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KSGNS TYPE REPRESENTATIONS ON KREIN C^*-MODULES ASSOCIATED WITH PROJECTIVE J- COVARIANT (α) - COMPLETELY POSITIVE MAPS

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In this paper we construct a projective covariant J-representation associated to a unital projective covariant completely positive linear map. We give the projective covariant version of KSGNS type representation on a Krein C^*-module associated to a unital projective covariant α-completely positive map and then we prove that there is a projective covariant J-representation of a crossed product of a C^*-algebra by a locally compact group.

Keywords: Hilbert C^*- modules, Krein C^*- modules, C^*- algebras, C^*- dynamical systems, J- representations, projective J- unitary representations, projective covariant J- representations, projective J- covariant completely α- positive linear maps, crossed products.

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1. Introduction

The GNS (Gel'fand-Naimark-Segal) representation theorem is one of the most useful theorems frequently applied to mathematical physics. The GNS construction applied to an invariant state, gives a cyclic covariant representation with an invariant cyclic vector. To every positive linear functional on a C^*-algebra A can be associated a cyclic representation on a Hilbert space H by GNS construction. In [19], Stinespring extended this theorem for a completely positive linear map from A into $\mathcal{B}(H)$, the C^*-algebra of linear bounded operators on a Hilbert space H, in order to obtain a representation of A on another Hilbert space K. Stinespring's dilation theorem is one of the fundamental and important results for the study of operator algebras and mathematical physics. In particular, this theorem is the basic structure theorem for quantum channels: it states that any quantum channel arises from a unitary evolution on a larger system. On the other hand, Paschke [16] (respectively, Kasparov [12]) showed that a completely positive linear map from A to another C^*-algebra of all adjointable operators on the Hilbert C^*-module H_B) induces a $*$-representation of A on a Hilbert B-module. Kaplan introduced in [11] the notion of multi-positive (or n-positive) linear functional on a C^*-algebra A and proved that a multi-positive linear functional on a C^*-algebra induces a $*$-representation of this C^*-algebra on a Hilbert space in terms of the GNS construction. Representations on Hilbert spaces

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are naturally generalized to representations on Hilbert C^*-modules. Heo, combined in [5] these two constructions to obtain a representation of A on a Hilbert C^*-module for completely multi-positive linear maps from A to another C^*-algebra. Using this, he obtained a representation on a Hilbert C^*-module associated with completely bounded linear maps. By KSGNS (Kasparov-Stinespring-Gel'fand-Naimark-Segal) construction [15], to a strictly completely positive map ρ from a C^*-algebra A on a Hilbert C^*-module F over a C^*-algebra B can be associated a triple $(F_\rho, \pi_\rho, v_\rho)$ consisting of a Hilbert B-module F_ρ, a $*$-homomorphism $\pi_\rho\colon A \to \mathcal{L}_B(F_\rho)$ and an adjointable operator $v_\rho\colon F \to F_\rho$ which is unique up to a unitary equivalence. If $F = B = \mathbb{C}$, then the KSGNS construction reduces to the classical GNS construction. If $B = \mathbb{C}$ (so F is a Hilbert space), then we get the Stinespring construction. In the context of Hilbert C^*-modules the construction was given by Kasparov. In [10], Joiţa extended KSGNS construction for strict continuous completely multi-positive linear maps from a locally C^*-algebra A to $\mathcal{L}_B(E)$, the C^*-algebra of all adjointable B-module morphisms from E into E, and showed in Theorem 4.3, [10] a covariant version of this construction. In Theorem 1, [2] we prove a projective generalization of this construction. We found a projective covariant representation of a unital C^*-dynamical system (G, A, α) on a Hilbert C^*-module associated with a unital completely positive projective u-covariant linear map, extending Stinespring's theorem to Hilbert C^*-modules and we constructed a projective covariant representation on a Hilbert C^*-module associated with a completely multi-positive projective u-covariant linear map. In [3] we proved a covariant projective version of the Stinespring theorem in terms of Hilbert C^*-modules. We also presented an extension of a projective u-covariant completely positive linear map on the twisted crossed product to a unique completely positive linear map and we proved the KSGNS construction associated with a projective u-covariant completely positive linear map. Krein spaces arise naturally in situations where the indefinite inner product has an analytically useful property (such as Lorentz invariance) which the Hilbert inner product lacks. It is known that in massless quantum field theory the state space may be a space with an indefinite metric. Motivated by this physical fact, many people extended the GNS construction to Krein spaces. More generally, Heo, Hong and Ji introduced in [6] the notion of α-completely positive linear maps between C^*-algebras as a natural generalization of completely positive maps between C^*-algebras and constructed a Stinespring type covariant representation for a pair of a covariant completely positive map ρ and a covariant ρ-map. They established the KSGNS construction on Krein C^*-modules for a C^*-algebra with a α-completely positive map of which the construction leads to J-representation of a C^*-algebra. In [7], Heo, Ji and Kim constructed a KSGNS type covariant representation for a pair of a covariant α-completely positive map ρ on a C^*-algebra and a covariant ρ-map on Krein C^*-modules associated to α-completely positive maps. They also gave a covariant J-representation of a crossed product of a C^*-algebra by a locally compact group and a covariant map on the crossed product of a Hilbert C^*-module by a locally compact group. As a generalization of covariant completely positive maps, Heo, Ji and Kim [8] considered (projective) covariant α-completely positive maps between locally C^*-algebras. They studied (projective) covariant J-representations of locally C^*-algebras on Krein modules over locally C^*-algebras, constructed a covariant KSGNS type representation associated with a covariant α-completely positive map on a locally C^*-algebra and studied extensions to a locally C^*-crossed product of α-completely positive maps on a locally C^*-algebra. The results provided (projective) covariant representations of a locally C^*-crossed product on a Krein module over a locally C^*-algebra. In [9], Heo introduced the notion of a (covariant) α-completely positive map of a topological group into a (locally) C^*-algebra, which is a counterpart of a (covariant) α-completely positive linear map between (locally) C^*-algebras. He constructed a (covariant) KSGNS type representation of a group on a Krein module over a (locally) C^*-algebra, which is associated to a (covariant) α-completely positive map of a group (system).

