]

Dixon has proved that each semivariety can be characterized by homogeneous polynomials [6]. *Also, it has been proved that each variety of Banach algebras between the variety of all IQ-algebras and the variety of all IR-algebras can not be charactrized by homogeneous polynomials alone* [7]. *In this paper, according to the structure and definition of varieties, *-varieties of C*-algebras have been introduced. In addition, it will be see that each *-variety of C*-algebras can be characterized by homogeneous polynomials alone. It is shown that each *-variety of C*-algebras has a unique core. Also, we shall introduce the cores of some well-known *-varieties.*

Keywords: Banach algebra, Variety, *-variety, cores, IQ-algebra and IR-algebras.
MSC2010: Primary 08B15; Secondary 08B26, 46H20

1. Introduction and Preliminaries

Let $\mathcal{A}$ be a Banach algebra and $\delta > 0$. Then,

$$\|p\|_{\mathcal{A},\delta} = \sup\{\|p(x_1, ..., x_n)\| : x_i \in \mathcal{A}, \|x_i\| \leq \delta, 1 \leq i \leq n\}.$$

When $\delta = 1$, $\|p\|_{\mathcal{A},\delta}$ is denoted by $\|p\|_{\mathcal{A}}$, where $p = p(X_1, ..., X_n)$ is a polynomial [8]. However, a polynomial in this paper is non-commuting and there is no constant term for it. A product (or direct sum) of a family $\{\mathcal{A}_i\}_{i\in I}$ of Banach algebras, is defined as follows

$$\oplus_{i\in I}\mathcal{A}_i = \{(a_i) \in \prod_{i\in I} \mathcal{A}_i : \|(a_i)\| < \infty\}$$

where $\|(a_i)\| = \sup\{\|a_i\| : i \in I\}$ [6].
Obviously, $\oplus_{i\in I}\mathcal{A}_i$ is a Banach algebra under these pointwise operations
i) $(a_i) + (b_i) = (a_i + b_i)$
ii) $\mu(a_i) = (\mu a_i)$
iii) $(a_i)(b_i) = (a_i b_i)$.

A formal expression $\|p\| \leq K$ is called a law, where $K \in \mathbb{R}$ and p is a polynomial. A Banach algebra $\mathcal{A}$ satisfies the recent law if $\|p\|_{\mathcal{A}} \leq K$ and $\|p\|_{\mathcal{A}} \leq K$ is a homogeneous law if p is a homogeneous polynomial.
When we talk about universal algebras, a non-empty class V of complex associative algebras is a variety if it is closed under taking subalgebras, quotient algebras, direct sum and isomorphic images. Regarding [1] (or [3], p.169-170, Theorem1.3), Birkhoff has proved that a non-empty class of complex associative algebras V is a variety if and only if there is a set L of polynomials such that,

$$V = \{\mathcal{A} : p(x_1, ..., x_n) = 0, (x_1, ..., x_n \in \mathcal{A}), \forall p \in L\}.$$

[1]Mathematics Department, Bu-Ali Sina University, 65174-4161, Hamadan, Iran, e-mail: gh_khalilzadeh@yahoo.com
[2]Mathematics Department, Bu-Ali Sina University, 65174-4161, Hamadan, Iran,e-mail: amiri_takin@yahoo.com

Now, we talk about varieties of Banach algebras and introduce some useful results. Dixon defined varieties of Banach algebras and proved an analogue of Birkhoff's Theorem for Banach algebras.
If there exists a non-negative real-valued function on the set of all polynomials P as $p \longmapsto f(p)$ and V is precisely a class of Banach algebras $\mathcal{A}$ such that $\|p\|_{\mathcal{A}} \leq f(p)$, then V is said to be a variety where $p \in P$ [6]. A non-empty class V of Banach algebras is a variety if and only if it is closed under taking closed subalgebras, quotient algebras, products (or direct sums) and images under isometric isomorphisms [6].

Supose V is a varity and p a polynomial. Then, $|p|_V$ has been defined by [10] as follows

$$|p|_V = \sup\{\|p\|_{\mathcal{A}} : \mathcal{A} \in V\}.$$

Also, for a variety V and each polynomial p, there is found the following definition

$$|p|_V = \inf\{K_p : V \textit{ can be obtained by the laws } \{\|p\| \leq K_p\}_p \ \}$$

where $\{|p|_V\}_p$ is a family of laws which determine V.[10]

These results have been provided in [7]. An H-variety is a variety that is generated by a family of homogeneous laws. There exists an $\mathcal{A} \in V$ such that for all polynomial p, we have $|p|_V = \|p\|_{\mathcal{A}}$ where V is a variety and this theorem shows that $|p|_V$ is always obtaind and also, we have each variety of Banach algebras is singly generated. Let V_1, V_2 be two varietirs. Then, $V_1 \subseteq V_2$ if and only if $|p|_{V_1} \leq |p|_{V_2}$ for all polynomials p.

We note that, partially ordered by inclusion, the class of all varieties is a complete lattice.
Each variety is determined by a family of laws, but among such families one is particularly noteworthy; namely, the family of laws with minimal right-hand sides K. The function giving these right-hand sides is,

$$p \longmapsto |p|_V$$

we can compare the elements of the lattice of all varieties,

$$V = \{\mathcal{A} : \|p\|_{\mathcal{A}} \leq K_p, \textit{for all polynomial } p\}.$$

which is an analogue of Birkhoff's Theorem. We shall denote the unit ball of $\mathcal{A}$ by $\mathcal{A}_1$, where $\mathcal{A}$ is a Banach algebra. The class of all varieties is a complete lattice, and we shall denote the variety of all Banach algebras by 1. So, it is clear that if V is a variety of algebras, then $V \cap 1$ is a variety of Banach algebras. Take C as a class of Banach algebras, the intersection of all varieties containing C is called the variety generated by C and denoted by $V(C)$. If C has exactly single member as $\mathcal{A}$, then $V(C)$ is singly generated and denoted by $V(\mathcal{A})$.

If $\{\mathcal{A}_\alpha\}_\alpha$ is a family of Banach algebra and $\mathcal{A} = \prod_{\alpha \in I} \mathcal{A}_\alpha$ Then, the supremum of $\{V(\mathcal{A}_\alpha) : \alpha \in I\}$ in the lattice of varieties is $V(\mathcal{A})$.

A Banach algebra which is bi-continuously isomorphic with the quotient of a uniform algebra by a closed ideal of it, is a Q-algebra. If the isomorphism is isometric, then it is said to be an IQ-algebra. An R-algebra is a Banach algebra which is bicontinuously isomorphic to a closed subalgebra of $\mathcal{B}(H)$, where H is a separable infinite dimentional Hilbert space. When the isomorphic is an isometriy, it is said to be an IR-algebra.

Let $\mathcal{A}$ be a closed subalgebra (regarding the uniform norm) of the C*-algebra $C(X)$ where X is a compact Hausdorff topological space and $C(X)$ is the set of all continuous complex functions on X. If all of the constant functions are contained in $\mathcal{A}$ and for every distinct $x, y \in X$ there is $f \in \mathcal{A}$ with $f(x) \neq f(y)$ (in the other words,$\mathcal{A}$ separate points in X), then $\mathcal{A}$ is a uniform algebra.
As a closed subalgebra of the commutative Banach algebra $C(X)$, a uniform algebra itself is a unital commutative Banach algebra (when equipped with the uniform norm). Hence, it is a Banach function algebra. A considerable property about varieties is that they can be

compared by their laws. Also, it has been proved that each variety V has a Banach algebra $\mathcal{A} \in V$ as a generator such that for any polynomial p we have $\|p\|_{\mathcal{A}} = |p|_V$. So, V is singly generated meaning that each element of V is a quotient of a closed subalgebra of a direct sum of some copies of $\mathcal{A}$, up to isometric isomorphic). Each variety has many generators, but just one of them has maximum property, that is, a maximal algebra of the variety by the name core .
If C is a class of Banach algebras, then $S(C)$, $Q(C)$ and $P(C)$ are the classes of all Banach algebras which are isometrically isomorphic to closed subalgebras of Banach algebras in C, quotient algebras of Banach algebras in C and products of families of Banach algebras in C. Let C be a non-empty class of Banach algebras. Then, $V(C) = QSP(C)$.

For a non-empty class of Banach algebras as U, we denote the class of all Banach algebras which are bi-continuously isomorphic to members of U by $\widehat{U}$ [10].

The variety of all IQ-algebras and IR-algebras are generated by complex numbers ($\mathbb{C}$)and $\mathcal{B}(H)$, respectively where H is a separable infinite dimensional Hilbert space [6] (p.483–484).

The following concepts are required for our work, so they've been listed from[11]. For a conjugate-linear map $a \mapsto a^*$ on $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$ we have,

$$a^{**} = a$$

and

$$(ab)^* = b^* a^*.$$

where $\mathcal{A}$ the pair $(\mathcal{A}, *)$ is called an involution algebra, or *-algebra. A Banach *-algebra is a *-algebra $\mathcal{A}$ with a complete sub-multiplicative norm such that,

$$\|a^*\| = \|a\| \qquad (a \in \mathcal{A}).$$

A C*-algebra is a Banach *-algebra $\mathcal{A}$ with this property that,

$$\|a^* a\| = \|a\|^2 \qquad (a \in \mathcal{A}).$$

A closed *-subalgebra of a C*-algebra is obviously a C*-algebra. So, we shall call it a C*-subalgebra.
If $\{\mathcal{A}_i\}_{i \in I}$ is a family of C*-algebras, then the direct sum $\oplus_{i \in I} \mathcal{A}_i$ is a C*-algebra with the pointwise-defined involution. Let V be a family of C*-algebras. We say that V is a *-variety (variety of C*-algebras), if it is closed under taking direct sums, C*-subalgebras, quotients (by closed ideals) and *-isomorphisms. These results are chosen from [10]. Let V^* be a non-empty class of C*-algebras. Then, V^* is a *-variety if and only if there exists a family of laws $\{\|p\| \leq K_p\}_p$ such that

$$V^* = \{\mathcal{A} : \mathcal{A} \text{ is a } C^*\text{-algebra and } \|p\|_{\mathcal{A}} \leq K_p \ for\ all\ p\}.$$

Let V^*, W^* be two *-varieties. Then,
(i) V is the smallest variety of Banach algebras such that $V^* = V \cap 1^*$ where 1^* is the *-variety of all C*-algebras.
(ii) $V^* \subseteq W^*$ if and only if $V \subseteq W$. Let $\mathcal{A}$ be a C*-algebra and W be a variety of Banach algebras such that $W \subseteq V(\mathcal{A})$ but $W \neq V(\mathcal{A})$. Then, there exists a C*-algebra $\mathcal{B}$ such that, $\mathcal{B} \in V(\mathcal{A}) \backslash W$. Hereinafter, even we say *-variety it means a variety of C*-algebras.