In this paper we extend Theorem 3, [2] on Krein C^*-modules and construct a projective covariant J-representation associated to a unital projective covariant completely positive linear map. In Section 3 we give the projective covariant version of KSGNS type representation on a Krein C^*-module associated to a unital projective covariant α-completely positive map (Theorem 3.1), as a generalization of Theorem 2.4, [7]. In Section 4 we construct a projective covariant J-representation of a crossed product of a C^*-algebra (Theorem 4.1).

2. The projective covariant version of KSGNS type representation on a Krein C^*-module associated to a unital projective covariant completely positive map

We remind some definitions and notations that will be used throughout the paper.

Definition 2.1. *([19], [1]) Let A and B be two C^*-algebras and let $M_n(A)$, respectively $M_n(B)$ denote the $*$-algebra of all $n \times n$ matrices over A, respectively B with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A, respectively B. A linear map $\rho\colon A \to B$ is* ***completely positive*** *if the linear map $\rho^{(n)}\colon M_n(A) \to M_n(B)$, defined by $\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$ is positive for all positive integers n. We say that ρ is* ***unital*** *if $\rho(1_A) = 1_B$, where 1_A, respectively 1_B is the unit of A, respectively B.*

Definition 2.2. *([14]) Let G be a locally compact group with identity e and let $\mathbb{T}$ be the group of complex numbers of modulus one. A* ***multiplier*** *ω of G is a function $\omega\colon G \times G \to \mathbb{T}$ with the properties :*

i) $\omega(x, e) = \omega(e, x) = 1$ *for all* $x \in G$;
ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ *for all* $x, y, z \in G$.

Definition 2.3. *([13]) A multiplier is* ***normalized*** *if $\omega(x, x^{-1}) = 1$ for all $x \in G$.*

Let G be a locally compact group and let A be a C^*-algebra. Let E be a Hilbert C^*-module over A and let $\mathcal{L}_A(E)$ be the Banach space of all adjointable module homomorphisms from E to E.

Definition 2.4. *([2]) A* ***projective unitary representation*** *of G on E with multiplier ω is a map $u\colon G \to \mathcal{L}_A(E)$ such that:*

i) u_s *is a unitary element in* $\mathcal{L}_A(E)$ *for all* $s \in G$*;*
ii) $u_{st} = \omega(s,t)u_s u_t$ *for all* $s,t \in G$*.*

If A is a C^*-algebra, an automorphism of A is an isomorphism of complex vector spaces $\nu\colon A \to A$ such that

(1) $\nu(a^*) = \nu(a)^*$ for all $a \in A$;
(2) $\nu(ab) = \nu(a)\nu(b)$ for all $a, b \in A$;
(3) $\nu(I) = I$, where I is the identity of A.

The set of all automorphisms of A is denoted by $Aut(A)$; it is a group in a natural way, the group operation being a composition of mappings, named the group of automorphisms of A. If A is a C^*-algebra, then A is semisimple with unique topology in the sense of [17] and thus any automorphism of A is continuous in the norm topology of A. Hence $Aut(A) \subset \mathcal{L}(A)$ and may equip $Aut(A)$ with the relative topology, the uniform topology. In this topology $Aut(A)$ becomes a topological group. ([18])

Definition 2.5. *(*[20]*) A C^*-**dynamical system** is a triple (G, A, θ), where G is a locally compact group, A is a C^*- algebra and θ is a continuous action of G on A, i.e. a continuous homomorphism $\theta\colon G \to Aut(A)$.*

Definition 2.6. *(*[2]*) Let (G, A, θ) be a C^*-dynamical system and let u be a projective unitary representation of G on a Hilbert B-module E with multiplier ω. We say that a completely positive linear map ρ from A into $\mathcal{L}_B(E)$ is **projective u-covariant** with respect to the C^*-dynamical system (G, A, θ) if $\rho(\theta_s(a)) = u_s\rho(a)u_s^*$ for all $a \in A$ and $s \in G$.*

Definition 2.7. *(*[6]*) Let A be a C^*-algebra and let E be a Hilbert A-module. Suppose that a symmetric operator J on E (i.e. $J = J^* = J^{-1}$) is given to produce an A-valued indefinite inner product $\langle x, y\rangle_J = \langle x, Jy\rangle$, $x, y \in E$. (E, J) is called a **Krein A-module**.*

For each $T \in \mathcal{L}(E)$, there is an operator $T^J \in \mathcal{L}(E)$ such that $\langle T\xi, \eta\rangle_J = \langle \xi, T^J\eta\rangle_J$, $\xi, \eta \in E$ and then T^J is called the **J-adjoint** of T. It can easily seen that $T^J = JT^*J$. Let B be a C^*-algebra and let (F, J) be a Krein B-module. We denote by $\mathcal{U}_J(F)$ the set of all J-unitary operators in $\mathcal{L}(F)$, i.e for each $s \in G$, $v_s^J v_s = v_s v_s^J = I$, which is equivalent to $v_s^* = Jv_{s^{-1}}J$ or $v_s^J = v_{s^{-1}}$. $\mathcal{U}_J(F)$ is called **the J-unitary group**.

Definition 2.8. *A **projective J-unitary representation** of a locally compact group G into the J-unitary group $\mathcal{U}_J(F)$ is a map $v\colon G \to \mathcal{U}_J(F)$ that satisfies the following properties:*

(i) $v_s^* = Jv_{s^{-1}}J$ *or* $v_s^J = v_{s^{-1}}$;
(ii) $v_{st} = \omega(s,t)v_s v_t$ *for all* $s, t \in G$

Definition 2.9. *Let (G, A, θ) be a C^*-dynamical system, B a C^*-algebra, (F, J) a Krein B-module and u a projective J-unitary representation of G into $\mathcal{U}_J(F)$. We say that a completely positive linear map ρ from A to $\mathcal{L}(F)$ is **projective J-covariant** with respect to the C^*-dynamical system (G, A, θ) if $\rho(\theta_g(a)) = u_g\rho(a)u_g^J$ for all $a \in A$ and $g \in G$.*

Definition 2.10. *([6]) Let A and B be C^*-algebras, F a Hilbert B-module and (F, J) a Krein B-module. A homomorphism $\pi\colon A \to \mathcal{L}(F)$ is called a* ***representation*** *of A on the Hilbert C^*-module F. A* ***$*$-representation*** *$\pi\colon A \to \mathcal{L}(F)$ of A on the Hilbert B-module F is a representation of A such that $\pi(a^*) = \pi(a)^*$, $a \in A$. A representation $\pi\colon A \to \mathcal{L}(F)$ of A on the Hilbert B-module F is called a* ***J-representation*** *on the Krein B-module (F, J) if π is a representation of A on the Hilbert C^*-module F and $\pi(a^*) = \pi(a)^J = J\pi(a)^*J$, $a \in A$.*

Definition 2.11. *A* ***projective covariant J-representation*** *of a C^*-dynamical system (G, A, θ) on a Krein B-module (F, J) is a triple $(\pi, v, (F, J))$, where π is a J-representation of A on (F, J) and v is a projective J-unitary representation of G into $\mathcal{U}_J(F)$ such that the (θ, v)-covariance property holds: $\pi(\theta_s(a)) = v_s\pi(a)v_s^J$ for all $a \in A$ and $s \in G$.*

The following result is the extension on Krein modules of Theorem 1, [2].

Theorem 2.1. *Let (G, A, θ) be a unital C^*-dynamical system such that $\theta_s = I$(=the identity map on A), for all $s \in G$, B a C^*-algebra, (F, J) a Krein B-module and u a projective unitary representation of G on F with normalized multiplier ω. If $\rho\colon A \to \mathcal{L}(F)$ is a unital projective u-covariant completely positive linear map, then there are a Krein module (K, J), a J-representation π of A on the Krein module (K, J), a projective J-unitary representation v of G into $\mathcal{U}_J(K)$ with multiplier ω and an isometry $V : F \to K$ such that:*

i) *$\rho(a) = V^*\pi(a)V$ for all $a \in A$;*
ii) *$u_s = V^*v_sV$ for all $s \in G$;*
iii) *the (θ, v)-covariance property holds: $\pi(\theta_s(a)) = v_s\pi(a)v_s^J$, for all $a \in A$ and $s \in G$.*

Proof. Following the proof of Lemma 4.1, [6], we form the algebraic tensor product $A \otimes_{alg} F$ and endow it with a pre-inner product by setting $\langle a \otimes \xi, b \otimes \eta\rangle_{A\otimes_{alg}F} = \langle \xi, \rho(\theta_s(a^*)b)\eta\rangle_F$. To obtain K we divide $A \otimes_{alg} F$ by the kernel $N = \left\{z \in A \otimes_{alg} F | \langle z, z\rangle_{A\otimes_{alg}F} = 0\right\}$ of $\langle\cdot,\cdot\rangle_{A\otimes_{alg}F}$ and complete. K becomes a Hilbert B-module. For notational convenience, we use the notation $a\dot{\otimes}\xi$ for the element $a\otimes\xi+N$ of $A\otimes_{alg}F/N$. Now we define a B-valued indefinite inner-product $[\cdot,\cdot]$ on $A\otimes_{alg}F/N$ by $[a\dot{\otimes}\xi, b\dot{\otimes}\eta] = \langle\xi, \rho(a^*b)\eta\rangle_F$ for any $a, b \in A$ and $\xi, \eta \in F$. By construction, for any $f, g \in A\otimes_{alg}F$, we have $\langle(\theta_s \otimes I)f, g\rangle_{A\otimes_{alg}F} = \langle(\theta_s \otimes I)(a \otimes \xi), b \otimes \eta\rangle_{A\otimes_{alg}F} = \langle\theta_s \otimes \xi, b \otimes \eta\rangle_{A\otimes_{alg}F} = \langle\xi, \rho(\theta_s(\theta_s(a)^*)b)\eta\rangle_F = \langle\xi, \rho(\theta_s(\theta_s(a^*))b)\eta\rangle_F = \langle\xi, \rho(a^*b)\eta\rangle_F = [a\dot{\otimes}\xi, b\dot{\otimes}\eta] = [f, g]$ and by the continuity of θ, the inner product $\langle\cdot,\cdot\rangle_K$ and the indefinite inner product $[\cdot,\cdot]$, we have $[f, g] = \langle Jf, g\rangle_K$, $f, g \in K$ for $J = \theta_s \otimes I$. Therefore, J is symmetric on K, so (K, J) is a Krein B-module. The isometry $V\colon F \to K$ is defined by $V\xi = 1_A\dot{\otimes}\xi$ for all $\xi \in F$. It is easy to check that $V^*\colon K \to F$ is given by $V^*(a\dot{\otimes}\xi) = \rho(a)\xi$. The representation π of A on K is defined by $\pi(a)(b\dot{\otimes}\xi) = (ab)\dot{\otimes}\xi$ for all $\xi \in F, a, b \in A$. For any $a', b \in A$ and $\xi, \eta \in F$, we obtain $\left\langle\pi(a)(a'\dot{\otimes}\xi), b\dot{\otimes}\eta\right\rangle_K = \left\langle(aa')\dot{\otimes}\xi), b\dot{\otimes}\eta\right\rangle_K = \langle(aa') \otimes \xi), b \otimes \eta\rangle_{A\otimes_{alg}F} = \langle\xi, \rho(\theta_s((aa')^*)b)\eta\rangle_F = \left\langle\xi, \rho(\theta_s(a'^*)\theta_s(a^*)\theta_s^2(b))\eta\right\rangle_F = \langle\xi, \rho(\theta_s(a'^*)\theta_s(a^*\theta_s(b)))\eta\rangle_F =$