2. Characterization of *-varieties

Definition 2.1. *Let Ω be a class of C*-algebras. Then:*
(i) $S^*\Omega$ is the class of all C-algebras that are *-isomorphic to C*-subalgebras of C*-algebras in Ω.*
(ii) $Q^*\Omega$ is the class of all C-algebras that are *-isomorphic to quotient algebras of C*-algebras in Ω.*

(iii) $P^*\Omega$ is the class of all C-algebras that are *-isomorphic to direct sums of families of C*-algebras in Ω.*

Lemma 2.1. *Let Ω be a non-empty class of C*-algebras. Then,*
(i)$S^*S^*\Omega = S^*\Omega$
(ii)$Q^*Q^*\Omega = Q^*\Omega$
(iii)$P^*P^*\Omega = P^*\Omega$.

Proof. (i) Straightforward.
(ii) By definition we have $\Omega \subseteq Q^*\Omega$ thus $Q^*\Omega \subseteq Q^*Q^*\Omega$. For reverse of inclusion, if $\mathcal{A} \in Q^*Q^*\Omega$, then, there exist $\mathcal{B} \in Q^*\Omega$ and closed ideal $I \trianglelefteq \mathcal{B}$. Also, there exists an *-isomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}/I$. Since $\mathcal{B} \in Q^*\Omega$ there is $E \in \Omega$, closed ideal $J \subseteq E$ and *-isomorphism $g : \mathcal{B} \longrightarrow E/J$. Since I is a closed ideal of $\mathcal{B}$, it is cocluded that $g(I)$ is also a closed ideal of E/J. Take

$$H = \{h \in E : h + J \in g(I)\}.$$

Then, H is a C*-subalgebra of E. If $\Pi : E \longrightarrow E/J$ is the quotient map, then it is continuous. Since Π is continuous, $H = \Pi^{-1}(g(I))$ is closed and it is an ideal of E. Let $h : E/H \longrightarrow \mathcal{A}$ be defined by,

$$h(e + H) = f^{-1}(g^{-1}(e + J) + I).$$

It is obvious that h is well-defined and an *-isomorphism of E/H onto $\mathcal{A}$, so $\mathcal{A} \in Q^*\Omega$.
(iii) Since $\Omega \subseteq P^*\Omega$, we have $P^*\Omega \subseteq P^*P^*\Omega$. If $\mathcal{A} \in P^*P^*\Omega$, then there exist $\{\mathcal{A}_i\}_{i\in I}$ a family of elements of $P^*\Omega$ and an *-isomorphism $f : \mathcal{A} \longrightarrow \oplus_{i\in I}\mathcal{A}$. Take $i \in I$, since $\mathcal{A} \in P^*\Omega$, there exists $\{\mathcal{A}_{(i,j)}\}_{j\in I_i}$ a family of elements of Ω and an *-isomorphism $g_i : \mathcal{A}_i \longrightarrow \oplus_{j\in I_i}\mathcal{A}_{(i,j)}$. Take $J = \{(i.j) : i \in I, j \in I_i\}$ and $h : \oplus_{(i,j)\in J}\mathcal{A}_{(i,j)} \longrightarrow \mathcal{A}$ is defined as follows:

$$h((a_{(i,j)})_{(i,j)\in J}) = f^{-1}(g_i^{-1}(a_{(i,j)})_{j\in I_i})_{i\in I}.$$

It is clear that h is an *- isomorphism of $\oplus_{(i,j)\in J}\mathcal{A}_{(i,j)}$ onto $\mathcal{A}$. So, $\mathcal{A} \in P^*\Omega$. □

Lemma 2.2. *Let Ω be a non-empty class of C*-algebras. Then,*
(i) $P^*S^*\Omega \subseteq S^*P^*\Omega$.
(ii) $P^*Q^*\Omega \subseteq Q^*P^*\Omega$.
(iii) $S^*Q^*\Omega \subseteq Q^*S^*\Omega$.