$\langle a'\otimes\xi,\theta_s(a^*\theta_s(b))\otimes\eta\rangle_{A\otimes_{alg}F}=\left\langle a'\dot{\otimes}\xi,\theta_s(a^*\theta_s(b))\dot{\otimes}\eta\right\rangle_K=$
$\left\langle a'\dot{\otimes}\xi,(\theta_s\otimes I)(a^*\theta_s(b)\dot{\otimes}\eta\right\rangle_K=\left\langle a'\dot{\otimes}\xi,(\theta_s\otimes I)\pi(a^*)\theta_s(b)\dot{\otimes}\eta\right\rangle_K=$
$\left\langle a'\dot{\otimes}\xi,(\theta_s\otimes I)\pi(a^*)(\theta_s\otimes I)(b\dot{\otimes}\eta\right\rangle_K$ and so $\pi(a)$ is adjointable and $\pi(a)^*=(\theta_s\otimes I)\pi(a^*)(\theta_s\otimes I)=J\pi(a^*)J$, $a\in A$, hence π is a J-representation. Let $a\in A$ and $\xi\in F$. We have $V^*\pi(a)V\xi=V^*\pi(a)(1_A\dot{\otimes}\xi)=V^*(a1_A\dot{\otimes}\xi)=V^*(a\dot{\otimes}\xi)=\rho(a)\xi$. So i) is proved. We define $v\colon G\to\mathcal{U}_J(K)$ by setting $v_s(a\dot{\otimes}\xi)=\theta_s(a)\dot{\otimes}u_s\xi$ for all $a\in A,s\in G,\xi\in F$. For $a,b\in A$ and $\xi,\eta\in F$, we have $\left\langle v_s(a\dot{\otimes}\xi),b\dot{\otimes}\eta\right\rangle_K=\left\langle\theta_s(a)\dot{\otimes}u_s\xi,b\dot{\otimes}\eta\right\rangle_K=$
$\langle\theta_s(a)\otimes u_s\xi,b\otimes\eta\rangle_{A\otimes_{alg}F}=$
$\langle u_s\xi,\rho(\theta_s(\theta_s(a)^*)b)\eta\rangle_F=\langle u_s\xi,\rho(\theta_s(\theta_s(a^*))\theta_s(\theta_s(b)))\eta\rangle_F=$
$\langle\xi,u_s^*\rho(\theta_s(\theta_s(a^*)\theta_s(b)))\eta\rangle_F=\langle\xi,u_s^*\rho(\theta_s(\theta_s(a^*b)))\eta\rangle_F=\langle\xi,\rho(\theta_s(a^*b))u_{s^{-1}}\eta\rangle_F$. On the other hand, we have $\left\langle a\dot{\otimes}\xi,Jv_{s^{-1}}J(b\dot{\otimes}\eta)\right\rangle_K=$
$\langle a\otimes\xi,Jv_{s^{-1}}J(b\otimes\eta)\rangle_{A\otimes_{alg}F}=\langle a\otimes\xi,(\theta_s\otimes I)v_{s^{-1}}(\theta_s\otimes I)(b\otimes\eta)\rangle_{A\otimes_{alg}F}=$
$\langle a\otimes\xi,(\theta_s\otimes I)v_{s^{-1}}(\theta_s(b)\otimes\eta)\rangle_{A\otimes_{alg}F}=$
$\langle a\otimes\xi,(\theta_s\otimes I)(\theta_{s^{-1}}(\theta_s(b))\otimes u_{s^{-1}}\eta)\rangle_{A\otimes_{alg}F}=$
$\langle a\otimes\xi,(\theta_s\otimes I)(b\otimes u_{s^{-1}}\eta)\rangle_{A\otimes_{alg}F}=\langle a\otimes\xi,\theta_s(b)\otimes u_{s^{-1}}\eta\rangle_{A\otimes_{alg}F}=$
$\langle\xi,\rho(\theta_s(a^*)\theta_s(b))u_{s^{-1}}\eta\rangle_F=\langle\xi,\rho(\theta_s(a^*b))u_{s^{-1}}\eta\rangle_F$. Therefore, $v_s^*=Jv_{s^{-1}}J$, which means that v_s is a J-unitary representation. We show now that v is a projective representation with multiplier ω. Let $a\in A,s,t\in G,\xi\in F$. Since θ is a group homomorphism and u is a projective representation with the multiplier ω, we have $v_{st}(a\dot{\otimes}\xi)=\theta_{st}(a)\dot{\otimes}u_{st}\xi=\theta_s(a)\theta_t(a)\dot{\otimes}\omega(s,t)u_su_t\xi=\omega(s,t)\theta_s(\theta_t(a))\dot{\otimes}u_s(u_t\xi)=\omega(s,t)v_s(\theta_t(a)\dot{\otimes}u_t\xi)=\omega(s,t)v_sv_t(a\dot{\otimes}\xi)$. So we proved that v is a projective representation with multiplier ω. We verify now condition ii). Let $s\in G$ and $\xi\in F$. We have $V^*v_sV\xi=V^*v_s(1_A\dot{\otimes}\xi)=V^*(\theta_s(1_A)\dot{\otimes}u_s\xi)=V^*(1_A\dot{\otimes}u_s\xi)=\rho(1_A)u_s\xi=I_Fu_s\xi=u_s\xi$, because ρ is unital. We prove condition iii). Let $a,b\in A,s\in G,\xi\in F$. Then $v_s\pi(a)v_s^J(b\dot{\otimes}\xi)=v_s\pi(a)v_{s^{-1}}(b\dot{\otimes}\xi)=v_s\pi(a)(\theta_{s^{-1}}(b)\dot{\otimes}u_{s^{-1}}\xi)=$
$v_s(a\theta_{s^{-1}}(b)\dot{\otimes}u_{s^{-1}}\xi)=$
$\theta_s(a\theta_{s^{-1}}(b))\dot{\otimes}(u_su_{s^{-1}}\xi)=\theta_s(a)\theta_s(\theta_{s^{-1}}(b))\dot{\otimes}\overline{\omega(s,s^{-1})}u_{ss^{-1}}\xi=\theta_s(a)b\dot{\otimes}I_F\xi=$
$\theta_s(a)b\dot{\otimes}\xi=\pi(\theta_s(a))(b\dot{\otimes}\xi)$, so the (θ,v)-covariance property holds. □