Proof. (i) Straightforward.
(ii) If $\mathcal{A} \in P^*Q^*\Omega$, then there exists a family $\{\mathcal{A}_i\}_{i\in I}$ of elements of $Q^*\Omega$ and an *-isomorphism $f : \mathcal{A} \longrightarrow \oplus_{i\in I}\mathcal{A}_i$. Take $i \in I$. Since $\mathcal{A}_i \in Q^*\Omega$, there exist $\mathcal{B}_i \in \Omega$, closed ideal $D_i \in \mathcal{B}_i$ and *-isomorphism $g_i : \mathcal{A}_i \longrightarrow \mathcal{B}_i/D_i$. Since D_i is a closed ideal of $\mathcal{B}_i$, it is concluded that, $\oplus_{i\in I}D_i$ is a closed ideal of $\oplus_{i\in I}\mathcal{B}_i$. Let $h : \oplus\mathcal{B}_i/\oplus D_i \longrightarrow \mathcal{A}$ be defined as follows,

$$h((b_i)_{i\in I} + \oplus D_i) = f^{-1}(g_i^{-1}(b_i + D_i))_{i\in I}.$$

Then, h is an *-isomorphism of $\oplus\mathcal{B}_i/\oplus D_i$ onto $\mathcal{A}$. So $\mathcal{A} \in Q^*P^*\Omega$.
(iii) Straightforward. □

Definition 2.2. *For each non-empty class of C*-algebras Ω, we denote by $V^*(\Omega)$ the smallest *-variety containing Ω.*

The following theorm shows that each *-variety can be characterized by means of the operators S^*, Q^* and P^*.

Proposition 2.1. *Let Ω be a non-empty class of C*-algebras. Then,*

$$V^*(\Omega) = Q^*S^*P^*\Omega.$$

Proof. By Lemmas 2.2 and 2.3, we have

$$Q^*Q^*S^*P^*\Omega = Q^*S^*P^*\Omega$$

and

$$\begin{aligned} S^*Q^*S^*P^*\Omega &\subseteq Q^*S^*S^*P^*\Omega \\ &= Q^*S^*P^*\Omega \end{aligned}$$

also

$$\begin{aligned} P^*Q^*S^*P^*\Omega &\subseteq Q^*P^*S^*P^*\Omega \\ &\subseteq Q^*S^*P^*P^*\Omega \\ &= Q^*S^*P^*\Omega. \end{aligned}$$

Therfore, $Q^*S^*P^*\Omega$ is a *-variety. But, if W^* is a *-variety containing Ω, then

$$Q^*S^*P^*\Omega \subseteq W^*$$

therefore

$$V^*(\Omega) = Q^*S^*P^*\Omega.$$

□

Corollary 2.1. *Every C*-algebra is a quotient of a *-subalgebra of a direct sum of some copies of* $\mathcal{B}(H)$.

Proof. We have

$$V^*(\mathcal{B}(H)) = V(\mathcal{B}(H)) \cap 1^* = 1^*$$

by [5]. So, by theorem 2.5, we have

$$1^* = Q^*S^*P^*(\mathcal{B}(H)).$$

□

3. C*-algebras and homogeneous polynomials

Each variety of Banach algebras is not a semi-variety. Also, each variety is not a homogeneous variety (that could be generated by homogeneous laws). But it is proved that each *-variety is a semi-variety, and also is a homogeneous variety that is a considerable difference between the variety of Banach algebras and the *-variety.

Theorem 3.1. *Let* $V^* = V^*(\mathcal{B})$ *be a *-variety. Then,*
(i) $V^* = \{\mathcal{A} : \mathcal{A} \textit{is a } C^*\textit{-algebra and } \|p\|_{\mathcal{A}} \le \|p\|_{\mathcal{B}} \textit{ for all } p\}$
(ii) $V^* = \{\mathcal{A} : \mathcal{A} \textit{ is a } C^*\textit{-algebra and there are } M, \delta > 0, \|p\|_{\mathcal{A},\delta} \le M.\|p\|_{\mathcal{B}} \textit{ for all } p\}$
(iii) $V^* = \{\mathcal{A} : \mathcal{A} \textit{ is a } C^*\textit{-algebra and there are } M, \delta > 0, \|p\|_{\mathcal{A},\delta} \le M.\|p\|_{\mathcal{B}} \textit{ when } p$ *is a homogeneous polynomial*$\}$
(iv) $V^* = \{\mathcal{A} : \mathcal{A} \textit{ is a } C^*\textit{-algebra and } \|p\|_{\mathcal{A}} \le \|p\|_{\mathcal{B}} \textit{ for all homogeneous } p\}$

Proof. (i) It is clear.
(ii) Let $\mathcal{A}$ be a C*-algebra such that for some $M, \delta > 0$, $\|p\|_{\mathcal{A},\delta} \le M.\|p\|_{\mathcal{B}}$ *for all p*. It is proved that $\mathcal{A} \in V^*$ (by the following modification of the proof of theorem 3.1). Take $X = \mathcal{B}_1^{\mathcal{A}_\delta}$ and $\Gamma = \mathcal{B}^X$. Let ξ_a, ξ_a^* and $\theta : U_0 \longrightarrow \mathcal{A}_\delta$ be defined as before (for each $a \in \mathcal{A}_\delta$). We have

$$\|p(a_1', ..., a_n')\| \le M.\|p(\xi_{a_1}', ..., \xi_{a_n}')\|.$$

So, θ is well-defined. Let $U = \overline{U}_0$, so $U/ker\theta$ is *-isomorphic with $\mathcal{A}$. Hence $\mathcal{A} \in V^*$. The converse is evident.
(iii) If p be a polynomial, we write p_i for the part of p which is homogeneous of degree i.Thus, we can write $p = p_1 + ... + p_m$ for some positive integer m. Let $\mathcal{A}$ be a C*-algebra such that

for some $M, \delta > 0$ and all homogeneous polynomials p, $\|p\|_{\mathcal{A},\delta} \leq M.\|p\|_{\mathcal{B}}$. We shall prove that $\mathcal{A} \in V^*$. Let p be a polynomial and i is a positive integer. We have $\|p_i\|_{\mathcal{A},\delta} \leq M.\|p_i\|_{\mathcal{B}}$. But $\|p_i\|_{\mathcal{B}} \leq \|p\|_{\mathcal{B}}$ (see[5]). So, $\|p_i\|_{\mathcal{A},\delta} \leq M.\|p\|_{\mathcal{B}}$, or $\|p_i\|_{\mathcal{A},\delta/2} \leq 2^{-i}M.\|p_i\|_{\mathcal{B}}$. Hence

$$\|p\|_{\mathcal{A},\delta/2} \leq \sum_{1}^{m} \|p_i\|_{\mathcal{A},\delta/2} \leq M.\|p\|_{\mathcal{B}}.$$

Thus $\mathcal{A} \in V^*$. The converse is evident.
(iv) Observe that we have
$V^* = \{\mathcal{A} : \mathcal{A}$is a C^*-algebra and $\|p\|_{\mathcal{A}} \leq \|p\|_{\mathcal{B}}$for all polynomials $p\}$
$\subseteq \{\mathcal{A} : \mathcal{A}$is a C^*-algebra and $\|p\|_{\mathcal{A}} \leq \|p\|_{\mathcal{B}}$for all homogeneous polynomials $p\}$
$\subseteq \{\mathcal{A} : \mathcal{A}$ is a C^*-algebra, there are $M, \delta > 0$ such that $\|p\|_{\mathcal{A},\delta} \leq M.\|p\|_{\mathcal{B}}$for all homogeneous polynomials $p\} = V^*$. Thus, the theorem is proved. □

Corollary 3.1. *Let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{B} \in \widehat{V(\mathcal{A})} \backslash V(\mathcal{A})$. Then $\mathcal{B}$ is not a C^*-algebra.*

Proof. By theorem 3.1, we have

$$V^*(\mathcal{A}) = V(\mathcal{A}) \cap 1^* = \widehat{V(\mathcal{A})} \cap 1^*.$$

□

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4. **cores of *-varieties**

In this section, we introduce the core of *-varieties and its uniqueness, like the one proposed in [9] for varieties of Banach algebras. Also, we give some examples of *-varieties and find their cores.

Definition 4.1. *Let V be a *-variety. A C^*-algebra $\mathcal{A}$ in V is its core if there are sequences, like $\{a_i\}_{i=1}^{\infty}$ and $\{a_i^*\}_{i=1}^{\infty}$ of members of $\mathcal{A}_1$ (where $\mathcal{A}_1$ is the unit ball of $\mathcal{A}$) such that the C^*-subalgebra generated by $\{a_1, a_2, ...\} \cup \{a_1^*, a_2^*, ...\}$ is dense in $\mathcal{A}$ and*

$$|p|_V = \|p(a_1, ..., a_n)\| = \|p(a_1^*, ..., a_n^*)\|$$

for all polynomial $p = p(X_1, ..., X_n)$.

Definition 4.2. *Suppose that $\mathcal{A}$ is a C^*-algebra. Then, we say that $\mathcal{A}$ has maximum property (m-property) if there are sequences $\{a_i\}_{i=1}^{\infty}$ and $\{a_i^*\}_{i=1}^{\infty}$ of members of $\mathcal{A}_1$ such that for each polynomial $p = p(X_1, ..., X_n)$ it is concluded that,*

$$\begin{aligned}\|p\|_{\mathcal{A}} &= \sup\{\|p(x_1, ..., x_n)\| : x_i \in \mathcal{A}\} \\ &= \|p(a_1, ..., a_n)\| \\ &= \|p(a_1^*, ..., a_n^*)\|\end{aligned}$$

Lemma 4.1. *Let $\mathcal{A}$ be a C^*-algebra, then*
*(i) If $\mathcal{A}$ is a core of any *-variety like V, then $\mathcal{A}$ has m-property.*
(ii) If $\mathcal{A}$ has m-property, then there exists a C^*-subalgebra of $\mathcal{A}$ which is a core of $V(\mathcal{A})$

Proof. (i) Suppose that $\mathcal{A}$ is a core of *-variety V. Then, there are sequences $\{a_i\}_{i=1}^{\infty}$ and $\{a_i^*\}_{i=1}^{\infty}$ of members of $\mathcal{A}_1$ such that for each polynomial $p = p(X_1, ..., X_n)$ we have

$$\|p\|_{\mathcal{A}} = \|p(a_1, ..., a_n)\| = \|p(a_1^*, ..., a_n^*)\|.$$

So,

$$\begin{aligned}\|p\|_{\mathcal{A}} &\leq |p|_V \\ &= \|p(a_1,...,a_n)\| = \|p(a_1^*,...,a_n^*)\| \\ &\leq \|p\|_{\mathcal{A}}.\end{aligned}$$