3. The projective covariant version of KSGNS type representation on a Krein C^*-module associated to a unital projective covariant α-completely positive map

Definition 3.1. ([6]) *Let A and B be two C^*-algebras. A map $\rho\colon A\to B$ is a Hermitian map if $\rho(a^*)=\rho(a)^*$.*

Definition 3.2. ([6]) *Let A be a C^*-algebra and let F be a Hilbert B-module. A Hermitian map ρ defined on A into $\mathcal{L}(F)$ is called **α-completely positive** if there is a bounded Hermitian map $\alpha\colon A\to A$ such that:*

(i) *$\alpha^2=I$ (the identity map on A);*
(ii) *for any approximate unit $\{f_i\}_{i\in I}$ for A, $\{\alpha(f_i)\}_{i\in I}$ is also an approximate unit;*
(iii) *$\rho(ab)=\rho(\alpha(a)\alpha(b))=\rho(\alpha(ab))$ for any $a,b\in A$;*
(iv) *$\sum_{i,j=1}^{n}\langle x_i,\rho(\alpha(a_i)^*a_j)x_j\rangle\geq 0$ for any $n\geq 1,a_1,\ldots,a_n\in A$ and $x_1,\ldots,x_n\in F$;*

(v) *for each* $a \in A$, *there is a constant* $c(a) \geq 0$ *such that* $\rho(\alpha(a_i a)^* a a_j) \leq c(a)\rho(\alpha(a_i)^* a_j)$ *for any* $n \geq 1, a_1, \ldots, a_n \in A$;

(vi) *there is a strictly continuous positive linear map* $\phi\colon M_n(A) \to \mathcal{L}(F)$ *and a constant* $k \geq 0$ *such that* $\rho(\alpha(a)^* a) \leq k\phi(a^* a), a \in A$.
If A *is a unital* C^**-algebra with unit* ***1****, the condition (ii) is replaced by* $\alpha(\mathbf{1}) = \mathbf{1}$.

The following result is the extension to the projective case of Theorem 2.4, [7].

Theorem 3.1. *Let* (G, A, θ) *be a unital* C^**-dynamical system, let* u *be a projective unitary representation of* G *on a Hilbert* B*-module* E*. If* $\rho\colon A \to \mathcal{L}(E)$ *is a unital projective* (θ, u)*-covariant* α*-completely positive map, then there are a projective covariant* J*-representation* $(\pi, v, (F, J))$ *of* (G, A, θ) *and an isometry* $V \in \mathcal{L}(E, F)$ *such that*

(i) $\rho(a) = V^*\pi(a)V$, *for any* $a \in A$;
(ii) $\pi(\theta_s(a)) = v_s\pi(a)v_s^J$ *for any* $s \in G$ *and* $a \in A$;
(iii) $Vu_s = v_sV$, *for any* $s \in G$.

Proof. By Theorem 4.4, [6], there are a Krein B-module (F, J), a J-representation $\pi\colon A \to \mathcal{L}(F)$ and an operator $V \in \mathcal{L}(E, F)$ such that $\rho(a) = V^*\pi(a)V$, for any $a \in A$. It is enough to construct a projective J-unitary representation v of G on F satisfying (ii) and (iii). We define $v\colon G \to \mathcal{L}(F)$ by $v_s = \theta_s \otimes u_s$ on $F = A \otimes_\rho E$ (see Lemma 4.1, [6]).

For any $a_i, a'_j \in A$ and $\xi_i, \xi'_j \in E$ ($i = \overline{1,n}, j = \overline{1,m}$), using relation (4.1) in [6] and conditions (i) and (iii) in Definition 3.2, we have:

$$\left\langle v_s(\sum_{i=1}^{n} a_i \otimes \xi_i), \sum_{j=1}^{m} a'_j \otimes \xi'_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \theta_s(a_i) \otimes u_s(\xi_i), a'_j \otimes \xi'_j \right\rangle =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle u_s(\xi_i), \rho(\alpha(\theta_s(a_i)^*)a'_j)\xi'_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \xi_i, \rho(\alpha(\theta_s(a_i^*)a'_j)u_s^*(\xi'_j) \right\rangle =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \xi_i, \rho(\alpha(\theta_s(a_i^*)\alpha(\alpha(a'_j)))u_s^*(\xi'_j) \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \xi_i, \rho(\alpha(\theta_s(a_i^*\alpha(a'_j)))u_s^*(\xi'_j) \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \xi_i, \rho(\alpha(a_i^*\theta_{s^{-1}}(\alpha(a$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle \xi_i, \rho(\alpha(a_i^*)\alpha(\theta_{s^{-1}}(\alpha(a'_j))))u_s^*(\xi'_j) \right\rangle =$$
$$\left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{j=1}^{m} \alpha(\theta_{s^{-1}}(\alpha(a'_j))) \otimes u_s^*(\xi'_j) \right\rangle =$$
$$\left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, ((\alpha \circ \theta_{s^{-1}} \circ \alpha) \otimes u_s^*)(\sum_{j=1}^{m} a'_j \otimes \xi'_j) \right\rangle$$

Hence $v_s^* = (\alpha \circ \theta_{s^{-1}} \circ \alpha) \otimes u_s^*$, $s \in G$, so that $v_{s^{-1}} = v_s^J = \theta_{s^{-1}} \otimes u_s^*$, because $v_s^* = (\alpha \circ \theta_{s^{-1}} \circ \alpha) \otimes u_s^* = (\alpha \otimes I)(\theta_{s^{-1}} \otimes u_s^*)(\alpha \otimes I) = (\alpha \otimes I)(\theta_{s^{-1}} \otimes u_{s^{-1}})(\alpha \otimes I) = = Jv_{s^{-1}}J$ (by Lemma 4.1, [6]). This means that v is a J-unitary representation of G on F.