Thus, it is concluded that, $\|p\|_{\mathcal{A}} = \|p(a_1,...,a_n)\| = \|p(a_1^*,...,a_n^*)\|$ and it means that $\mathcal{A}$ has m-property.
(ii) If $\mathcal{A}$ has the m-property and $\{a_i\}_{i=1}^{\infty}, \{a_i^*\}_{i=1}^{\infty}$ are the sequences in definition5.2 , then we have

$$\begin{aligned}|p|_{V(\mathcal{A})} &= \|p\|_{\mathcal{A}} \\ &= \|p(a_1,...,a_n)\| = \|p(a_1^*,...,a_n^*)\|.\end{aligned}$$

Let $\mathcal{A}_0$ be the normed subalgebra of $\mathcal{A}$ generates by $\{a_i\}_{i=1}^{\infty}$ and $\{a_i^*\}_{i=1}^{\infty}$ and $\overline{\mathcal{A}_0}$ be its closure. Then, $\overline{\mathcal{A}_0} \in V(\mathcal{A})$. In addition, for all polynomial p we will have,

$$\begin{aligned}\|p(a_1,...,a_n)\|_{\overline{\mathcal{A}_0}} &= \|p(a_1^*,...,a_n^*)\|_{\overline{\mathcal{A}_0}} \\ &= \|p(a_1,...,a_n)\|_{\mathcal{A}} \\ &= \|p(a_1^*,...,a_n^*)\|_{\mathcal{A}} \\ &= |p|_{V(\mathcal{A})}\end{aligned}$$

So, $\overline{\mathcal{A}_0}$ is a core of $V(\mathcal{A})$. □

Theorem 4.1. *Each *-variety has a unique core (up to an isometric *-isomorphism).*

Proof. For each *-variety V, there exist C*-algebra $\mathcal{A}$ and sequences $\{a_i\}_{i=1}^{\infty}, \{a_i^*\}_{i=1}^{\infty}$ in $\mathcal{A}_1$ such that, for any polynomial $p(X_1,...,X_n)$ we see that,

$$\begin{aligned}\|p(a_1^*,...,a_n^*)\| &= \|p(a_1,...,a_n)\| \\ &\leq \|p\|_{\mathcal{A}} \\ &\leq \sup\{\|p\|_{\mathcal{A}} : \mathcal{A} \in V\} \\ &= \|p(a_1,...,a_n)\| = \|p(a_1^*,...,a_n^*)\|.\end{aligned}$$

therefore, $\mathcal{A}$ has m-property. Then, by the lemma 5.3, it has a core. Now suppose that V is a *-variety with two cores $\mathcal{A}$ and $\mathcal{B}$, so by definition 5.1, there are sequences $\{a_i\}_{i=1}^{\infty}, \{a_i^*\}_{i=1}^{\infty}$ for $\mathcal{A}$ and $\{b_i\}_{i=1}^{\infty}, \{b_i^*\}_{i=1}^{\infty}$ for $\mathcal{B}$ such that for each polynomial $p = p(X_1,...,X_n)$, we have,

$$|p|_V = \|p(a_1,...,a_n)\| = \|p(a_1^*,...,a_n^*)\|$$

and

$$|p|_V = \|p(b_1,...,b_n)\| = \|p(b_1^*,...,b_n^*)\|.$$

Let $\mathcal{A}_0, \mathcal{B}_0$ be normed subalgebras determined , respectively, by $\{a_i\}_{i=1}^{\infty}, \{a_i^*\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$, $\{b_i^*\}_{i=1}^{\infty}$ and mapping $Q : \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ defined by,

$$\begin{aligned}Q(p(a_1,...,a_n)) &= p(b_1,...,b_n) \\ Q(p(a_1^*,...,a_n^*)) &= p(b_1^*,...,b_n^*)\end{aligned}$$

where $p(X_1,...,X_n)$ is a polynomial. The mapping Q is well-defined because, if $p(a_1,...,a_n) = 0$ and $p(a_1^*,...,a_n^*) = 0$, then we have $\|p(b_1,...,b_n)\| = 0$ and $\|p(b_1^*,...,b_n^*)\| = 0$, hence $p(b_1,...,b_n) = 0$ and $p(b_1^*,...,b_n^* = 0$. Also, this mapping is a homomorphism of $\mathcal{A}_0$ onto $\mathcal{B}_0$ and it is clear that Q is an isometry, so $\mathcal{A}_0, \mathcal{B}_0$ are isometrically isomorphic. But we know $\overline{\mathcal{A}_o} = \mathcal{A}$ and $\overline{\mathcal{B}_o} = \mathcal{B}$, thus $\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$. □

Example 4.1. *Let $\mathbf{N}_n$ be the *-variety defined by the law*

$$\|X_1...X_n\| = \|X_1^*...X_n^*\| = 0$$

for each $n \in \mathbb{N}$, namely the set of all nilpotent C-algebras of class n. Let $\mathcal{F}_n$ be the free C*-algebra on symbols $O_1, O_2, ...$ and $O_1^*, O_2^*, ...$ subjected to the relations*

$$\|O_{i_1}...O_{i_n}\| = \|O_{i_1}^*...O_{i_n}^*\| = 0$$

for all $i_1, ..., i_n \in \mathbb{N}$. Let $\{U_n\}_{n=1}^{\infty}, \{U_n^*\}_{n=1}^{\infty}$ be the fixed enumeration of the non-zero words in $\{O_1, O_2, ...\}$ and $\{O_1^*, O_2^*, ...\}$ and $\mathcal{A}_n$ be the C-algebra of all infinite series $x = \sum \alpha_n U_n$ and $y = \sum \beta_n U_n^*$ where $\sum |\alpha_n| < \infty$ and $\sum |\beta_n| < \infty$. Then, $\mathcal{A}_n$ is a nilpotent C*-algebra of class n with norm defined by*

$$\|x\| = \sum |\alpha_n|, \|y\| = \sum |\beta_n|.$$

If there are sequences $\{O_n\}_{n=1}^{\infty}, \{O_n^*\}_{n=1}^{\infty}$ of elements of $(\mathcal{A}_n)_1$ such that, for any polynomial $p = p(X_1, ..., X_n)$ we have,

$$|p|_{\mathbf{N}_n} = \|p(O_1, ..., O_m)\| = \|p(O_1^*, ..., O_m^*)\|.$$

whereas, members of $\mathcal{A}_n$ are finite serie, Therefore $\mathcal{A}_n$ is the core of $\mathbf{N}_n$.

Example 4.2. *Suppose that 1^* is the *-variety of all C*-algebras. Also, let F be the free algebra on symbols $X_1, X_2, ..., X_1^*, X_2^*, ...$ and $\{w_n\}_{n=1}^{\infty}, \{w_n^*\}_{n=1}^{\infty}$ be fixed enumeration of the words on the alphabet $\{X_1, X_2, ...\}$ and $\{X_1^*, X_2^*, ...\}$. Let $\mathcal{A}$ be the C*-algebra of all infinite series $x = \sum \alpha_n w_n$, $y = \sum \beta_n w_n^*$ such that, $\sum |\alpha_n| < \infty$, $\sum |\beta_n| < \infty$. Then, $\mathcal{A}$ with the norm defined by*

$$\|x\| = \sum |\alpha_n|, \|y\| = \sum |\beta_n|$$

is a C-algebra. Then $\{X_n\}_{n=1}^{\infty}$ and $\{X_n^*\}_{n=1}^{\infty}$ are sequences of elements of $\mathcal{A}_1$ such that for all polynomials $p(Y_1, ..., Y_n)$ we have,*

$$|p|_{1^*} = \|p(X_1, ..., X_n)\| = \|p(X_1^*, ..., X_n^*)\|.$$

*If $\mathcal{A}_0$ is a *-subalgebra of $\mathcal{A}$ that is generated by $\{X_1, X_2, ...\}$ and $\{X_1^*, X_2^*, ...\}$ then $\mathcal{A}_0$ is dense in $\mathcal{A}$. It is seen that, any *-variety is a homogeneous variety and we can suppose*

$$p(X_1, X_2, ...) = \sum a_{i_1 i_2 ...} X_{i_1} X_{i_2} ...$$

is an element of $\mathcal{A}$, so we have $\sum |a_{i_1 i_2 ...}| < \infty$.
If $\epsilon > 0$, then, we have $N > 0$ such that for all $k > N$, $\sum_{m>k} |a_{i_1 ... i_k}| < \epsilon$.
If $p_k(X_1, X_2, ...) = \sum ai_1...i_k X_{i_1}...X_{i_k}$, then for all $k \in \mathbb{N}$ we have $p_k \in \mathcal{A}_0$ and $\|p - p_k\| \longrightarrow 0$. So $p_k \longrightarrow p$ and this means that $\overline{\mathcal{A}_0} = \mathcal{A}$, therefore $\mathcal{A}$ is the core of 1^*.

Corollary 4.1. *The cores of *-varieties of IR-algebras and IQ-algebras are defined as above.*

It is seen that, the core of any *-variety is unique and it has the maximum property. Now, we try to find more about cores of *-varieties, so at first, we present the following theorem that shows the relationship between a generator and the core of a *-variety.

Theorem 4.2. *Let $\mathcal{A}$ be a C*-algebra, and $V^* = V^*(\mathcal{A})$ (the *-variety generated by $\mathcal{A}$). Then, the core of V^* is a closed C*-subalgebra of direct sum of some copies of $\mathcal{A}$ (up to an isometric *-isomorphism).*