Now we show that v is a projective representation. Let $s_1, s_2 \in G$, $a_i \in A$ and $\xi_i \in E$ $(i = \overline{i,n})$. We have $v_{s_1 s_2}(\sum_{i=1}^{n} a_i \otimes \xi_i) = \sum_{i=1}^{n} \theta_{s_1 s_2}(a_i) \otimes u_{s_1 s_2}(\xi_i) =$
$\sum_{i=1}^{n} \theta_{s_1}(\theta_{s_2}(a_i)) \otimes \omega(s_1, s_2) u_{s_1}(u_{s_2}(\xi_i)) = \sum_{i=1}^{n} \omega(s_1, s_2) \theta_{s_1}(\theta_{s_2}(a_i)) \otimes u_{s_1}(u_{s_2}(\xi_i)) =$
$\omega(s_1, s_2) \sum_{i=1}^{n} \theta_{s_1}(\theta_{s_2}(a_i)) \otimes u_{s_1}(u_{s_2}(\xi_i)) = \omega(s_1, s_2) v_{s_1}(\sum_{i=1}^{n} \theta_{s_2}(a_i) \otimes u_{s_2}(\xi_i)) = \omega(s_1, s_2) v_{s_1} v_{s_2}(\sum_{i=1}^{n} a_i \otimes \xi_i)$. So, v is projective.

Since $V(y) = 1 \otimes y$ (see Remark 4.5, [6]) and ρ is unital, we have $v_s V(\sum_{i=1}^{n} a_i \otimes \xi_i) = v_s(1 \otimes \sum_{i=1}^{n} a_i \otimes \xi_i) = \theta_s(1) \otimes u_s(\sum_{i=1}^{n} a_i \otimes \xi_i) = 1 \otimes u_s(\sum_{i=1}^{n} a_i \otimes \xi_i) =$
$V(u_s(\sum_{i=1}^{n} a_i \otimes \xi_i)) = V u_s(\sum_{i=1}^{n} a_i \otimes \xi_i)$, so that $v_s V = V u_s$.

For all $s \in G$, $a_i \in A$ and $\xi_i \in E$ $(i = \overline{i,n})$, let $y = \sum_{i=1}^{n} a_i \otimes \xi_i$ and we have, by relation (4.5), [6], that $\pi(\theta_s(a))(y) = \sum_{i=1}^{n} \theta_s(a) a_i \otimes \xi_i =$
$\sum_{i=1}^{n} \theta_s(a) \theta_s(\theta_{s^{-1}}(a_i)) \otimes u_s u_{s^{-1}}(\xi_i) = \sum_{i=1}^{n} \theta_s(a \theta_{s^{-1}}(a_i)) \otimes u_s u_{s^{-1}}(\xi_i) =$
$v_s(\sum_{i=1}^{n} a \theta_{s^{-1}}(a_i) \otimes u_{s^{-1}}(\xi_i)) = v_s \pi(a)(\sum_{i=1}^{n} \theta_{s^{-1}}(a_i) \otimes u_{s^{-1}}(\xi_i)) =$
$v_s \pi(a) v_{s^{-1}}(\sum_{i=1}^{n} a_i \otimes \xi_i) = v_s \pi(a) v_s^J(y)$. Hence we proved (ii). □

Let F, L be two Hilbert B-modules. Then $\mathcal{L}(F, L)$ can be regarded as a Hilbert $\mathcal{L}(F)$-module with the following operations

(i) $\mathcal{L}(F, L) \times \mathcal{L}(F) \ni (T, S) \longmapsto TS \in \mathcal{L}(F, L)$

(ii) $\mathcal{L}(F, L) \times \mathcal{L}(F, L) \ni (T_1, T_2) \longmapsto \langle T_1, T_2 \rangle = T_1^* T_2 \in \mathcal{L}(F)$

Definition 3.3. *([7]) Let E be a Hilbert A-module and let $\rho\colon A \to \mathcal{L}(F)$ be a linear map. A linear map $\phi\colon E \to \mathcal{L}(F, L)$ is called a **ρ-map** if $\langle \phi(x), \phi(y) \rangle = \rho(\langle x, y \rangle)$, $x, y \in E$.*

If (F, J_F) is a Krein B-module and ρ is a nondegenerate J_F-representation, then a ρ-map $\phi\colon E \to \mathcal{L}(F, L)$ is linear and satisfies the relation $\phi(xa) = \phi(x) J_F \rho(a) J_F$, for any $x \in E$ and $a \in A$.

Definition 3.4. *([7]) Let (G, A, θ) be a C^*-dynamical system and let (E, J_E) be a Krein A-module. A group homomorphism $\tau\colon G \to \mathcal{U}_{J_E}(E)$ such that for any $s \in G, a \in A$ and $x, x' \in E$, we have*

(i) $\tau_s(xa) = \tau_s(x) \theta_s(a)$

(ii) $\langle \tau_s(x), \tau_s(x') \rangle_{J_E} = \theta_s(\langle x, x' \rangle_{J_E})$
is called a θ-***compatible action*** *of* G *on* (E, J_E).

Definition 3.5. *Let* (F, J_F) *and* (L, J_L) *be Krein* B*-modules. For a* θ*-compatible action* τ *of* G *on* (E, J_E) *and a map* $\phi\colon E \to \mathcal{L}(F, L)$, *if there are a projective* J_F*-unitary representation* $v\colon G \to \mathfrak{U}_{J_F}(F)$ *and a projective* J_L*-unitary representation* $w\colon G \to \mathfrak{U}_{J_L}(L)$ *such that* $\phi(\tau_s(x)) = w_s \phi(x) v_s^{J_F}$ *for any* $x \in E$ *and* $s \in G$, *then* ϕ *is called* ***projective*** (τ, w, v)***-covariant***.

Proposition 3.1. *Let* E *be a Hilbert* A*-module and let* F, L *be Hilbert* B*-modules. If* $\rho\colon A \to \mathcal{L}(F)$ *is a unital projective* (θ, u)*-covariant* α*-completely positive map and if* $\phi\colon E \to \mathcal{L}(F, L)$ *is a projective* (τ, w, u)*-covariant* ρ*-map such that the closure* $[\phi(E)F]$ *is orthogonal complemented in* L, *then there is a pair* $((\pi, V, (E', J)), (\Pi, W, F'))$ *such that*

(i) (E', J) *is a Krein* B*-module and* F' *is a Hilbert* B*-module;*
(ii) $\pi\colon A \to \mathcal{L}(E')$ *is a* J*-representation;*
(iii) $\Pi\colon E \to \mathcal{L}(E', F')$ *is a* $J \circ \pi$*-map;*
(iv) $V \in \mathcal{L}(F, E')$ *is an isometry and* $W \in \mathcal{L}(L, F')$ *is a projection satisfying the conditions (i)-(iii) in Theorem 4.4,* [6] *and* $\phi(x) = W^*\Pi(x)V$, *for all* $x \in E$

Moreover, there are a projective J*-unitary representation* v *and a map* $w'\colon G \to \mathfrak{U}(F')$ *such that*

(1) $(\pi, v, (E', J))$ *is a projective covariant* J*-representation of* (G, A, θ)*;*
(2) Π *is projective* (τ, w', v)*-covariant.*

Proof. The proof follows the proof of Theorem 3.2, [7] and Theorem 3.1. □

Remark 3.1. *A pair* $((\pi, V, (E', J)), (\Pi, W, F'))$ *satisfying conditions (i) and (ii) in Theorem 4.4,* [6] *and* $\phi(x) = W^*\Pi(x)V$, $x \in E$ *is called a* ***KSGNS type representation*** *for a pair* (ρ, ϕ). *Such a representation is said* ***to be minimal*** *if* $E' = [\pi(A)V(F)]$ *and* $F' = [\phi(E)(F)]$. *The pair* $((\pi, V, (E', J)), (\Pi, W, F'))$ *constructed in the Corollary 3.1 is minimal and it is unique up to a unitary equivalence, by Theorem 3.5,* [7].

4. Extension on the crossed product $A \times_\theta G$ of a projective covariant α-completely map

Let G be a locally compact group with left Haar measure dt. By uniqueness of left Haar measure, there is a function $\Delta\colon G \to (0, \infty)$ such that $d(ts) = \Delta(s)dt$ and $d(t^{-1}) = \Delta(t)^{-1}dt$. Let (G, A, θ) be a unital C^*-dynamical system and let $C_c(G, A)$ be the set of all continuous functions from G into A with compact support. The set $C_c(G, A)$ is a linear space with the multimplication, involution and norm of $C_c(G, A)$ as follows: $(f \star g)(s) = \int_G f(t)\theta_t(g(t^{-1}s))dt$, $f^*(s) = \Delta(s)^{-1}[\theta_s(f(s^{-1}))]^*$ and $\|f\|_1 = \int_G \|f(t)\|dt$. The completion of $C_c(G, A)$ with respect to $\|\cdot\|_1$ becomes a Banach $*$-algebra denoted by $L^1(G, A)$. We define a norm on $L^1(G, A)$ by $\|f\| = \sup_{\pi}\|\pi(f)\|$, where π ranges over all Hilbert space representations of $L^1(G, A)$. This norm $\|\cdot\|$ becomes a C^*-norm. The completion of $L^1(G, A)$ with respect to this norm is called the **crossed product of** A **by** G, denoted by $A \times_\theta G$. Let E be a Hilbert A-module.

The linear space $C_c(G,E)$ is a pre-Hilbert $A\times_\theta G$-module with the action of $A\times_\theta G$ on $C_c(G,E)$ and the inner product given by $(\xi\cdot f)(s)=\int_G \xi(t)\theta_t(f(t^{-1}s))dt,\ \xi\in C_c(G,E),\ f\in C_c(G,A)$ and $\langle\xi,\xi'\rangle(s)=\int_G\theta_{t^{-1}}(\langle\xi(t),\xi'(ts)\rangle)dt,\ \xi,\xi'\in C_c(G,E)$. The **crossed product** $E\times_\tau G$ **of** E **by** G is defined by completion of $C_c(G,E)$ with respect to the inner product. Then $E\times_\tau G$ becomes a Hilbert $A\times_\theta G$-module (see Proposition 3.5, [4]). Let E be a Hilbert A-module and let F,L be Hilbert B-modules. Suppose that $\rho\colon A\to\mathcal{L}(F)$ is a unital projective (θ,u)-covariant α-completely positive linear map and that $\phi\colon E\to\mathcal{L}(F,L)$ is a projective (τ,w,u)-covariant ρ-map such that $[\phi(E)F]$ is orthogonal complemented in L. By Corollary 3.1, there is a KSGNS type representation $((\pi,V,(E',J)),(\Pi,W,F'))$ for a pair (ρ,ϕ) such that $(\pi,v,(E',J))$ is a projective covariant J-representation of (G,A,θ) and Π is projective (τ,w',v)-covariant. The bounded maps $\pi\times v\colon C_c(G,A)\to\mathcal{L}(E')$ and $\Pi\times v\colon C_c(G,E)\to\mathcal{L}(E',F')$ defined by $(\pi\times v)(f)=\int_G\pi(f(s))v_sds,\ f\in C_c(G,A)$ and $(\Pi\times v)(\xi)=\int_G\Pi(\xi(s))v_sds,\ \xi\in C_c(G,E)$ can be extended to $A\times_\theta G$ and $E\times_\tau G$, respectively and the extensions are denoted by the same symbols. For each $f\in C_c(G,A)$ and $x\in E$, we have that $(\Pi\times v)(\xi)=\int_G\Pi(xf(s))v_sds=\Pi(x)(\pi\times v)(f),\ \xi=xf\in C_c(G,E)$ and $((\Pi\times v)(\xi))^*=((\pi\times v)(f))^*(\Pi(x))^*$. We obtain that the closed linear span $[(\Pi\times v)(E\times_\tau G)E']$ is equal to the Hilbert B-module F', which means that $\Pi\times v$ is nondegenerate. We define the maps $\widetilde{\alpha}\colon A\times_\theta G\to A\times_\theta G$ and $\widetilde{\rho}\colon A\times_\theta G\to\mathcal{L}(F)$ by $\widetilde{\alpha}(f)(s)=\alpha(f(s))$ and $\widetilde{\rho}(f)=V^*(\pi\times v)(f)V,\ f\in C_c(G,A),\ s\in G$. We define a group action $\widetilde{\theta}$ of G on $A\times_\theta G$ by $(\widetilde{\theta}_t(f))(s)=\theta_t(f(t^{-1}st)),\ f\in C_c(G,A)$, where π and V are given by Theorem 4.4, [6] and v is given as in Theorem 3.1.