Proof. By theorem 5.4, V^* has a unique core, namely M. Let $\{a_n\}_{n=1}^{\infty}, \{a_n^*\}_{n=1}^{\infty}$ be sequences of elements of M_1 such that, for each polynomial $p = p(X_1, ..., X_n)$ we have

$$\overline{M_0} = \overline{\{p(a_1, ..., a_n)\}_p} = M$$

and

$$\overline{M_0} = \overline{\{p(a_1^*, ..., a_n^*)\}_p} = M$$

also we have,

$$|p|_{V^*} = \|p(a_1, ..., a_n)\|$$

and

$$|p|_{V^*} = \|p(a_1^*, ..., a_n^*)\|.$$

Let $X = {\mathcal{A}_1}^{M_0}$ and $\Gamma = \mathcal{A}^X$. Since $\mathcal{A} \in X$, for each $1 \leq i < \infty$ we have $\Gamma \in V$. Now suppose that $\xi_{a_i} : X \longrightarrow \mathcal{A}$ are defined by,

$$\xi_{a_i}(x) = x(a_i) \qquad (x \in X)$$

and $\xi_{a_i^*} : X \longrightarrow \mathcal{A}$ are defined by

$$\xi_{a_i^*}(x) = x(a_i^*) \qquad (x \in X).$$

So, we have $\xi_{a_i}, \xi_{a_i^*} \in \Gamma$ and $\|\xi_{a_i}\|_\infty, \|\xi_{a_i^*}\|_\infty \leq 1$ for each $1 \leq i < \infty$. Let U_0 be the *-subalgebra of Γ determined by $\{\xi_{a_i}\}_{i=1}^\infty$ and $\{\xi_{a_i^*}\}_{i=1}^\infty$. If $\Theta : U_0 \longrightarrow M_0$ is the *-homomorphism defined by,

$$\Theta(\xi_{a_i}) = a_i$$

and

$$\Theta(\xi_{a_i^*}) = a_i^*.$$

Take $u = p(\xi_{a_1}, ..., \xi_{a_n}), u^* = p(\xi_{a_1^*}, ..., \xi_{a_n^*})$. Then, $\Theta(u) = p(a_1, ..., a_n), \Theta(u^*) = p(a_1^*, ..., a_n^*)$, and also we have,

$$\begin{aligned}\|\Theta(u)\| &= \|p(a_1, ..., a_n)\|_{M_0} \\ &= |p|_{V^*} \\ &= \|p\|_{\mathcal{A}} \\ &= \sup\{\|p(x(a_1), ..., x(a_n))\|_{\mathcal{A}} : x \in X\} \\ &= \sup\{\|p(\xi_{a_1}, ..., \xi_{a_n})(x)\| : x \in X\} \\ &= \sup\{\|u(x)\| : x \in X\} \\ &= \|u\|\end{aligned}$$

and

$$\begin{aligned}\|\Theta(u^*)\| &= \|p(a_1^*, ..., a_n^*)\|_{M_0} \\ &= |p|_{V^*} \\ &= \|p\|_{\mathcal{A}} \\ &= \sup\{\|p(x(a_1^*), ..., x(a_n^*))\|_{\mathcal{A}} : x \in X\} \\ &= \sup\{\|p(\xi_{a_1^*}, ..., \xi_{a_n^*})(x)\| : x \in X\} \\ &= \sup\{\|u^*(x)\| : x \in X\} \\ &= \|u^*\|.\end{aligned}$$

So, $\Theta : U_0 \longrightarrow M_0$ is an isometrically isomorphism. Thus, $\overline{U_0}$ and $\overline{M_0} = M$ are isometrically isomorphic. We saw that $\overline{U_0}$ is a closed C*-subalgebra of $\mathcal{A}^X$ and this completes the proof. □

Corollary 4.2. *Let V be a *-variety and $D \neq \{0\}$ be the core of V. Then, we can choose $\{a_i\}_{i=1}^\infty, \{a_i^*\}_{i=1}^\infty$ of elements of D_1 such that, for all $i \in \mathbb{N}$, we have $\|a_i\| = 1$, $\|a_i^*\| = 1$ and for all polynomial $p = p(X_1, ..., X_n)$,*

$$|p|_V = \|p(a_1, ..., a_n)\| = \|p(a_1^*, ..., a_n^*)\|$$

and also $\overline{D_0} = D$, where D_0 is the C-subalgebra generated by $\{a_i\}_{i=1}^\infty$ and $\{a_i^*\}_{i=1}^\infty$.*

Proof. By the theorem 4.8, if $\mathcal{A} \neq \{0\}$, then $\|\xi_{a_i}\|_\infty = 1$ for each $1 \leq i < \infty$. □

Corollary 4.3. *Let V be the variety of all IQ-algebras. Then, there exists a set X such that, the core of V is a *-subalgebra of $\mathbb{C}^X$, where $\mathbb{C}$ is the set of all complex numbers.*

Proof. By theorem 4.8 and this fact that $V = V(\mathbb{C})$, the proof is obvious. □

Corollary 4.4. *For each C^*-algebra $\mathcal{A}$, there exists a *-subalgebra $\mathcal{B}$ of direct sum of some copies of $\mathcal{A}$ and sequences $\{a_i\}_{i=1}^{\infty}, \{a_i^*\}_{i=1}^{\infty}$ of members of $\mathcal{B}_1$ (or with norms one) such that for each polynomial $p(X_1, ..., X_n)$, we have*

$$\sup\{\|p(x_1, ..., x_2)\| : x_i \in \mathcal{B}_1\} = \|p(a_1, ..., a_n)\| = \|p(a_1^*, ..., a_n^*)\|.$$

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