Theorem 4.1. *If α and θ are equivariant, i.e. $\theta_s\circ\alpha=\alpha\circ\theta_s$ $(s\in G)$, $\alpha(0)=0$, u is a projective J-unitary representation with the multiplier ω, which satisfies $\omega(s,t)=\omega(t,s)$, for all $s,t\in G$ and $\rho\colon A\to\mathcal{L}(F)$ is a unital projective (θ,u)-covariant α-completely positive linear map, then*

1. *$\widetilde{\rho}$ is a projective $(\widetilde{\theta},u)$-covariant $\widetilde{\alpha}$-completely positive map such that $\widetilde{\rho}(f)=\int_G\rho(f(s))u_sds,\ f\in C_c(G,A)$*
2. *the triple $(\pi\times v,v,(E',J))$ is a projective covariant J-representation of $(A\times_\theta G,G,\widetilde{\theta})$.*

Proof. (1) Following the proof of Theorem 4.2, [7] and applying Theorem 3.1, we get that $\widetilde{\rho}$ is $\widetilde{\alpha}$-completely positive map and $\widetilde{\rho}(f)=\int_G\rho(f(s))u_sds$.

We prove now that $\widetilde{\rho}$ is projective $(\widetilde{\theta},u)$-covariant.

Let $f\in C_c(G,A)$ and $t\in G$. Then $\widetilde{\rho}(\widetilde{\theta}_t(f))=\int_G\rho(\widetilde{\theta}_t(f)(s))u_sds=$
$\int_G\rho(\theta_t(f(t^{-1}st)))u_sds=\int_G u_t\rho(f(t^{-1}st))u_t^Ju_sds=\int_G u_t\rho(f(t^{-1}st))u_{t^{-1}}u_sds=$
$\int_G u_t\rho(f(t^{-1}st))u_{t^{-1}}u_su_{tt^{-1}}ds=\int_G u_t\rho(f(t^{-1}st))u_{t^{-1}}u_s\omega(t,t^{-1})u_tu_{t^{-1}}ds=$
$\int_G u_t\rho(f(t^{-1}st))\overline{\omega(t^{-1},s)}\omega(t,t^{-1})u_{t^{-1}s}u_tu_t^Jds=$
$\int_G u_t\rho(f(t^{-1}st))\overline{\omega(t^{-1},s)}\omega(t,t^{-1})\overline{\omega(t^{-1}s,t)}u_{t^{-1}st}u_t^Jds=$
$u_t(\int_G\rho(f(t^{-1}st))\overline{\omega(t^{-1},s)}\omega(t,t^{-1})\overline{\omega(t^{-1}s,t)}u_{t^{-1}st}ds)u_t^J=u_t\widetilde{\rho}(f)u_t^J$,
because $\overline{\omega(t^{-1},s)\omega(t^{-1}s,t)\omega(t,t^{-1})}=\overline{\omega(s,t^{-1})\omega(t,t^{-1}s)\omega(t,t^{-1})}=$
$\overline{\omega(s,t^{-1})}\omega(t^{-1},s)\overline{\omega(t,t^{-1})\omega(tt^{-1},s)}\omega(t,t^{-1})=|\omega(t,t^{-1})|^2\omega(e,s)|\omega(t^{-1},s)|^2=1$

(2) Let $t \in G$ and $f \in C_c(G, A)$. Since π is projective (θ, v)-covariant, we have by (ii), Theorem 3.1 :

$$(\pi \times v)(\widetilde{\theta}_t(f)) = \int_G \pi(\widetilde{\theta}_t(f)(s))v_s ds = \int_G \pi(\theta_t(f(t^{-1}st)))v_s ds =$$
$$\int_G v_t \pi(f(t^{-1}st))v_t^J v_s ds = \int_G v_t \pi(f(t^{-1}st))v_{t^{-1}} v_s v_{tt^{-1}} ds =$$
$$\int_G v_t \pi(f(t^{-1}st))v_{t^{-1}} v_s \omega(t, t^{-1}) v_t v_{t^{-1}} ds =$$
$$\int_G v_t \pi(f(t^{-1}st))\overline{\omega(t^{-1}, s)} v_{t^{-1}s} \omega(t, t^{-1}) v_t v_t^J ds =$$
$$\int_G v_t \pi(f(t^{-1}st))\overline{\omega(t^{-1}, s)} \omega(t, t^{-1}) \overline{\omega(t^{-1}s, t)} v_{t^{-1}st} v_t^J ds =$$
$$\int_G v_t \pi(f(t^{-1}st)) v_{t^{-1}st} v_t^J ds = v_t (\pi \times v)(f) v_t^J$$

Hence $\pi \times v$ is $(\widetilde{\theta}, v)$-covariant. By Corollary 3.1 and Theorem 4.2, [7], we proved that $(\pi \times v, v, (E', J))$ is a projective covariant J-representation of $(A \times_\theta G, G, \widetilde{\theta})$. □

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